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Julia sets of quartic polynomials and polynomial semigroups

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Abstract

For a polynomial of degree two or more, the Julia set and the filled-in Julia set are either connected or else have uncountably many components. If the Julia set is totally disconnected, then the polynomial is topologically conjugate to the shift map. In the case of neither connected nor totally disconnected Julia set of a quartic polynomial, there exists a homeomorphism between the set of all components of the filled-in Julia set and some subset of the corresponding symbol space. Furthermore the polynomial is topologically conjugate to the shift map with respect to the homeomorphism. Moreover there exists a homeomorphism between the Julia set of the polynomial and that of a certain polynomial semigroup.

1 Preparations and the main results

Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere and let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a polynomial of degree $d \geq 2$. The *filled-in Julia set* K_f is defined as

$$K_f = \{z \in \mathbb{C} : \{f^n(z)\}_{n=0}^{\infty} \text{ is bounded}\}.$$

The topological boundary of K_f is called the *Julia set* J_f, and its complement $\hat{\mathbb{C}} \setminus J_f$ is called the *Fatou set* F_f. In this case, ∞ is a superattracting fixed point. We call $A_f(\infty) = \hat{\mathbb{C}} \setminus K_f$ the *basin of attraction.*

Definition 1.1. A *rational semigroup* G is a semigroup generated by a family of non-constant rational functions $\{g_1, g_2, \dots, g_n, \dots\}$ defined on $\hat{\mathbb{C}}$. We denote this situation by

$$G = \langle g_1, g_2, \dots, g_n, \dots \rangle.$$

A rational semigroup G is called a *polynomial semigroup* if each $g \in G$ is a polynomial.

Definition 1.2. Let G be a rational semigroup. The *Fatou set* F_G of G is defined as

$$F_G = \{z \in \hat{\mathbb{C}} : G \text{ is normal in a neighborhood of } z\}.$$

Its complement $\hat{\mathbb{C}} \setminus F_G$ is called the *Julia set* J_G of G. Note that $F_{\langle g \rangle} = F_g$ and $J_{\langle g \rangle} = J_g$.

Definition 1.3. Let $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ be the set of non-negative integers and let $\Sigma_q = \{1, 2, \ldots, q\}^{\mathbb{N}_0}$ be the symbol space of q-symbols. For $s = (s_n)$ and $t = (t_n)$ in Σ_q, a metric ρ on Σ_q is defined as

$$\rho(s,t) = \sum_{n=0}^{\infty} \frac{\delta(s_n, t_n)}{2^n}, \quad \text{where } \delta(k,l) = \begin{cases} 1 & \text{if } k \neq l, \\ 0 & \text{if } k = l. \end{cases}$$

Then Σ_q is a compact metric space. We define the *shift map* $\sigma : \Sigma_q \to \Sigma_q$ as

$$\sigma((s_0, s_1, s_2, \ldots)) = (s_1, s_2, \ldots).$$

The shift map σ is continuous with respect to the metric ρ.

In the case of a polynomial of degree two or more, the connectivity of the Julia set is affected by the behavior of finite critical points.

Theorem 1.4 ([1]). *Let f be a polynomial of degree $d \geq 2$. If all finite critical points of f are in $A_f(\infty)$, then J_f is totally disconnected and $J_f = K_f$. Furthermore $f|_{J_f}$ is topologically conjugate to the shift map $\sigma|_{\Sigma_d}$. On the other hand, if all finite critical points of f are in K_f, then J_f and K_f are connected.*

Definition 1.5. The *Green's function* associated with K_f is defined as

$$G(z) = \lim_{n \to \infty} \frac{1}{d^n} \log^+ |f^n(z)|,$$

where $\log^+ x = \max\{\log x, 0\}$. $G(z)$ is zero for $z \in K_f$ and $G(z)$ is positive for $z \in \mathbb{C} \setminus K_f$. Note the identity $G(f(z)) = dG(z)$.

Definition 1.6. We call the triple (f, U, V) of bounded simply connected domains U and V such that $\overline{U} \subset V$ and a holomorphic proper map $f : U \to V$ of degree d a *polynomial-like map* of degree d. The *filled-in Julia set* K_f of a polynomial-like map (f, U, V) is defined as

$$K_f = \{z \in U : \{f^n(z)\}_{n=0}^{\infty} \subset U\}.$$

Definition 1.7. Let (X, d) be a metric space. For a compact subset $A \subset X$ and $\delta > 0$, let $A[\delta]$ be a δ-neighborhood of A. For compact subsets $A, B \subset X$, we define the *Hausdorff metric* d_H as

$$d_H(A, B) = \inf\{\delta : A \subset B[\delta] \text{ and } B \subset A[\delta]\}.$$

Situation : Let f be a quartic polynomial and let c_1, c_2 and c_3 be finite critical points of f. G is the Green's function associated with the filled-in Julia set K_f. Suppose that $G(c_1) = G(c_2) = 0$ and $G(c_3) > 0$, that is, $c_1, c_2 \in K_f$ and $c_3 \in A_f(\infty)$.

