数理解析研究所講究録
1494 巻 2006 年 9-12

On critical orbits of holomorphic maps in the complex projective plane

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This note is the abstract of my talk in the conference held at RIMS, January 2006.

1 Introduction

We study the dynamics of holomorphic self-maps of the complex projective plane. We introduce a concept of complete saddle structure and show some properties of maps which possess it. Besides, we deal with maps with its critical limit set pluripolar. For such maps, without any hyperbolicity condition, it is shown that the support of the measure of maximal entropy equals the closure of the set of repelling periodic points.

2 Complete saddle structure

2.1 Preliminaries

We denote the complex projective plane by $\mathbb{P}^2$ and the normalized Fubini-Study form by ω. Let f be a holomorphic self-map of $\mathbb{P}^2$. We define the degree of f by $d := \int f^*\omega \wedge \omega$. We suppose that $d \geq 2$. Then, f is an analytic covering and the topological degree is d^2. The critical set C is of degree $3d-3$.

We consider a natural invariant positive closed (1,1) current T which is obtained by

$$T := \lim_{n\to\infty} \frac{1}{d^n}(f^*)^n\omega$$

where we take the limit in the sense of currents ([S]). Since T is of the form

$$T = \omega + dd^c u$$

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where u is a continuous function in $\mathbb{P}^2$, it follows that $||T|| = \int T \wedge \omega = 1$. By construction, $f^*T = dT$. We call T Green (1,1) current. Since T has locally continuous potentials, we can obtain an invariant probability measure $\mu := T \wedge T$.

Definition 2.1. We set $\mathcal{J}_1 := \mathrm{supp}(T)$ and $\mathcal{J}_2 := \mathrm{supp}(\mu)$. They are called *the 1st Julia set* and *the 2nd Julia set* respectively. We denote by $\mathcal{F}$ the maximal open subset in $\mathbb{P}^2$ on which the sequence $\{f^n\}_{n\geq 1}$ is normal in the sense of Montel. We call $\mathcal{F}$ *the Fatou set* and the complement $\mathcal{J}$ of $\mathcal{F}$ *the Julia set.*

The current T is extremely related to the normality of $\{f^n\}_{n\geq 1}$. The following theorem is fundamental.

Theorem 2.2. *([FS],[U1])* $\mathcal{J}_1 = \mathcal{J}$

In this study, we focus on the following set:

$$\Lambda := \{x \in \mathbb{P}^2 : \text{there is a sequence of points } \{x_{-i}\}_{i\geq 0}$$
$$\text{such that } x_0 = x,\ f(x_{-i}) = x_{-i+1} \text{ and } x_{-i} \in \mathcal{J} \cap \mathcal{PC}_\infty \cap \Omega \text{ for all i} \geq 0\},$$

where Ω denote the nonwandering set for f and

$$\mathcal{PC}_\infty := \bigcap_{n\geq 1} \overline{\bigcup_{k\geq n} f^k(C)}.$$

We can show that Λ is compact and surjectively forward invariant by f.

Definition 2.3. We say that f has *complete saddle structure* if Λ is a hyperbolic set with unstable dimension one and has local product structure.

Let us see an example. When f is Axiom A, we denote by Ω_i the set of nonwandering points which have unstable dimension i.

Proposition 2.4. *Let f be a holomorphic self-map of $\mathbb{P}^2$ of degree at least 2. If f is Axiom A and $\mathcal{PC}_\infty \cap \Omega_2 = \emptyset$, then f has complete saddle structure.*

It follows that when Ω_2 is completely invariant, f has complete saddle structure.

2.2 The Fatou set

If f has complete saddle structure, we can get an information about the Fatou set.

Theorem 2.5. *Let f be a holomorphic self-map of $\mathbb{P}^2$ of degree at least 2. Suppose that f has complete saddle structure. Then, the Fatou set $\mathcal{F}$ for f consists of the basins of attraction for finitely many attracting periodic orbits.*

Corollary 2.6. *Let f be the same as in Theorem 2.5. Then, all Fatou components are taut.*

2.3 The unstable manifolds

By the following theorem, when f satisfies the condition in Proposition 2.4, we can draw the unstable manifold for Ω_1 by chasing the critical orbit.

Theorem 2.7. *Let f be a holomorphic self-map of $\mathbb{P}^2$ of degree at least 2. Suppose that f has complete saddle structure. Then,*

$$\mathcal{PC}_\infty = \{\text{attracting periodic points}\} \cup W^u(\Lambda).$$

3 Sparse critical orbits and repelling periodic points

It is known that for any holomorphic self-map of $\mathbb{P}^2$ of degree at least 2, the repelling periodic points are dense in $\mathcal{J}_2$. However, in general, they need not be contained in $\mathcal{J}_2$. So here we consider a sufficient condition by which they are contained in $\mathcal{J}_2$.

Theorem 3.1. *Let f be a holomorphic self-map of $\mathbb{P}^2$ of degree at least 2. Suppose that $\mathcal{PC}_\infty$ is pluripolar. Then,*

$$\mathcal{J}_2 = \overline{\{\text{repelling periodic points}\}}.$$

By this, in case when f is Axiom A and $\mathcal{PC}_\infty$ is pluripolar, Ω_2 is completely invariant because it coincides with $\mathcal{J}_2$.

4 Critically finite maps and Axiom A

At present, the relation between Axiom A and complete saddle structure is not clear. So here we shall examine it in a special case. We say that a holomorphic map f is *critically finite* if each irreducible component of C is eventually periodic.

Theorem 4.1. *Let f be a critically finite map. Then, the following $(a) - (c)$ are equivalent.*

(a) f is Axiom A.

(b) f has complete saddle structure.

(c) $\mathcal{J} \cap \mathcal{PC}_\infty$ is a hyperbolic set with unstable dimension one.

If one of these is satisfied, each irreducible component of $\mathcal{PC}_\infty$ is in a critical cycle.

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