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Semi-hyperbolicity of entire functions

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Abstract

In this paper, we investigate a condition for semi-hyperbolicity of (transcendental) entire functions (Theorem A). As an application of the main theorem, we show a result on a measure theoretical property for the dynamics of entire functions (Theorem B). In particular, we give a sufficient condition which guarantees that $\{\infty\}$ is a metric global attractor (Corollary C).

1 Preliminaries

Let f be an entire function and f^n denote the n-th iterate of f. Recall that the *Fatou set* F_f and the *Julia set* J_f of f are defined as follows:

$$\begin{aligned} F_f &:= \{z \in \mathbb{C} \mid \{f^n\}_{n=1}^{\infty} \text{ is a normal family in a neighborhood of } z\}, \\ J_f &:= \mathbb{C} \setminus F_f. \end{aligned}$$

By definition, F_f is open and J_f is closed in $\mathbb{C}$. Also J_f is compact if f is a polynomial, while it is non-compact if f is transcendental. This is due to the fact that ∞ is an essential singularity of f. A connected component U of F_f is called *a Fatou component of* f. U is called a *wandering domain* if $f^m(U) \cap f^n(U) = \emptyset$ for every $m, n \in \mathbb{N}$ $(m \neq n)$. If there exists an $n_0 \in \mathbb{N}$ with $f^{n_0}(U) \subseteq U$, U is called a *periodic component of period* n_0 and it is well known that there are four possibilities, namely, an *attracting basin*, a *parabolic basin*, a *Siegel disk* and a *Baker domain*.

A *critical value* is a point $p := f(c)$ for a point c with $f'(c) = 0$. This is a singularity of f^{-1}. For polynomials we have only to consider this type of singularities but there can be another type of singularities called an *asymptotic value* for transcendental entire functions. A point p is called an *asymptotic value* if there exists a continuous curve $L(t)$ $(0 \le t < 1)$ (which is called an *asymptotic path*) with

$$\lim_{t\to 1} L(t) = \infty \quad \text{and} \quad \lim_{t\to 1} f(L(t)) = p.$$

A point p is called a *singular value* if it is either a critical or an asymptotic value and we denote the set of all singular values by $\mathrm{sing}(f^{-1})$. Also we define

$$P(f) := \overline{\bigcup_{n=0}^{\infty} f^n(\mathrm{sing}(f^{-1}))}$$

and call it the *post-singular set* of f.

The following are some basic concepts from dynamical system theory:

Definition 1.1. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function and $z \in \mathbb{C}$.

(1) The *forward orbit* of a point z is the set

$$O^+(z) := \{z,\ f(z),\ \cdots, f^n(z), \cdots\}.$$

(2) We define

$$\omega(z) := \{w \mid w = \lim_{n_i \nearrow \infty} f^{n_i}(z),\ {}^{\exists}n_1 < n_2 < \cdots\}$$

and call it the *ω-limit set of z.*

(3) A point z is called *recurrent* if $z \in \omega(z)$, that is, the forward orbit of z passes through an arbitrary small neighborhood of z infinitely often. Otherwise, it is called *non-recurrent.*

(4) f is called *ergodic* if any measurable set A satisfying $f^{-1}(A) = A$ has zero or full measure in $\mathbb{C}$.

2 The Mañé's Theorem —Semi-hyperbolicity—

The following is a part of the Mañé's theorem, which was proved in 1993.

Theorem 2.1 (Mañé, [M]). *Let f be a rational function and $x \in J_f$. Suppose that*

(i) *x is not a parabolic periodic point and*

(ii) $x \notin \bigcup_{c \in \mathrm{Rec} \cap J_f} \omega(c)$,

where

$$\mathrm{Rec} = \{\textit{recurrent critical points of } f\}.$$

Then for every $\varepsilon > 0$, there exists a neighborhood U of x which satisfies the following:

(1) *For every $n \in \mathbb{N}$ and every connected component V of $f^{-n}(U)$,*

$$\mathrm{diam}_{\mathrm{sph}}\ (V) \leq \varepsilon$$

holds, where $\mathrm{diam}_{\mathrm{sph}}$ *denotes the spherical diameter on $\widehat{\mathbb{C}}$.*

(2) *There exists an $N \in \mathbb{N}$ such that for any connected component V of $f^{-n}(U)$ $({}^{\forall}n)$, $f^n|_V : V \to U$ satisfies*

$$\deg(f^n|_V : V \to U) \leq N.$$

Taking this result into account, we define the semi-hyperbolicity of f at a point $x_0 \in J_f$ as follows:

