Bivariate Chebyshev maps of $\mathbb{C}^2$ and their dynamics

東海大学・理学部 内村桂輔 (Keisuke Uchimura)
Department of Mathematics
Tokai University

Abstract
We study the properties of bivariate (two-dimensional) Chebyshev maps $T_n(x, y)$ from $\mathbb{C}^2$ to $\mathbb{C}^2$ and study the properties and dynamics of the maps.
(A) The properties of $T_n$.
(1) Solutions of $T_n(x, y) = (0, 0)$ are obtained.
(2) A critical set $\det(DT_n) = 0$ is written in a simple formula.

These properties are similar to those of Chebyshev maps of $\mathbb{C}$.

(B) The dynamics of $T_n$.
(1) $T_n$ is strictly critically finite.
(2) Any periodic point of $T_n$ is repelling.
(3) The exact form of the invariant probability measure $\mu$ of maximal entropy associated with $T_n$ is obtained.
(4) External rays for $J_2(T_n)$ and foliations of $J_1(T_n)$ are studied.

These properties are also similar to those of Chebyshev maps of $\mathbb{C}$.

1 Bivariate Chebyshev maps

The Chebyshev map is a typical chaotic map. Generalized Chebyshev maps are studied by several researchers, Koornwinder [1974], Lidle [1975], Veselov [1987] and Hoffman & Withers [1988].

In this paper, we study bivariate Chebyshev maps $T_n$ from $\mathbb{C}^2$ to $\mathbb{C}^2$, $n \in \mathbb{Z}$.

$$T_n(x, y) = (g^{(n)}(x, y), g^{(n)}(y, x)).$$

This definition is due to [V]. Here $g^{(n)}(x, y)$ is a generalized Chebyshev polynomial defined by Lidle [L].
Let
\[ x = t_1 + t_2 + t_3, \quad y = t_1 t_2 + t_1 t_3 + t_2 t_3, \quad 1 = t_1 t_2 t_3. \]
Then
\[ g^{(n)}(x, y) := t_1^n + t_2^n + t_3^n. \]
So
\[ g^{(n)}(y, x) = (1/t_1)^n + (1/t_2)^n + (1/t_3)^n = g^{(-n)}(x, y). \]
For instance,
\[ T_2(x, y) = (x^2 - 2y, y^2 - 2x), \]
\[ T_3(x, y) = (x^3 - 3xy + 3, y^3 - 3xy + 3), \]
\[ T_4(x, y) = (x^4 - 4x^2y + 2y^2 + 4x, y^4 - 4xy^2 + 2x^2 + 4y). \]
\{g^n(x, y)\} satisfy the following recurrence equation:
\[ g^{(n)}(x, y) = xg^{(n-1)}(x, y) - yg^{(n-2)}(x, y) + g^{(n-3)}(x, y). \]

First, we show a branch covering over \( \mathbb{C}^2 \).
The following diagram is commutative.
\[
\begin{array}{ccc}
(C - \{0\})^2 & \overset{g_n}{\rightarrow} & (C - \{0\})^2 \\
\downarrow \psi & & \downarrow \psi \\
\mathbb{C}^2 & \overset{T_n}{\rightarrow} & \mathbb{C}^2 \\
\end{array}
\]
where
\[ g_n(u, v) = (u^n, v^n), \]
and
\[ (x, y) = \Psi(u, v) = (u + v + \frac{1}{uv}, \frac{1}{u} + \frac{1}{v} + uv). \]
The covering map
\[ \Psi : \mathbb{C}^2 - \Psi^{-1}(D) \rightarrow \mathbb{C}^2 - D \]
is a 6-sheated covering map. Branch locus \( D \) of \( \Psi \) is written as
\[ x^2y^2 - 4x^3 - 4y^3 + 18xy - 27 = 0. \]
In the case \( n = 2 \), Ueda[Ue] showed this diagram.
\( T_n(x, y) \) restricted on \( \{x = \overline{y}\} \) is a Chebyshev polynomial defined by Koornwinder [K]
\[ P_{n,0}^{-\frac{1}{2}}(z, \overline{z}) = e^{i\sigma} + e^{-i\tau} + e^{i(n\tau - n\sigma)}. \]
Set
\[ z(\sigma, \tau) := e^{i\sigma} + e^{-i\tau} + e^{i(\tau - \sigma)} = u + iv. \]
The mapping
\[ z : (\sigma, \tau) \rightarrow (u, v) \]
is a diffeomorphism from $R$ onto $S$. See Koornwinder [K].

