The dynamics on Teichmüller spaces
induced by holomorphic self-coverings

Ege Fujikawa (Sophia University)
藤川 英華 (上智大学)
Katsuhiko Matsuzaki (Okayama University)
松崎 克彦 (岡山大学)
Masahiko Taniguchi (Nara Women's University)
谷口 雅彦 (奈良女子大学)

1 Introduction

The Teichmüller space $T(R)$ of a Riemann surface $R$ is the set of equivalence classes $[f]$ of quasiconformal homeomorphisms $f$ on $R$. Here we say that two quasiconformal homeomorphisms $f_1$ and $f_2$ on $R$ are equivalent if there exists a conformal homeomorphism $h : f_1(R) \to f_2(R)$ such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity. All homotopies are considered to be relative to the ideal boundary at infinity. A distance between two points $[f_1]$ and $[f_2]$ in $T(R)$ is defined by $d([f_1],[f_2]) = (1/2) \log K(f)$, where $f$ is an extremal quasiconformal homeomorphism in the sense that its maximal dilatation $K(f)$ is minimal in the homotopy class of $f_2 \circ f_1^{-1}$. Then $d$ is a complete distance on $T(R)$ which is called the Teichmüller distance.

We assume that a Riemann surface $R$ is of hyperbolic type. Namely, it is represented by a quotient space $\mathbb{H}^+ / \Gamma$ of the upper half-plane $\mathbb{H}^+ = \{ z \in \mathbb{C} \mid \text{Im } z > 0 \}$ by a torsion free Fuchsian group $\Gamma$. Let $R' = \mathbb{H}^- / \Gamma$ be the complex conjugate of $R$ where $\mathbb{H}^- = \{ z \in \mathbb{C} \mid \text{Im } z < 0 \}$, and $B(R')$ the complex Banach space of all bounded holomorphic quadratic differentials on $R'$ with the hyperbolic supremum norm. Then the Teichmüller space $T(R)$ is a complex Banach manifold modeled on $B(R')$. In fact, $T(R)$ is embedded in $B(R')$ as a bounded contractible domain. Hence it is equipped with the Kobayashi distance. If $R$ is a Riemann surface whose fundamental group is infinitely generated, then the Teichmüller space is infinite dimensional. For details, see [4] and [8]. It was proved in [3] that the Teichmüller distance and the Kobayashi distance are coincident for all Riemann surfaces.

We consider a holomorphic map of $T(R)$ into $T(R)$. Every quasiconformal automorphism of a Riemann surface $R$ induces a biholomorphic automorphism of $T(R)$. Then this is an isometry with respect to the Teichmüller-Kobayashi distance. Furthermore, the converse is also true, namely every biholomorphic automorphism of $T(R)$ is induced by a quasiconformal automorphism of the
Riemann surface. This is a combination of results of [1] and [5]. In [2], we have considered the dynamics of isometric automorphisms in general metric spaces as well as that of biholomorphic automorphisms of the Teichmüller space.

In this paper, we consider a Riemann surface $R$ in which there exists a non-injective unramified holomorphic self-covering $f : R \to R$. Then the fundamental group of $R$ is infinitely generated. For example, we can obtain such a surface by a Fatou component of the complex dynamics on the Riemann sphere. The holomorphic self-covering $f$ is locally isometric with respect to the hyperbolic metric on $R$, and it induces a holomorphic self-map

$$f^* : T(R) \to T(R).$$

Then $f^*$ is non-expanding with respect to the Teichmüller-Kobayashi distance $d$ and not surjective. We investigate the dynamics of $f^*$ on $T(R)$.

## 2 Dynamics of holomorphic self-maps

**Definition 1** We define the full cluster set of $f^*$ by

$$C(f^*) = \lim_{k \to \infty} \bigcup_{n=k}^{\infty} (f^*)^n(T(R)) = \bigcap_{n=1}^{\infty} (f^*)^n(T(R)).$$

The full cluster set $C(f^*)$ is the maximal closed and completely invariant set under the action of $f^*$.

**Definition 2** For a point $x \in T(R)$, it is said that $y \in T(R)$ is a $\omega$-limit point of $x$ for $f^*$ if there exists a sequence $\{n_i\} \subset \mathbb{Z}_+$ of positive integers such that $\lim_{i \to \infty} d((f^*)^{n_i}(x), y) = 0$. The set of all $\omega$-limit points of $x$ for $f^*$ is called the $\omega$-limit set of $x$ for $f^*$ and is denoted by $\Lambda(f^*, x)$. It is said that $x \in T(R)$ is a recurrent point for $f^*$ if $x \in \Lambda(f^*, x)$. The set of all recurrent points for $f^*$ is called the recurrent set for $f^*$ and is denoted by $\text{Rec}(f^*)$. The $\omega$-limit set for $f^*$ is defined by $\Lambda(f^*) = \bigcup_{x \in T(R)} \Lambda(f^*, x)$. The set of all periodic points for $f^*$ is denoted by $\text{Per}(f^*)$.

The following properties make the definitions for a non-expanding map simple.

**Proposition 3** The recurrent set $\text{Rec}(f^*)$ is a subset of the full cluster set $C(f^*)$, and the recurrent set $\text{Rec}(f^*)$ is coincident with the limit set $\Lambda(f^*)$. Moreover $\text{Rec}(f^*)$ is closed, and so is $\Lambda(f^*)$.

However $\text{Rec}(f^*)$ is not coincident with $C(f^*)$. In fact, we have the following.