Let U be a bounded component of $\mathbb{C} \setminus G^{-1}(G(f(c_3)))$. Suppose that U_A and U_B be bounded components of $\mathbb{C} \setminus G^{-1}(G(c_3))$ such that $c_1 \in U_A$ and $c_2 \in U_B$. Then U_A and U_B are proper subsets of U. Furthermore $(f|_{U_A}, U_A, U)$ and $(f|_{U_B}, U_B, U)$ are polynomial-like maps of degree 2. We set $f_1 = f|_{U_A}$ and $f_2 = f|_{U_B}$.

Under this situation, we define the *A-B kneading sequence* $(\alpha_n)_{n\geq 0}$ of c_i as

$$\alpha_n = \begin{cases} A & \text{if } f^n(c_i) \in U_A, \\ B & \text{if } f^n(c_i) \in U_B. \end{cases}$$

We assume that the A-B kneading sequence of c_1 is $(AAA\cdots)$ and the A-B kneading sequence of c_2 is $(BBB\cdots)$. Note that K_{f_1} and K_{f_2} are connected (see [3]).

Let $\mathrm{Comp}(K_f)$ be the set of all components of K_f. Since $G(c_3) > 0$, $\mathrm{Comp}(K_f)$ is an uncountable set. $\mathrm{Comp}(K_f)$ becomes a metric space with the Hausdorff metric d_H. We define a map $F : \mathrm{Comp}(K_f) \to \mathrm{Comp}(K_f)$ as $F(K) = f(K)$ for $K \in \mathrm{Comp}(K_f)$. This map F is continuous with respect to the Hausdorff metric d_H.

Let $\Sigma_6 = \{1, 2, 3, 4, A, B\}^{\mathbb{N}_0}$ be the symbol space. We define a subset Σ of Σ_6 as follows: $s = (s_n) \in \Sigma$ if and only if

1. $s_n = A \Rightarrow s_{n+1} = A$,
2. $s_n = B \Rightarrow s_{n+1} = B$,
3. $s_n = A$ and $s_{n-1} \neq A \Rightarrow s_{n-1} = 3$ or 4,
4. $s_n = B$ and $s_{n-1} \neq B \Rightarrow s_{n-1} = 1$ or 2,
5. if $s \in \Sigma_4 = \{1, 2, 3, 4\}^{\mathbb{N}_0}$, then there exist subsequences $(s_{n(k)})_{k=1}^{\infty}$ and $(s'_{n(l)})_{l=1}^{\infty}$ such that $s_{n(k)} = 1$ or 2 for all $k \in \mathbb{N}$ and $s'_{n(l)} = 3$ or 4 for all $l \in \mathbb{N}$.

It is our goal to prove the following theorems.

Theorem 1.8. *Let f be a quartic polynomial. Suppose that its finite critical points $c_1, c_2 \in K_f$ and $c_3 \in A_f(\infty)$ differ mutually and suppose that J_f is disconnected but not totally disconnected. Moreover, suppose that the A-B kneading sequence of c_1 is $(AAA\cdots)$ and the A-B kneading sequence of c_2 is $(BBB\cdots)$. Then there exists a homeomorphism $\Lambda : \mathrm{Comp}(K_f) \to \Sigma$ such that $\Lambda \circ F = \sigma \circ \Lambda$.*

Theorem 1.9. *Under the assumption of Theorem 1.8, there exist quadratic polynomials g_1 and g_2 and a homeomorphism h on K_f such that*

$$h(J_f) = J_G,$$

where $G = \langle g_1, g_2 \rangle$ is a polynomial semigroup.

2 Proof of Theorem 1.8

A conformal map Ψ with the following properties exists (see [6, p.88]): there exist $r > 1$ and $W \subset \mathbb{C} \setminus K_f$ with $c_3 \in \partial W$ and $\mathbb{C} \setminus \overline{W} = U_A \cup U_B$ such that $\Psi : \mathbb{C} \setminus \overline{\mathbb{D}}_r \to W$ is conformal and $\Psi^{-1} \circ f \circ \Psi(z) = z^4$, where $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$. For $t \in [0, 1)$, $R(t) = \Psi(\{z \in \mathbb{C} : |z| > r \text{ and } \arg(z) = 2\pi t\})$ is called the *external ray* with angle t for K_f.