**Proposition 1.** There are $n^2$ solutions of $T_n(x, y) = (0, 0)$. All solutions lie in the closed domain $S$ in $\{x = \overline{y}\}$. They are written in the $(\sigma, \tau)$ coordinate.

1.\[(\sigma, \tau) = \left(\frac{2(1+j+h)\pi}{3n}, \frac{2(1+2j+h)\pi}{3n}\right)\]
   \[j = 0, 1, \ldots, n-1, \text{and} \quad h = 0, 1, \ldots, j.\]

2.\[(\sigma, \tau) = \left(\frac{2(2+j+h)\pi}{3n}, \frac{2(2+2j-h)\pi}{3n}\right)\]
   \[j = 0, 1, \ldots, n-2, \text{and} \quad h = 0, 1, \ldots, j.\]

**Proof.** By definition,

\[T_n(x, y) = (g^{(n)}(x, y), g^{(n)}(y, x)).\]

$g^{(n)}(x, y)$ and $g^{(n)}(y, x)$ are polynomials of degree $n$ with no common components. We can find $n^2$ zeros on $S$. See Uchimura [Uc1].

We see that the zeros of $T_n$ and $T_{n+1}$ "mutually separate each other". Next we consider critical set of $T_n(x, y)$.

\[C_n := \{(x, y) \in \mathbb{C}^2 : \text{det}(DT_n) = 0\}.\]

**Proposition 2.** Let $n \in \mathbb{Z}$. Assume that

\[x = t_1 + t_2 + t_3, \quad y = t_1 t_2 + t_1 t_3 + t_2 t_3, \quad t_1 t_2 t_3 = 1.\]

Then

\[\text{det}(DT_n) = n^2 \frac{t_1^n t_2^n}{t_1 - t_2} \cdot \frac{t_1^n t_3^n}{t_1 - t_3} \cdot \frac{t_2^n t_3^n}{t_2 - t_3}.\]

**Proof.**

\[\text{det}(DT_n) = \frac{\text{det}(D(T_n \circ \Psi))}{\text{det}(D\Psi)}.\]

The similar result is holds for generalized Chebyshev maps from $\mathbb{C}^n$ to $\mathbb{C}^n$.

**Corollary 1.** Any irreducible component of $C_n$ is a rational curve of degree 2 or 4.

**Proof.** From Proposition 2, we have

\[x = t + \epsilon^k t + \frac{1}{\epsilon \epsilon^{k+2}}, \quad \epsilon = e^{\frac{2\pi i}{n}}\]
$y = \frac{1}{t} + \frac{1}{\epsilon^k t} + \epsilon^k t^2.$

When $\epsilon^k = -1$, the degree of the rational curve is 2.

We see that $C_n$ and $C_{n+1}$ "mutually separate each other", and

$C_n \cap S \neq \phi$ \hspace{1cm} ($S = J_2(T_n))$.

Note that $\{T_m : m \in \mathbb{Z}\}$ is a semigroup satisfying

$T_m \circ T_n = T_{mn}$.

2 Dynamics of Bivariate Maps

We study the dynamics of $T_n(x, y)$. Let

\[ K(T_n) := \{(x, y) : \{T_n^m(x, y)\} \text{is bounded for any } m\}. \]

In our setting we have the following proposition.

Proposition 3. \hspace{1cm} $K(T_n) = \{|t_1| = |t_2| = 1\} = S \subset \{x = \overline{y}\}$.

Proof

\[
\begin{array}{ccc}
(t_1, t_2) & \xrightarrow{g_n} & (t_1^n, t_2^n) \\
\downarrow{\psi} & & \downarrow{\psi} \\
(x, y) & \xrightarrow{T_n} & (g^{(n)}, g^{(-n)})
\end{array}
\]

$\square$ $f$ is called critically finite if each irreducible component of the critical set of $f$ is periodic or preperiodic. Dihn and Sibony [DS] show that generalized Chebyshev maps are critically finite. Here using proposition 2, we give a direct proof.