**Theorem 4** (i) For every point $x \in C(f^*)$, the orbit $O(x) = \{(f^*)^n(x) \mid n \in \mathbb{Z}_+\}$ is not dense in $C(f^*)$. (ii) The following inclusion relations are proper;

$$C(f^*) \supset \text{Rec}(f^*) \supset \overline{\text{Per}(f^*)} \supset \text{Per}(f^*) \supset \text{Fix}(f^*).$$

(iii) The recurrent set $\text{Rec}(f^*)$ is nowhere dense in $C(f^*)$. 

3 Geometry of holomorphic self-map

Next, we consider the non-expanding property of $f^*$ more closely. The injective holomorphic map $f^*$ induces an injective holomorphic map

$$\hat{f}^*: T(T(R)) \rightarrow T(T(R))$$

of the holomorphic tangent bundle $T(T(R))$ of $T(R)$ such that $f^*$ sends $(p, v)$ to $(f^*(p), (df^*)_p(v))$. Then we define the magnification of a tangent vector $v$ at $p$ by

$$r(p, v) := \frac{||((df^*)_p(v))||_{T_{f^2(p)}(T(R))}}{||v||_{T_p(T(R))}}.$$

If a covering $f : R \rightarrow R$ is amenable, then $r(p, v) = 1$ for every $(p, v) \in T(T(R))$ (see [6]). Namely, $f^*$ is an isometry on $T(R)$. Thus hereafter we assume that $f$ is a non-amenable cover. In this case, we see that there are a lot of tangent vectors in $T(T(R))$ that are actually contracted by $f^*$.

**Theorem 5** The set \{$(p, v) \in T(T(R)) \mid r(p, v) < 1$\} is dense in $T(T(R))$.

This theorem is also followed by [6] combined with the fact that the Reich-Strebel functionals (tangent vectors) are dense in each tangent space $T_p(T(R))$.

However, we know that the magnification $r(p, v)$ is not uniformly bounded, for otherwise, the fixed point theorem says that the full cluster set $C(f^*)$ should be a unique fixed point of $f^*$.

**Theorem 6** For every point $(p, v) \in T(T(R))$, we have

$$\lim_{n \rightarrow \infty} r((f^*)^n(p, v)) = 1.$$

Actually, there exists some tangent vector $(p, v)$ such that $r(p, v) = 1$.

**Theorem 7** (i) For every point $p \in \text{Per}(f^*)$, there exists a tangent vector $v \in T_p(T(R))$ such that $r(p, v) = 1$. (ii) For every point $p \in \text{Rec}(f^*)$, we have $\sup_{v \in T_p(T(R))} r(p, v) = 1$.

4 Dynamics on the base surface

We prove these theorems by the following structure theorem on the dynamics of a holomorphic self-covering on a Riemann surface. A similar result was proved also by McMullen and Sullivan [7].

**Theorem 8 (Structure theorem I)** Suppose that there exist a Riemann surface $R$ and a non-injective unramified holomorphic self-covering $f : R \rightarrow R$. Then there exist a Riemann surface $S$, a holomorphic covering $\pi : R \rightarrow S$ and a
biholomorphic automorphism $g : S \to S$ of infinite order such that the following diagram commutes:

$$
\begin{array}{ccc}
R & \xrightarrow{f} & R \\
\downarrow{\pi} & & \downarrow{\pi} \\
S & \xrightarrow{g} & S \\
\end{array}
$$

This theorem insists that the action of $f^*$ is very similar to the isometry $g^*$.

**Remark 9** The grand orbit of $x \in R$ under $f$ is the set of points $y \in R$ such that $f^n(x) = f^m(y)$ for some $n, m \geq 0$. Furthermore, the small orbit of $x \in R$ under $f$ is the set of points $y \in R$ such that $f^n(x) = f(y)$ for some $n \geq 0$. We define $R/f$ as the quotient space by the grand orbit relation, and $R/(f)$ as the quotient space by the small orbit relation. The Riemann surface $S$ as in Theorem 8 is coincident with $R/(f)$ and the quotient surface $S/\langle g \rangle$ is coincident with $R/f$.

Finally we consider another application obtained by the structure theorem.

**Definition 10** For a holomorphic self-covering $f : R \to R$, we say that a subset $U \subset R$ is an absorbing domain if $f(U) \subset U$ and if, for every point $x \in R$, there exists $n \in \mathbb{N}$ such that $f^n(x) \in U$. If $f$ is injective in the absorbing domain $U$, then we call $U$ simple. Furthermore we say that the absorbing domain $U$ is escaping if, for every compact subset $K \subset R$, the number of integers $n$ satisfying $f^n(U) \cap K \neq \emptyset$ is finite.

**Theorem 11** For every non-injective holomorphic self-covering $f : R \to R$, there exists a simple, escaping, absorbing domain.

**Corollary 12** (Denjoy-Wolff type theorem) For a non-injective holomorphic self-covering $f : R \to R$, there exists a unique topological end $e$ of $R$ such that $f^n(x) \to e$ for every $x \in R$.

In fact, there exists a unique analytical end which is determined by a fixed point of a lift of $g$ to $\mathbb{H}$.

On the last of this section, we mention the existence of holomorphic self-coverings.

**Theorem 13** (Structure theorem II) For every Riemann surface $S$ and for every biholomorphic automorphism $g : S \to S$ of infinite order, there exist a holomorphic covering $\pi : R \to S$ and a holomorphic self-covering $f : R \to R$ such that $\pi \circ f = g \circ \pi$. 
References