Definition 2.2. f is *semi-hyperbolic at $x \in J_f$* if there exists a neighborhood U of x such that the condition (2) in Theorem 2.1 holds. In the case that f is transcendental, we add the following property:

$$f^n|_V : V \to U \text{ is proper for every } V.$$

Recall that $f : X \to Y$ is called *proper* if $f^{-1}(K) \subset X$ is compact for every compact subset $K \subset Y$. Note that this property is automatically satisfied when f is a polynomial or rational. We say f *is semi-hyperbolic* if f is semi-hyperbolic at any point $x_0 \in J_f$.

The converse of Theorem 2.1 is also true. That is, if x is a parabolic periodic point or $x \notin \bigcup_{c \in \mathrm{Rec} \cap J_f} \omega(c)$, then f is not semi-hyperbolic at $x \in J_f$. In this paper we investigate a condition for semi-hyperbolicity for transcendental entire functions. In transcendental case, a new phenomena can occur. For example, Bergweiler and Morosawa ([**BM**]) constructed an example of f with no parabolic periodic point and no recurrent critical point, but has a point $x_0 \in J_f$ at which f is not semi-hyperbolic.

3 Main Result

Define the sets Rec, Non-Rec and AV as follows:

$$\begin{aligned}\text{Rec} &:= \{c \mid c \text{ is a recurrent critical point of } f\}\\ \text{Non-Rec} &:= \{c \mid c \text{ is a non-recurrent critical point of } f\}\\ \text{AV} &:= \{c \mid c \text{ is an asymptotic value of } f\}.\end{aligned}$$

Then the main result of this paper is the following:

Theorem A (Mañé's Theorem for entire functions). *Let f be a (transcendental) entire function and $z_0 \in J_f$. Then f is semi-hyperbolic at z_0 if and only if $z_0 \notin Z$, where the set Z is defined as follows:*

$$Z = \overline{\left(\bigcup_{i=1}^{3} X_i\right) \cup \left(\bigcup_{j=1}^{5} Y_j\right)},$$

where

$$\begin{aligned}
X_1 &= \overline{\{p \mid p \text{ is a parabolic periodic point of } f\}},\\
X_2 &= \text{derived set of } \{p \mid p \text{ is a attracting periodic point of } f\},\\
X_3 &= \{p \mid f^{n_i}|_W \to p\ (n_i \to \infty) \text{ for some wandering domain } W\},\\
Y_1 &= \overline{\bigcup_{c \in \text{Rec} \cap J_f} \omega(c)}, \quad Y_2 = \overline{\bigcup_{n=0}^{\infty} f^n(\text{AV}) \cap J_f},\\
Y_3 &= \{p \mid p = \lim_{i \to \infty} f^{n_i}(c_i),\ c_i \in \text{Non-Rec} \cap J_f\ (i \in \mathbb{N}) \text{ are mutually}\\
&\qquad \text{different and order of } c_i \to \infty\ (i \to \infty)\},\\
Y_4 &= \Big\{p \mid p = \lim_{i \to \infty} f^{n_i}(c_i),\ c_i \in \text{Non-Rec} \cap J_f\ (i \in \mathbb{N}) \text{ are mutually}\\
&\qquad \text{different with } \sup_i (\text{order of } c_i) < \infty \text{ and for any } \varepsilon > 0\\
&\qquad \text{let } N_i(\varepsilon) := {}^{\#}\{c \mid c : \text{critical point}, O^+(c_i) \cap U_\varepsilon(c) \neq \emptyset\}\\
&\qquad \text{then } \sup_i N_i(\varepsilon) = \infty\Big\},\\
Y_5 &= \Big\{p \mid p = \lim_{i \to \infty} f^{n_i}(c_i),\ c_i \in \text{Non-Rec} \cap J_f\ (i \in \mathbb{N}) \text{ are mutually}\\
&\qquad \text{different with } \sup_i (\text{order of } c_i) < \infty \text{ and let}\\
&\qquad \delta_i(n) := \sup\{\delta \mid {}^{\#}\{O^+(c_i) \cap (U_\delta(c_i) \setminus \{c_i\})\} \leq n\}\\
&\qquad \text{then } \inf_i \delta_i(n) = 0 \text{ for } {}^{\forall}n\Big\}.
\end{aligned}$$

4 Outline of the proof of Theorem A

Suppose $z_0 \in J_f$, $z_0 \notin Z$, then take a neighborhood U of z_0 with $\overline{U} \cap Z = \emptyset$.