Remark 2.1. W is an unbounded component of $\mathbb{C} \setminus G^{-1}(G(c_3))$ and its boundary ∂W is $G^{-1}(G(c_3))$.

Let R be the intersection of the external ray passes through $f(c_3)$ and $\mathbb{C} \setminus \overline{U}$. Two of four rays $f^{-1}(R)$ have a limit point c_3. $\Psi^{-1}(f^{-1}(R))$ is four half-lines extended from $\partial \mathbb{D}_r$ with adjacent angles $\pi/2$. There are three invariant half-lines extended from the unit circle under $z \mapsto z^4$ and their angles are $0, 1/3$ and $2/3$. At least two of three invariant half-lines do not overlap with $\Psi^{-1}(f^{-1}(R))$. Let $\widetilde{R}_1$ be the intersection of one of these invariant half-lines and $\mathbb{C} \setminus \overline{\mathbb{D}}_r$. Let R_1 be the image of $\widetilde{R}_1$ under Ψ. We extend R_1 to become the invariant ray under f. Let R_0 be a component of $f^{-1}(R_1)$ which satisfies $R_1 \cap R_0 \neq \emptyset$. Then $R_1 \subset R_0$ and f maps $J_0 = R_0 \setminus R_1$ onto $J_1 = R_1 \cap \overline{U}$. Inductively, let R_{-n} be a component of $f^{-1}(R_{-(n-1)})$ which satisfies $R_{-(n-1)} \cap R_{-n} \neq \emptyset$. Then $R_{-(n-1)} \subset R_{-n}$ and f maps J_{-n} onto $J_{-(n-1)}$, where

$$J_{-n} = \begin{cases} R_{-n} \setminus R_{-(n-1)} & \text{if } n \geq 0, \\ R_1 \cap \overline{U} & \text{if } n = -1. \end{cases}$$

At this time, a ray

$$R_\infty = \bigcup_{n=0}^{\infty} R_{-n} = R_1 \cup \left(\bigcup_{n=0}^{\infty} J_{-n} \right)$$

is invariant under f.

Lemma 2.2 ([8]). *Let F be a rational map and let X denote the closure of the union of the postcritical set and possible rotation domains of F. Suppose that $\gamma : (-\infty, 0] \to \hat{\mathbb{C}} \setminus X$ is a curve with*

$$F^{nk}(\gamma(-\infty, -k]) = \gamma(-\infty, 0]$$

for all positive integers k. Then $\lim_{t \to -\infty} \gamma(t)$ exists and is a repelling or parabolic periodic point of F whose period divides n.

We can apply Lemma 2.2 to $R_\infty \setminus R_1 = \bigcup_{n=0}^{\infty} J_{-n}$, setting γ such that $\gamma(-(k+1), -k] = J_{-k}$ for all positive integers k. Therefore R_∞ lands at a repelling or parabolic fixed point of f. If R_∞ lands at a point on K_{f_1}, then we describe R_∞ with R_{A1}. Similarly, if R_∞ lands at a point on K_{f_2}, then we describe R_∞ with R_{B1}. In fact, we can obtain both R_{A1} and R_{B1} by choosing $\widetilde{R}_1$ well.

To the next, let R_{A2} and R_{B2} be components of $f^{-1}(R_{A1})$ and $f^{-1}(R_{B1})$ which satisfy $R_{A2} \cap U_A \neq \emptyset$ and $R_{B2} \cap U_B \neq \emptyset$ and differ from R_{A1} and R_{B1} respectively. We set $V_A = U \setminus (K_{f_1} \cup R_{A1})$ and $V_B = U \setminus (K_{f_2} \cup R_{B1})$. Let I_1, I_2, I_3 and I_4 be branches of f^{-1} such that

$$I_1 : V_A \to U_1, \quad I_2 : V_A \to U_2,$$
$$I_3 : V_B \to U_3, \quad I_4 : V_B \to U_4,$$

where U_1 and U_2 are components of $U_A \setminus K_{f_1} \cup R_{A1} \cup R_{A2}$ respectively. Similarly, U_3 and U_4 are components of $U_B \setminus K_{f_2} \cup R_{B1} \cup R_{B2}$ respectively.

