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Remarks on Quasi-linear Wave Equations Related to Gas Dynamics

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dedicated to Professor Tai-Ping Liu on his 60th birthday

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1 Introduction

We consider the equation

$$y_{tt}-(G(y_x))_x=0 \tag{1}$$

on $0<x<1$ with the boundary conditions

$$y(t,0)=y(t,1)=0. \tag{2}$$

Here we suppose that $G(v)$ is real analytic in $|v|<\delta, G(0)=0, G'(0)=\gamma>0$ and $G'(v)>0$ for $|v|<\delta$. As example we keep in mind

$$G(v)=1-(1+v)^{-\gamma}, \qquad (|v|<1).$$

The linearized problem is

$$y_{1,tt}-\gamma y_{1,xx}=0, \qquad y_1(t,0)=y_1(t,1)=0,$$

which has smooth time periodic solutions

$$y_1 = \sum_{n=1}^{N} a_n \sin(n\pi\sqrt{\gamma}(t+\theta_n)) \sin n\pi x. \qquad (3)$$

Thus we have the problem: Are there time periodic solutions for (1)(2) near to the periodic solution of the linaerized problem? We have not yet obtained an answer to this question. In this report we give some related observations to this problem. Detailed discussion can be found in [2].

2 Derivation of the problem

We consider one-dimensional movement of polytropic gas without external force governed by the compressible Euler equation

$$\rho_t + (\rho u)_x = 0, \qquad (\rho u)_t + (\rho u^2 + P)_x = 0$$

on a fixed interval $0 < x < L$ with the boundary conditions

$$\rho u|_{x=0} = \rho u|_{x=L} = 0.$$

We assume that $P = A\rho^\gamma$, where A and γ are positive constants such that $1 < \gamma \leq 2$. Equilibria are constant densities $\rho = \bar{\rho} = Const. > 0, u = 0$.

Let us introduce the Lagrangean coordinate

$$m = \int_0^x \rho dx.$$

Then the equation is reduced to

$$x_{tt} + (A(x_m)^{-\gamma})_m = 0,$$

where $x = x(t, m)$ is new unknown function, while

$$u = \frac{\partial x}{\partial t}, \qquad \frac{1}{\rho} = \frac{\partial x}{\partial m}.$$

We consider the perturbation y near the equilibrium

$$x(t, m) = \bar{x} + y = \frac{L}{M}m + y,$$

where M is the total mass. Taking $\bar{x}$ as the independent variable and normalizing the variables, we have (1) and (2) with

$$G(v) = 1 - (1+v)^{-\gamma}.$$

3 Existence of smooth solutions on long time

Let us fix a smooth time periodic solution $y_1(t,x)$ of the form (3) of the linearized problem.

Theorem 1. *For any positive number T there are positive constants ε^* and C such that for any $0 < \varepsilon \le \varepsilon^*$ we have C^2-solution $y(t,x)$ of (1)(2) on $0 \le t \le T$ such that*

$$|y(t,x) - \varepsilon y_1(t,x)| \le C\varepsilon^2$$

for $0 \le t \le T$ and $0 \le x \le 1$. Here

$$y(0,x) = \varepsilon y_1(0,x), \qquad y_t(0,x) = \varepsilon y_{1,t}(0,x).$$

Before proving this theorem we consider the problem by extending solutions as

$$y(t,x) = -y(t,-x), \qquad y(t,x+2n) = y(t,x)$$

for any $n \in \mathbb{Z}$. Putting

$$u_1 = y_{1,t}, \quad v_1 = y_{1,x}, \qquad y_t = \varepsilon u_1 + U, \quad y_x = \varepsilon v_1 + V,$$

we have

$$\begin{aligned} &V_t - U_x = 0, \\ &U_t - G'(\varepsilon v_1 + V)V_x = (G'(\varepsilon v_1 + V) - \gamma)\varepsilon v_{1,x}. \end{aligned}$$

The variables

$$W = U + \hat{G}(\varepsilon v_1 + V) - \hat{G}(\varepsilon v_1), \qquad Z = U - \hat{G}(\varepsilon v_1 + V) + \hat{G}(\varepsilon v_1),$$

where

$$\hat{G}(v) = \int_0^v \sqrt{G'(s)}ds,$$

reduce the equation to the diagonalized equation

$$W_t - \Lambda(\varepsilon v_1 + V)W_x = L_-, \qquad Z_t + \Lambda(\varepsilon v_1 + V)Z_x = L_+,$$

where

$$\Lambda(v) = \sqrt{G'(v)},$$

and

$$L_\pm = (-\gamma + \Lambda(\varepsilon v_1 + V)\Lambda(\varepsilon v_1))\varepsilon v_{1,x} \pm (\Lambda(\varepsilon v_1) - \Lambda(\varepsilon v_1 + V))\varepsilon u_{1,x}.$$

We look for solutions W, Z such that

$$W(t,x) = -Z(t,-x), \qquad W(t,x+2n) = W(t,x).$$

For simplicity we consider solutions which satisfy the initial conditions

$$W(0,x) = Z(0,x) = 0.$$

We construct solutions by iterations. Given $V(t,x)$, we solve

$$\begin{aligned} \tilde{W}_t - \Lambda(\varepsilon v_1 + V)\tilde{W}_x &= L_-(t,x,V(t,x)), \\ \tilde{Z}_t + \Lambda(\varepsilon v_1 + V)\tilde{Z}_x &= L_+(t,x,V(t,x)), \end{aligned}$$

and find $\tilde{U}, \tilde{V}$ by

$$\tilde{W} = \tilde{U} + \hat{G}(\varepsilon v_1 + \tilde{V}) - \hat{G}(\varepsilon v_1), \qquad \tilde{Z} = \tilde{U} - \hat{G}(\varepsilon v_1 + \tilde{V}) + \hat{G}(\varepsilon v_1).$$

The solution of the above problem is given by the integral along the characteristic curves: if $\xi(\tau) = \xi(\tau; t, x)$ is the solution of

$$\frac{d\xi}{d\tau} = \Lambda(\varepsilon v_1 + V)(\tau, \xi(\tau)), \qquad \xi(t) = x,$$

then

$$\tilde{Z}(t,x) = \int_0^t L_+(\tau, \xi(\tau), V(\tau, \xi(\tau)))d\tau,$$

and so on. Through tedious computations we can prove the following lemmas.