Definition 4.1. For $z \in U$ let $S(z,\varepsilon)$ be a square centered at z with side length 2ε and with sides parallel to coordinate axes. We say $S(z,\varepsilon)$ is *admissible* if $S(z,3\varepsilon) \subset U$.

Lemma 4.2. *For a given $\varepsilon > 0$ and an $N \in \mathbb{N}$, there exists a $\delta > 0$ which satisfies the following: If $S(z,\delta)$ is an admissible square and S_n is a connected component of $f^{-n}(S(z,\delta))$ such that* $\deg(f^n|_{S_n}) \leq N$, *then*

$$\operatorname{diam}(f^{-n}(S(z,\frac{\delta}{2}))) \leq \varepsilon$$

holds for the same branch of f^{-n}.

(Proof of Lemma 4.2) : Suppose not, then there exist a $z_l \in U$ and admissible squares $S^l := S(z_l, 2^{-l})$ such that for some component V_l of $f^{-n_l}(S(z_l, 2^{-(l+1)})$ it holds that $\operatorname{diam} V_l \geq \varepsilon > 0$ and $\deg(f^{n_l}|_{S(z_l,2^{-l})}) \leq N$.

Now suppose there exist a subsequence $l_k \nearrow \infty$ and a disk $D_{l_k} \subset V_{l_k}$ with (spherical) radius $r > 0$ which is independent of l_k. Taking subsequence, if necessary, we have

$$D_{l_k} \to {}^{\exists}D \quad (k \to \infty).$$

Then $\{f^{n_{l_k}}|_D\}_{k=1}^{\infty}$ is bounded, since $f^{n_{l_k}}(D) \subset U$. Hence $\{f^{n_{l_k}}|_D\}_{k=1}^{\infty}$ is normal. So we have $D \subset F_f$ and let $D_{F_f} \supset D$ be the Fatou component containing D. On the other hand, taking subsequence, if necessary, we have

$$S^{l_k} \to {}^{\exists}z_{\infty} \in U \quad (k \to \infty).$$

Then

$$f^{n_{l_k}}|_D \to z_{\infty}.$$

Such a z_{∞} is either one of the following:

(i) attracting periodic point,

(ii) parabolic periodic point,

(iii) finite constant limit function on a wandering domain.

In other words, D_{F_f} is not a Siegel disk or a Baker domain. This is a contradiction by the assumption. Hence let D_l be the maximal disk in V_l, then it follows that $\operatorname{diam}(D_l) \to 0$. This again contradicts the following

Lemma 4.3 (cf. Carleson-Jones-Yoccoz, [CJY]). *Let $W \subset \mathbb{C}$ be a simply connected domain and let $g : W \to \mathbb{D}$, $g(\partial W) \subset \partial \mathbb{D}$ be degree N. Then there exists a constant $C > 0$ depending only on N such that*

$$B_{\mathbb{D}}(g(z), Cr) \subset g(B_W(z,r)) \subset B_{\mathbb{D}}(g(z), r).$$

□

Now since $z_0 \notin Z$, there is a neighborhood U of z_0 satisfying

(0) U does not contain attracting periodic points, parabolic periodic points, wandering domains, points in orbits of recurrent critical points or asymptotic values.

Moreover, U satisfies either one of the following:

(1) The number of critical points with $O^+(c) \cap U \neq \emptyset$ is finite (let us denote them by $c_1, c_2, \cdots, c_{N_0}$) and all of them are non-recurrent. Then for some $\varepsilon_0 > 0$ we have

$$(O^+(c_i) \setminus \{c_i\}) \cap U_{\varepsilon_0}(c_i) = \emptyset.$$

(2) The number of critical points with $O^+(c) \cap U \neq \emptyset$ is infinite (let us denote them by $c_1, c_2, \cdots$) and all of them are non-recurrent. There exists an $M_0 > 0$ such that

$$\text{order of } c_i \leq M_0, \text{ for } {}^\forall i \in \mathbb{N}.$$

Also there exists an $\varepsilon_1 > 0$ and an $N_0 \in \mathbb{N}$ such that

$${}^\#\{c \mid c : \text{critical point, } O^+(c_i) \cap U_{\varepsilon_1}(c) \neq \emptyset\} \leq N_0 < \infty$$

holds for every $i \in \mathbb{N}$. Furthermore there exists a $\delta_1 > 0$ and an $n_1 \in \mathbb{N}$ such that

$${}^\#\{O^+(c_i) \cap (U_{\delta_1}(c_i) \setminus \{c_i\})\} \leq n_1, \ {}^\forall i \in \mathbb{N}.$$

In this case, we put $\varepsilon_0 := \min(\varepsilon_1, \delta_1)$

Now let $N := (M_0 + 1)^{N_0(n_1+1)}$ and take $\varepsilon > 0$ with $\varepsilon < \varepsilon_0/36N$. Then there is a $\delta > 0$ which is determined by the previous Lemma 4.2.