We define a map $\Lambda : \mathrm{Comp}(K_f) \to \Sigma$ as follows: for $K \in \mathrm{Comp}(K_f)$,

$$[\Lambda(K)]_n = \begin{cases} i & \text{if } f^n(K) \subset U_i, \\ A & \text{if } f^n(K) = K_{f_1}, \\ B & \text{if } f^n(K) = K_{f_2}, \end{cases}$$

where $n \in \mathbb{N}_0$ and $i = 1, 2, 3, 4$.

Lemma 2.3. $\Lambda : \mathrm{Comp}(K_f) \to \Sigma$ *is continuous.*

Proof. For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $1/2^N < \varepsilon$. We take $K \in \mathrm{Comp}(K_f)$ arbitrarily and set $s = \Lambda(K) = (s_0, s_1, \ldots, s_N, \ldots)$. We consider the case of $s \in \Sigma \cap \Sigma_4$ first. By continuity of f, there exist $\delta_1, \ldots, \delta_N > 0$ such that $f^k(K[\delta_k]) \subset U_{s_k}$ for $k = 1, 2, \ldots, N$. Let δ be the minimum value of δ_k. Then $f^k(K[\delta]) \subset U_{s_k}$ for $k = 1, 2, \ldots, N$. Any component K' of K_f with $d_H(K, K') < \delta$ satisfies $K' \subset K[\delta]$ by the definition of the Hausdorff metric. Moreover any component $K' \subset K[\delta]$ of K_f satisfies $\Lambda(K') = (s_0, s_1, \ldots, s_N, t_{N+1}, \ldots)$. Therefore if any component K' of K_f satisfies $d_H(K, K') < \delta$, then

$$\rho(\Lambda(K), \Lambda(K')) = \sum_{k=N+1}^{\infty} \frac{\delta(s_k, t_k)}{2^k} \leq \sum_{k=N+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^N} < \varepsilon.$$

If $s_n = A$ and $s_{n-1} \neq A$ or $s_n = B$ and $s_{n-1} \neq B$, then s is an isolated point in Σ. Since corresponding K is also an isolated point in $\mathrm{Comp}(K_f)$, Λ is continuous at K. $\square$

We define a map $\widetilde{\Lambda} : \Sigma \to \mathrm{Comp}(K_f)$ as follows: for $s = (s_n) \in \Sigma$, if $s_n = A$ and $s_{n-1} \neq A$,

$$\widetilde{\Lambda}(s) = I_{s_0} \circ \cdots \circ I_{s_{n-1}}(K_{f_1}).$$

If $s_n = B$ and $s_{n-1} \neq B$,

$$\widetilde{\Lambda}(s) = I_{s_0} \circ \cdots \circ I_{s_{n-1}}(K_{f_2}).$$

If $s \in \Sigma_4$, there exists a subsequence $(s_{n(l)})_{l=1}^{\infty}$ such that $s_{n(l)} = 1$ or 2 and $s_{n(l)-1} = 3$ or 4. We set $K_s^{(l)} = I_{s_0} \circ \cdots \circ I_{s_{n(l)-1}}(\overline{U}_A)$. Then $K_s^{(l)} \supset K_s^{(l+1)}$. We define

$$\widetilde{\Lambda}(s) = \bigcap_{l=1}^{\infty} K_s^{(l)}.$$

Note that $\bigcap_{l=1}^{\infty} K_s^{(l)}$ is a one-point set since each I_k decreases the Poincaré distance on V_A or V_B.

Remark 2.4. We check that I_k decreases the Poincaré distance on V_A or V_B. For x and y in V_A, let γ be the Poincaré geodesic from x to y in V_A . Then there exists a constant $c < 1$ such that

$$\int_{I_1(\gamma)} ds_{V_A} \leq c \int_{I_1(\gamma)} ds_{U_1},$$

where ds_{V_A} and ds_{U_1} are the Poincaré metrics on V_A and U_1 respectively. Let γ' be the Poincaré geodesic from $I_1(x)$ to $I_1(y)$ in V_A. Then

$$\mathrm{dist}_{V_A}(I_1(x), I_1(y)) = \int_{\gamma'} ds_{V_A} \leq \int_{I_1(\gamma)} ds_{V_A},$$

where dist_{V_A} is the Poincaré distance. Since I_1 is conformal,

$$\int_{I_1(\gamma)} ds_{U_1} = \int_{\gamma} I_1^*(ds_{U_1}) = \int_{\gamma} ds_{V_A} = \mathrm{dist}_{V_A}(x, y).$$

As mentioned above,

$$\mathrm{dist}_{V_A}(I_1(x), I_1(y)) \leq c \cdot \mathrm{dist}_{V_A}(x, y).$$

Therefore I_1 decreases the Poincaré distance on V_A. It is similarly proved about I_2, I_3 and I_4.