Proposition 4. \hspace{1cm} $T_n$ is strictly critically finite.

Proof.

\[ C_n \xrightarrow{T_n} T_n(C_n) \xrightarrow{T_n} T_n(C_n) \]

\[
(t, \epsilon t) \quad (t^n, t^n) \quad (t^{n^2}, t^{n^2}) \]

$\square$

Next we study the second Julia set $J_2$ of $T_n(x, y)$.

Proposition 5. \hspace{1cm} All periodic points of $T_n$ lie on $S$ and are equidistributed in $S$. 
Proof. From [FS], we know that number of periodic points with period $k$ equals $n^{2k}$.
For the distribution of periodic points, see [Uc2].

Proposition 6. Any periodic point of $T_n$ is repelling.

To prove this proposition we consider the following function.

$$S_n := T_n \mid \{x = \bar{y}\} : \mathbb{R}^2 \to \mathbb{R}^2$$

\text{e.g.} \quad S_2(z) = z^2 - 2\bar{z} : (u, v) \mapsto (u^2 - 2u - v^2, 2uv + 2v).

Lemma 1. Let \( p \) be a periodic point of \( S_n \). Let \( \alpha \) and \( \beta \) be eigenvalues of \( D S_n(p) \).
Then
\[ |\alpha|, \quad |\beta| > 1.\]

Proposition 7. Let $f(x, y) \in \mathbb{R}[x, y]$.

$$T(x, y) := (f(x, y), f(y, x)) : \mathbb{C}^2 \to \mathbb{C}^2.$$ $t(z) := T \mid \{x = \bar{y}\} : \mathbb{R}^2 \to \mathbb{R}^2.$

Then
\[ U^{-1}DT(z, \bar{z})U = Dt(z), \]
where
\[ U = \frac{1}{2} \begin{pmatrix} 1 + i & -1 + i \\ 1 + i & 1 - i \end{pmatrix}. \]

From Lemma 1 and Proposition 7, Proposition 6 follows.

Next we study the invariant measure $\mu$ of maximal entropy for $T_n$.

Proposition 8. Under the above notation,

$$\text{supp } \mu = S.$$ $\mu = \left(\frac{2}{\pi}\right)^2 \frac{dx_1 dx_2}{\sqrt{-x^2 x^2 + 4x^3 + 4x^3 - 18x \bar{x} + 27}}.$$
\( (x = x_1 + ix_2)\)
This is an extension of invariant measure

$$\mu = \frac{1}{\pi} \frac{dx}{\sqrt{(x+1)(3-x)}}$$

for Chebyshev maps in one variable on \([-1, 3]\).

**Proof.** We prove this proposition in the following three steps.

1. Briand and Duval [BD] shows that let

   $$\mu_n := \frac{1}{dn_k} \sum_{f^n(y) = y, \text{repelling}} \delta_y,$$

   then

   $$\mu_n \to \mu$$ (weak convergence).

2. From Proposition 5, we see that the periodic points are repelling and equidistributed in the triangle on the \((s,t)\) plane (see [Uc2]).

3. Pullback of Lebesgue measure under \(\phi\).

Next we consider the properties of external rays of \(T_n(x, y)\). We use the definitions of external rays by Bedford and Jonsson [BJ]. We extend the map

$$T_n(x, y) : \mathbb{C}^2 \to \mathbb{C}^2$$

to

$$\hat{T}_n(x : y : z) : \mathbb{P}^2 \to \mathbb{P}^2.$$

Let \(\Pi := \mathbb{P}^2 - \mathbb{C}^2\) be the line at infinity.

Then

$$\hat{T}_n | \Pi : (x : y : 0) \to (x^n : y^n : 0).$$

Therefore

$$J_\Pi = \{(x : y : 0) : |x| = |y|\} \simeq S^1.$$

The stable set of \(J_\Pi\) for \(T_n\) is defined by

$$W^s(J_\Pi, T_n) := \{x \in \mathbb{P}^2 : d(T_n^jx, J_\Pi) \to 0, \ j \to \infty\}.$$

Bedford and Jonsson [BJ] state that there exists a Böttcher coordinate \(\Psi\) such that

$$\Psi : W^s(J_\Pi, f_n) \to W^s(J_\Pi, T_n)$$
conjuring $f_n$ to $T_n$, where

$$f_n(x, y) = (x^n, y^n).$$

They also show that $W^s(J_{II}, T_n)$ is foliated by stable disks $W_a$. They define a local stable manifold $W^s_{loc}(a)$, $(a \in J_{II})$ and then a stable disk $W_a \supset W^s_{loc}(a)$ and an external ray $R(a, \theta)$. They show that $J_0(T_n) = J_1(T_n)$ is laminated by stable disks $W_a$.