Lemma 4.4. *For any η with $0 < \eta \leq \delta$ and $n \in \mathbb{N}$, we have*

$$\operatorname{diam}(f^{-n}(S(z_0, \frac{1}{2}\eta))) \leq \varepsilon.$$

That is, the conclusion of Lemma 4.2 holds without the assumption on degree. □

Hence for any $\varepsilon > 0$ with $\varepsilon < \varepsilon_0/36N$ by taking $\sigma > 0$ sufficiently small, we have

$$\operatorname{diam}(f^{-n}(S(z_0, \sigma))) \leq \varepsilon, \ {}^{\forall}n.$$

With a little more argument, we can conclude

$$\deg(f^n|_{S(z_0,\sigma)}) < N = (M_0 + 1)^{N_0(n_1+1)}.$$

For the opposite implication, it is rather easy to check that $z_0 \in Z$ implies that f is not semi-hyperbolic at z_0. □

Remark. (1) Comparing Theorem A with the original Mañé's Theorem, in the case that f is rational, we have

$$Z = X_1 \cup Y_1$$

i.e. X_2, X_3, Y_2, Y_3, Y_4, Y_5 are all empty.

(2) Theorem A includes the following result:

Theorem 4.5 (Bergweiler-Morosawa (2002)). *Let f be entire. If f is semi-hyperbolic at $a \in \mathbb{C}$, then a is not a limit function of $\{f^n\}_{n=1}^{\infty}$ in any component of F_f.*

(3) Consider the following question:

Question : For each X_i $(i = 1 \sim 3)$ and Y_j $(j = 1 \sim 5)$, is there an f with $X_i \neq \emptyset$ or $Y_j \neq \emptyset$?

First, there are a lot of f with $X_1 \neq \emptyset$. But I do not know whether parabolic periodic points can accumulate to a finite point in $\mathbb{C}$. It is somehow surprising that there is an f with $X_2 \neq \emptyset$. We can construct such an example by using the similar method in [**KS**]. We omit the details. For X_3, Eremenko and Lyubich ([**EL**]) constructed an f with $X_3 \neq \emptyset$, that is, f has a wandering domain with (infinitely many) finite constant limit functions.

There are a lot of f with $Y_1 \neq \emptyset$ or $Y_2 \neq \emptyset$. It is not difficult to construct an f with $Y_3 \neq \emptyset$. For Y_4, Bergweiler and Morosawa ([**BM**]) showed the following example: Consider

$$f(z) = \frac{z}{2} - \frac{1}{2\pi}\sin \pi z + c(\cos \pi z - 1),$$

where $c = 0.467763\cdots$ is a solution of

$$\pi + 2\cos 2c\pi - 4c\pi \sin 2c\pi = 0.$$

Then, f has no asymptotic values, no parabolic periodic point and no recurrent critical point, but f is not semi-hyperbolic at $1 \in J_f$. This f has a sequence of critical points $\{c_i\}_{i=1}^{\infty}$ with

$$f(c_i) = c_{i-1} \quad (i = 2,\ 3, \cdots), \quad f(c_1) = 1$$

and $f(1)$ is a repelling fixed point of f so $1 \in J_f$. Hence $1 \in Y_4$ in this case. Finally we do not know an example of f with $Y_5 \neq \emptyset$.

5 Some applications of the main theorem

As an application of Theorem A, we can show the following result on a measure theoretical property for the dynamics of entire functions. This is a refinement of the result by Bock ([**B**]).

Theorem B. *Either one of the following* **(AT$\widehat{\mathbf{Z}}$)** *or* **(ERG)** *holds for an entire function f:*

(AT$\widehat{\mathbf{Z}}$) *Almost every point $z \in J_f$ is attracted to the set $\widehat{Z}$, that is,*

$$\lim_{n\to\infty} \mathrm{dist}_{\mathrm{sph}}(f^n(z), \widehat{Z}) = 0, \quad (\text{i.e. } \omega(z) \subset \widehat{Z})$$

holds for a.e. $z \in J_f$, where $\widehat{Z} := Z \cup \{\infty\}$.