Lemma 2.5. *$\widetilde{\Lambda}$ is the inverse map of Λ.*

Proof. What is necessary is just to prove that $\Lambda \circ \widetilde{\Lambda}$ and $\widetilde{\Lambda} \circ \Lambda$ are the identity maps. We take $s = (s_0, s_1, s_2, \dots) \in \Sigma$ arbitrarily. If $s_n = A$ and $s_{n-1} \neq A$, $\widetilde{\Lambda}(s) = I_{s_0} \circ \dots \circ I_{s_{n-1}}(K_{f_1})$. By definition, $f^k(\widetilde{\Lambda}(s)) = I_{s_k} \circ \dots \circ I_{s_{n-1}}(K_{f_1}) \subset U_{s_k}$. Then $[\Lambda(\widetilde{\Lambda}(s))]_k = s_k$. Therefore $\Lambda \circ \widetilde{\Lambda}(s) = s$. We can prove similarly in the case of $s_n = B$ and $s_{n-1} \neq B$. If $s \in \Sigma_4$,

$$f^k\left(\widetilde{\Lambda}(s)\right) = f^k\left(\bigcap_{l=1}^{\infty} K_s^{(l)}\right) \subset \bigcap_{l=1}^{\infty} f^k\left(K_s^{(l)}\right) \subset U_{s_k}.$$

Then $[\Lambda(\widetilde{\Lambda}(s))]_k = s_k$. Therefore $\Lambda \circ \widetilde{\Lambda}(s) = s$. As mentioned above, $\Lambda \circ \widetilde{\Lambda}$ is the identity map of Σ. It is clear that $\widetilde{\Lambda} \circ \Lambda$ is the identity map of $\mathrm{Comp}(K_f)$. □

Lemma 2.6. *$\Lambda^{-1} : \Sigma \to \mathrm{Comp}(K_f)$ is continuous.*

Proof. For any $s = (s_0, s_1, s_2, \dots) \in \Sigma$, we set $K = \Lambda^{-1}(s)$. If $s_n = A$ and $s_{n-1} \neq A$, $K = I_{s_0} \circ \dots \circ I_{s_{n-1}}(K_{f_1})$. Since K is an isolated point in $\mathrm{Comp}(K_f)$, Λ^{-1} is continuous at s. Similarly, if $s_n = B$ and $s_{n-1} \neq B$, then Λ^{-1} is continuous at s. We take $\varepsilon > 0$ arbitrarily. If $s \in \Sigma_4$,

$$\Lambda^{-1}(s) = \bigcap_{l=1}^{\infty} K_s^{(l)}.$$

Since $K_s^{(l)} \supset K_s^{(l+1)}$ and $\Lambda^{-1}(s)$ is a one-point set, there exists $l_0 \in \mathbb{N}$ such that

$$\Lambda^{-1}(s) \subset K_s^{(l_0)} \subset \Lambda^{-1}(s)[\varepsilon].$$

We set $\delta = 1/2^{n(l_0)-1}$. We consider $t \in \Sigma$ with $\rho(s,t) < \delta$. At this time, we can describe

$$t = (s_0, s_1, \ldots, s_{n(l_0)-1}, s_{n(l_0)}, t_{n(l_0)+1}, \ldots).$$

If $t \in \Sigma \setminus \Sigma_4$, by definition of $\Lambda^{-1}(t)$,

$$\Lambda^{-1}(t) \subset K_s^{(l_0)} \subset \Lambda^{-1}(s)[\varepsilon].$$

When $t \in \Sigma_4$, for the definition

$$\Lambda^{-1}(t) = \bigcap_{l=1}^{\infty} K_t^{(l)}$$

of $\Lambda^{-1}(t)$, it is clear that $K_t^{(l)} = K_s^{(l)}$ for $l = 1, 2, \ldots, l_0$. Then

$$\Lambda^{-1}(t) \subset K_s^{(l_0)} \subset \Lambda^{-1}(s)[\varepsilon].$$

Since $\Lambda^{-1}(s)$ is a one-point set, for $t \in \Sigma$ with $\rho(s,t) < \delta$,

$$d_H(\Lambda^{-1}(s), \Lambda^{-1}(t)) = \inf\{\varepsilon' : \Lambda^{-1}(t) \subset \Lambda^{-1}(s)[\varepsilon']\} < \varepsilon.$$

Therefore Λ^{-1} is continuous at s. □

Lemma 2.7. $\Lambda \circ F = \sigma \circ \Lambda$.