Nakane [N] shows the following results on $T_2(x, y)$:

(1) The map $\Psi$ defined by Ueda is essentially the inverse of Böttcher coordinate $\phi$.

$$\Psi(u, v) = \Psi(t, at), |t| > 1.$$ 

(2) The stable disk $W_a$ is the set of points $R(r, \phi, \theta)$

$$x = re^{-2\pi i \theta} + \frac{1}{r}e^{2\pi i (\theta - \phi)} + e^{2\pi i \phi},$$

$$y = re^{2\pi i (\phi - \theta)} + \frac{1}{r}e^{2\pi i \theta} + e^{-2\pi i \phi}, \quad a = e^{2\pi i \phi}, \quad (r > 1).$$

An external ray is written as

$$R(\phi, \theta) := \{R(r, \phi, \theta) : r > 1\}.$$ 

From this,

$$J_2 = S \subset \{x = \bar{y}\}.$$ 

(3) Each point $z \in S$ is the landing point of exactly 1, 3, or 6 external rays if $z$ is a cusp point on $\partial S$, $z$ is non-cusp point on $\partial S$ or $z \in \text{int}(S)$ respectively.

We can show that Nakane’s results are also true for any $T_n(x, y), \quad n \neq 0$.

Next we study the structure of foliations $W_a$ of

$$J_1(T_n) = W^s(J_{II}, T_n).$$

**Proposition 9.** For any point $z \in \text{int}(S)$, there exist three stable disks $W_a$ such that boundaries of these three disks intersect at $z$. At the point, two external rays on each $W_a$ land from opposite directions.

Metaphorically speaking, three mouths (stable disks) eat a sandwich (the second Julia set $S$).

Two external rays $R(\phi, \theta)$ and $R(\phi, \phi - \theta)$ lie on the stable disk

$$W_a \quad (a = e^{2\pi i \phi}).$$
Two points $R(r, \phi, \theta)$ and $R(r, \phi, \phi - \theta)$ are "symmetrical" about $\{x = y\}$ in the following sense.

(1) The midpoint of the segment $\overline{R(r, \phi, \theta)R(r, \phi, \phi - \theta)}$ lies on the plane $\{x = y\}$,

(2) The segment connecting two points is perpendicular to $\{x = y\}$.

We compare the external rays of $T_n(x, y)$ with those of Chebyshev map $T_n(z)$ in one variable. The external rays $T_n(z)$ is written as

$$R(r, \phi) : u = re^{2\pi i \phi} + \frac{1}{r} e^{2\pi i(-\phi)}, \quad (r > 1).$$

Clearly,

$$R(r, -\phi) : v = re^{2\pi i(-\phi)} + \frac{1}{r} e^{2\pi i\phi},$$

$$v = \bar{u}.$$  

It is well-known that $R(r, \phi)$ and $R(r, -\phi)$ are "symmetrical" about the real axis.

Note that symmetric group $S_2$ acts on external rays of $T_n(z)$. On the other hand, $S_3$ acts on external rays of $T_n(x, y)$.

Using the notations in Sect. 1, we can write

$$W^s(J_n, T_n) = \{\Psi(t_1, t_2) : |t_1| = \frac{1}{|t_2|} > 1\}.$$  

Then

$$C_n \cap W^s(J_n, T_n) = \phi.$$  

Lastly we consider periodic rays $R(\phi, \theta)$ of $T_n(x, y)$.

**Proposition 10.** If one periodic ray lands at the point $z_0 \in S$, all rays which land at $z_0$ are all periodic with the same period.

**References**


[DS] T. Dihn and N. Sibony, *Sur les endomorphismes holomorphes permutable*