(ERG) *$J_f = \mathbb{C}$ and f is ergodic.*

Furthermore, **(ERG)** *can be replaced by the following* **(IR)** *or* **(FOD)**:

(IR) *$J_f = \mathbb{C}$ and f is infinitely recurrent, i.e. for every $X \subset \mathbb{C}$ with* $\mathrm{Leb}(X) > 0$ *and every $z \in \mathbb{C}$,*

$$^{\#}\{n \in \mathbb{N} \mid f^n(z) \in X\} = \infty$$

holds, where $\mathrm{Leb}(\cdot)$ *denotes the Lebesgue measure on $\mathbb{C}$.*

(FOD) *$J_f = \mathbb{C}$ and for a.e. $z \in \mathbb{C}$, the forward orbit $O^+(z) \subset \mathbb{C}$ is dense.*

Corollary C. *Let f be an entire function with the following properties:*

(i) *Every critical point c of f is either preperiodic or satisfies $f^n(c) \to \infty$ $(n \to \infty)$.*

(ii) *Every asymptotic value is eventually periodic.*

(iii) *The post-singular set $P(f)$ is discrete in $\mathbb{C}$.*

Then either one of the following holds:

(MGA) *$\{\infty\}$ is a metric global attractor, that is, $f^n(z) \to \infty$ $(n \to \infty)$ for a.e. $z \in \mathbb{C}$ (i.e. $\omega(z) = \{\infty\}$).*

(FOD) *$J_f = \mathbb{C}$ and $O^+(z) \subset \mathbb{C}$ is dense for a.e. $z \in \mathbb{C}$ (i.e. $\omega(z) = \widehat{\mathbb{C}}$).*

In particular, if f satisfies the conditions (i) $\sim$ (iii) *and $J_f \neq \mathbb{C}$, then $\{\infty\}$ is a metric global attractor for f.*

(Proof): It follows from the assumptions (i) $\sim$ (iii) that every singular value p satisfies either $f^n(p) \to \infty$ or eventually lands on a repelling periodic point. If $F_f \neq \emptyset$, then only possible Fatou components are either Baker domains (or their preimages) or wandering domains. If there is a wandering domain U, then we have $f^n|_U \to \infty$, because in general a finite limit function on a wandering domain is a constant which belongs to the derived set of $P(f)$ (see [**BHKMT**]), which is empty by (iii) in our case.

Then either **(AT$\widehat{\mathbf{Z}}$)** or **(FOD)** holds by Theorem A. In the case of **(AT$\widehat{\mathbf{Z}}$)**, it follows that

$$\omega(z) \subset \widehat{Z} = Y_2 \cup \{\infty\}, \text{ for a.e. } z \in J_f.$$

On the other hand, Y_2 consists of repelling periodic points only and hence $O^+(z)$ cannot accumulate on Y_2. Therefore

$$\omega(z) = \widehat{Z} = \{\infty\}, \text{ i.e. } f^n(z) \to \infty \text{ for a.e. } z \in J_f,$$

which implies that $\{\infty\}$ is a metric global attractor.

In the case of **(FOD)**, it follows that $J_f = \mathbb{C}$ and $O^+(z) \subset \mathbb{C}$ is dense for a.e. $z \in \mathbb{C}$, which means that $\omega(z) = \widehat{\mathbb{C}}$. This completes the proof of Corollary C. □

Corollary D. *Let f be a semi-hyperbolic (transcendental) entire function with $J_f \neq \mathbb{C}$. Then,*

(1) $\mathrm{Leb}(J_f) = 0 \iff \mathrm{Leb}(J_f \cap I_f) = 0$, *where* $I_f := \{z \mid f^n(z) \to \infty\}$.

(2) $\mathrm{Leb}(J_f) > 0 \Longrightarrow f^n(z) \to \infty \ (n \to \infty)$ *for a.e.* $z \in J_f$

(Proof): Since f is semi-hyperbolic, we have $Z = \emptyset$ by Theorem A. Also **(AT$\widehat{\mathbf{Z}}$)** holds from Theorem B, because we assume that $J_f \neq \mathbb{C}$. This means that $f^n(z) \to \infty$ for a.e. $z \in J_f$. Now it is obvious to see that (1) and (2) hold. □

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