Proof. For $K \in \mathrm{Comp}(K_f)$, we set $\Lambda(K) = (s_0, s_1, s_2, \ldots)$. Then $\sigma \circ \Lambda(K) = (s_1, s_2, \ldots)$. On the other hand, $\Lambda \circ F(K) = \Lambda(f(K)) = (s_1, s_2, \ldots)$. Therefore $\Lambda \circ F = \sigma \circ \Lambda$. □

We have completed the proof of Theorem 1.8.

Remark 2.8. Various cases of the cubic polynomial are shown by [2].

3 Similar Results of Theorem 1.8

For a quartic polynomial, the following two cases are also considered. Theorem 3.1 and Theorem 3.2 are shown like the proof of Theorem 1.8. Suppose that the Julia set is disconnected but not totally disconnected.

Case 1 : Let f be a quartic polynomial and let c_1, c_2 and c_3 be finite critical points of f. Suppose that $G(c_1) = 0$ and $G(c_3) \geq G(c_2) > 0$, that is, $c_1 \in K_f$ and $c_2, c_3 \in A_f(\infty)$.

Let U be a bounded component of $\mathbb{C} \setminus G^{-1}(G(f(c_2)))$. Suppose that U_A, U_B and U_C be bounded components of $\mathbb{C} \setminus G^{-1}(G(c_2))$ such that $c_1 \in U_C$. Then U_A, U_B and U_C are proper subsets of U. Furthermore $(f|_{U_A}, U_A, U)$ and $(f|_{U_B}, U_B, U)$ are polynomial-like maps of degree 1 and $(f|_{U_C}, U_C, U)$ is a polynomial-like map of degree 2.

Under this situation, we define the kneading sequence $(\alpha_n)_{n\geq 0}$ of c_1 as

$$\alpha_n = \begin{cases} A & \text{if } f^n(c_1) \in U_A, \\ B & \text{if } f^n(c_1) \in U_B, \\ C & \text{if } f^n(c_1) \in U_C. \end{cases}$$

We assume that the kneading sequence of c_1 is $(CCC\cdots)$.

Let $\Sigma_5 = \{1, 2, 3, 4, C\}^{\mathbb{N}_0}$ be the symbol space. We define a subset Σ of Σ_5 as follows: $s = (s_n) \in \Sigma$ if and only if

1. $s_n = C \Rightarrow s_{n+1} = C$,
2. $s_n = C$ and $s_{n-1} \neq C \Rightarrow s_{n-1} = 1$ or 2,
3. if $s \in \Sigma_4 = \{1, 2, 3, 4\}^{\mathbb{N}_0}$, then there exists a subsequence $(s_{n(k)})_{k=1}^{\infty}$ such that $s_{n(k)} = 1$ or 2 for all $k \in \mathbb{N}$.

Theorem 3.1. *Let f be a quartic polynomial. Suppose that its finite critical points c_1, c_2 and c_3 satisfy $G(c_1) = 0$ and $G(c_3) \geq G(c_2) > 0$ and suppose that J_f is disconnected but not totally disconnected. Moreover, suppose that the kneading sequence of c_1 is $(CCC\cdots)$. Then there exists a homeomorphism $\Lambda : \mathrm{Comp}(K_f) \to \Sigma$ such that $\Lambda \circ F = \sigma \circ \Lambda$.*

Case 2 : Let f be a quartic polynomial and let c_1, c_2 and c_3 be finite critical points of f such that $c_1 = c_2$ and $c_1 \neq c_3$. Suppose that $G(c_1) = 0$ and $G(c_3) > 0$, that is, $c_1 \in K_f$ and $c_3 \in A_f(\infty)$.

Let U be a bounded component of $\mathbb{C} \setminus G^{-1}(G(f(c_3)))$. Suppose that U_A and U_B be bounded components of $\mathbb{C} \setminus G^{-1}(G(c_3))$ such that $c_1 \in U_B$. Then U_A and U_B are proper subsets of U. Furthermore $(f|_{U_A}, U_A, U)$ is a polynomial-like map of degree 1 and $(f|_{U_B}, U_B, U)$ is a polynomial-like map of degree 3. We assume that the kneading sequence of c_1 is $(BBB\cdots)$.

Let $\Sigma_5 = \{1, 2, 3, 4, B\}^{\mathbb{N}_0}$ be the symbol space. We define a subset Σ of Σ_5 as follows: $s = (s_n) \in \Sigma$ if and only if

1. $s_n = B \Rightarrow s_{n+1} = B$,

2. $s_n = B$ and $s_{n-1} \neq B \Rightarrow s_{n-1} = 1$,

3. if $s \in \Sigma_4 = \{1,2,3,4\}^{\mathbb{N}_0}$, then there exists a subsequence $(s_{n(k)})_{k=1}^{\infty}$ such that $s_{n(k)} = 1$ for all $k \in \mathbb{N}$.

Theorem 3.2. *Let f be a quartic polynomial. Suppose that its finite critical points c_1, c_2 and c_3 satisfy $c_1 = c_2$, $c_1 \in K_f$ and $c_3 \in A_f(\infty)$ and suppose that J_f is disconnected but not totally disconnected. Moreover, suppose that the kneading sequence of c_1 is $(BBB\cdots)$. Then there exists a homeomorphism $\Lambda : \mathrm{Comp}(K_f) \to \Sigma$ such that $\Lambda \circ F = \sigma \circ \Lambda$.*

4 Relevances with Polynomial Semigroups

In this section, we explore relevances of polynomials and polynomial semigroups. The following theorem about the polynomial-like map is important.

Theorem 4.1 **([3, 7]).** *Every polynomial-like map (f, U, V) of degree $d \geq 2$ is hybrid equivalent to a polynomial p of degree d. That is to say, there exist a polynomial p of degree d, a neighborhood W of K_f in U and a quasiconformal map $h : W \to h(W)$ such that*

1. *$h(K_f) = K_p$,*

2. *the complex dilatation μ_h of h is zero almost everywhere on K_f,*

3. *$h \circ f = p \circ h$ on $W \cap f^{-1}(W)$.*

If K_f is connected, p is unique up to conjugation by affine map.

Under the assumption of Theorem 1.8, (f_1, U_A, U) and (f_2, U_B, U) are polynomial-like maps of degree 2. Furthermore K_{f_1} and K_{f_2} are connected. By Theorem 4.1, there exist quadratic polynomials g_1 and g_2 with $K_{g_1} \cap K_{g_2} = \emptyset$, a neighborhood W_1 of K_{f_1} in U_A, a neighborhood W_2 of K_{f_2} in U_B and quasiconformal maps h_1 on W_1 and h_2 on W_2 such that $h_1(K_{f_1}) = K_{g_1}$ and $h_2(K_{f_2}) = K_{g_2}$.

We define branches $\tilde{I}_1$ and $\tilde{I}_2$ of g_1^{-1}. Since K_{g_1} is connected, there exists a conformal map $\Psi_1 : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus K_{g_1}$ such that $\Psi_1^{-1} \circ g_1 \circ \Psi_1(z) = z^2$. The external ray $R_1 = \Psi_1(\{z \in \mathbb{C} : |z| > 1 \text{ and } \arg(z) = 0\})$ lands at a fixed point of g_1. Let R_1' be the external ray which satisfies $g_1(R_1') = R_1$ and differs from R_1. At this time, we replace g_2 so that

$$R_1 \cap K_{g_2} = \emptyset \text{ and } R_1' \cap K_{g_2} = \emptyset.$$

Then we define branches $\tilde{I}_1$ and $\tilde{I}_2$ of g_1^{-1} as

$$\tilde{I}_1 : \mathbb{C} \setminus (K_{g_1} \cup R_1) \to \widetilde{U}_1 \text{ and } \tilde{I}_2 : \mathbb{C} \setminus (K_{g_1} \cup R_1) \to \widetilde{U}_2,$$

where $\widetilde{U}_1$ and $\widetilde{U}_2$ are components of $\mathbb{C} \setminus K_{g_1} \cup R_1 \cup R_1'$ respectively. Similarly, we take external rays R_2 and R_2'. Then we define branches $\tilde{I}_3$ and $\tilde{I}_4$ of g_2^{-1} as

$$\tilde{I}_3 : \mathbb{C} \setminus (K_{g_2} \cup R_2) \to \widetilde{U}_3 \text{ and } \tilde{I}_4 : \mathbb{C} \setminus (K_{g_2} \cup R_2) \to \widetilde{U}_4,$$

where $\widetilde{U}_3$ and $\widetilde{U}_4$ are components of $\mathbb{C} \setminus K_{g_2} \cup R_2 \cup R_2'$ respectively.

For $s \in \Sigma$, we set $K_s = \Lambda^{-1}(s)$ and $J_s = \partial K_s$. K_s is a component of K_f and J_s is a component of J_f. For $s = (s_0, s_1, s_2, \dots) \in \Sigma \setminus \Sigma_4$, we define a quasiconformal map h_s on a neighborhood of K_s. Let $n \in \mathbb{N}_0$ be the smallest number with $s_n = A$ and $s_{n-1} \neq A$ or $s_n = B$ and $s_{n-1} \neq B$. h_s is defined on $W_s = I_{s_0} \circ \cdots \circ I_{s_{n-1}}(W_i)$ as

$$h_s = \tilde{I}_{s_0} \circ \cdots \circ \tilde{I}_{s_{n-1}} \circ h_i \circ f^n, \text{ where } i = \begin{cases} 1 & \text{if } s_n = A \text{ and } s_{n-1} \neq A, \\ 2 & \text{if } s_n = B \text{ and } s_{n-1} \neq B. \end{cases}$$

We set $\widetilde{K}_s = h_s(K_s)$, $\widetilde{J}_s = \partial \widetilde{K}_s$ and $G = \langle g_1, g_2 \rangle$. If necessary, we replace g_1 and g_2 so that each $\widetilde{K}_s$ is disjoint. Since $\widetilde{J}_s = \partial \widetilde{K}_s = h_s(\partial K_s) = h_s(J_s)$ and J_G is backward invariant (see [4]), h_s maps J_s onto a component $\widetilde{J}_s$ of J_G. By definition, we turn out that $h_{(A,A,A,\dots)} = h_1$ and $h_{(B,B,B,\dots)} = h_2$.

Next, we define a homeomorphism

$$h : \bigcup_{s \in \Sigma \setminus \Sigma_4} K_s \to \bigcup_{s \in \Sigma \setminus \Sigma_4} \widetilde{K}_s$$

as $h|_{K_s} = h_s$.

Remark 4.2. For $s \in \Sigma \cap \Sigma_4$, a one-point component K_s of K_f is characterized using the Hausdorff topology. For $s = (s_0, s_1, s_2, \dots) \in \Sigma \cap \Sigma_4$, we set

$$t^{(n)} = \begin{cases} (s_0, s_1, \dots, s_{n-1}, A, A, \dots) & \text{if } s_{n-1} = 3 \text{ or } 4, \\ (s_0, s_1, \dots, s_{n-1}, B, B, \dots) & \text{if } s_{n-1} = 1 \text{ or } 2. \end{cases}$$

Then the sequence $\{t^{(n)}\}_{n=1}^{\infty}$ is in $\Sigma \setminus \Sigma_4$ and $t^{(n)} \to s$ as $n \to \infty$. Since Λ^{-1} is continuous,

$$K_s = \Lambda^{-1}(s) = \lim_{n \to \infty} \Lambda^{-1}\left(t^{(n)}\right) = \lim_{n \to \infty} K_{t^{(n)}}.$$

Finally, we extend h homeomorphically on $K_f = \bigcup_{s \in \Sigma} K_s$. For $s \in \Sigma \cap \Sigma_4$, we define $\widetilde{K}_s = h(K_s)$ as

$$h(K_s) = \lim_{n \to \infty} h(K_{t(n)}).$$

Note that each $\widetilde{I}_k$ decreases the Poincaré distance on $\mathbb{C} \setminus (K_{g_1} \cup R_1)$ or $\mathbb{C} \setminus (K_{g_2} \cup R_2)$. As mentioned above, h is a homeomorphism between $K_f = \bigcup_{s \in \Sigma} K_s$ and $\bigcup_{s \in \Sigma} \widetilde{K}_s$.

Lemma 4.3.

$$\partial \left(\bigcup_{s \in \Sigma} \widetilde{K}_s \right) = J_G.$$

Proof. Lemma 4.3 follows from the following lemma.

Lemma 4.4 ([4]). *If z is in $J_G \setminus E_G$, then*

$$\overline{O^-(z)} = J_G,$$

where $O^-(z) = \{w \in \hat{\mathbb{C}} : \text{there exists } g \in G \text{ such that } g(w) = z\}$ *is the backward orbit of z and* $E_G = \{z \in \hat{\mathbb{C}} : O^-(z) \text{ contains at most two points}\}$ *is the exceptional set of G.*

By Lemma 4.4,

$$\partial \left(\bigcup_{s \in \Sigma} \widetilde{K}_s \right) = \bigcup_{s \in \Sigma} \partial \widetilde{K}_s = \bigcup_{s \in \Sigma} \widetilde{J}_s = \overline{\bigcup_{s \in \Sigma \setminus \Sigma_4} \widetilde{J}_s} = J_G.$$

□

We have completed the proof of Theorem 1.9.

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