Lemma 1. *Thare exist $M_0 > 0, \varepsilon_0 > 0$ such that if $0 < \varepsilon \le \varepsilon_0$ and $||V|| \le \varepsilon^2 M_0$, then $||\tilde{U}|| \le \varepsilon^2 M_0, ||\tilde{V}|| \le \varepsilon^2 M_0$ and $\varepsilon||v_1|| + \varepsilon^2 M_0 \le \delta/2$.*

Here

$$||f|| = \sup\{|f(t,x)| \quad |0 \le t \le T, x \in \mathbb{R}\}.$$

Lemma 2. *There exist $0 < \varepsilon_1(\le \varepsilon_0), M_1$ such that if $0 \le \varepsilon \le \varepsilon_1$, $||V|| \le \varepsilon^2 M_0, ||V_x|| \le \varepsilon^2 M_1$, then $||\tilde{U}_x||, ||\tilde{U}_t||, ||\tilde{V}_x||, ||\tilde{V}_t|| \le \varepsilon^2 M_1$ and $\varepsilon^2 M_1 < 1$.*

Lemma 3. *There exist* $0 < \varepsilon_2(\leq \varepsilon_1), M_2$ *such that if* $0 < \varepsilon \leq \varepsilon_2$,

$$||V|| \leq \varepsilon^2 M_0, ||V_x|| \leq \varepsilon^2 M_1, ||V_{xx}|| \leq \varepsilon^2 M_2,$$

then $||\tilde{U}_{xx}||, ||\tilde{U}_{tx}||, ||\tilde{V}_{xx}||, ||\tilde{V}_{xt}||, ||\tilde{V}_{tt}|| \leq \varepsilon^2 M_2$ *and* $\varepsilon^2 M_2 < 1$.

Moreover, if V^1, V^0 satisfy the conditions of the lemmas, then we can prove that

$$||\tilde{V}^1 - \tilde{V}^0|| \leq \frac{1}{2}||V^1 - V^0||,$$

if $\varepsilon < \varepsilon^*(\leq \varepsilon_2)$, which is sufficiently small.

Thus we consider the iteration

$$V^{(0)} = 0, \qquad V^{(n+1)} = \widetilde{V^{(n)}}.$$

Then $V^{(n)}$ converges to the limit V and

$$y(t,x) = \varepsilon y_1(t,x) + \int_0^x V(t,s)ds$$

gives the required solution.

4 Non-existence of time global smooth solutions

Let us apply the arguemnet of [4].

Theorem 2. *Suppose that* $G''(v) < 0$ *for* $|v| < \delta$ *or* $G''(v) > 0$ *for* $|v| < \delta$. *If* $y(t,x) \in C^2([0,+\infty) \times [0,1])$ *is a solution of (1)(2) such that* $|y_x(t,x)| \leq \delta_1$ *for* $t \geq 0, x \in [0,1]$, *where* $0 < \delta_1 < \delta$, *then* $y = 0$ *identically.*

Proof. We can suppose that $y \in C^2([0,\infty) \times \mathbb{R})$ solves (1) and $y(t,x) = -y(t,-x), y(t,x+2n) = y(t,x)$ for $n \in \mathbb{Z}$.

Putting

$$w = y_t + \hat{G}(y_x), \qquad z = y_t - \hat{G}(y_x),$$

we reduce the equation to

$$w_t - \Lambda(y_x)w_x = 0, \qquad z_t + \Lambda(y_x)z_x = 0.$$

By the assumption, we have

$$\frac{1}{C} \leq \Lambda(y_x) \leq C.$$

Now let $x = x(t) = x(t; a)$ solve

$$\frac{dx}{dt} = \Lambda(y_x(t, x(t)), \qquad x(0) = a.$$

Consider

$$X(t) = \frac{\partial}{\partial a} x(t; a).$$

Then

$$X(t) = \exp\Big(\int_0^t \frac{\partial}{\partial x}\Lambda(y_x(\tau, x))d\tau\Big) > 0.$$

On the other hand we can prove that

$$X = \Big(\frac{\Lambda(y_x(t, x(t))}{\Lambda(y_x(0, a))}\Big)^{1/2}(1 + z_a(0, a)\int_0^t Q(\tau)d\tau),$$

where

$$Q(\tau) = -\Big(\frac{\Lambda(y_x(0, a))}{\Lambda(y_x(\tau, x(\tau))}\Big)^{1/2}\frac{1}{4}\frac{G''(y_x(\tau, x(\tau))}{G'(y_x(\tau, x(\tau))}.$$

If $G'' < 0$, then $Q \geq 1/c > 0$ and $\int_0^t Q(\tau)d\tau \to +\infty$ as $t \to \infty$. Thus we have $z_a(0, a) \geq 0$ for any a. Since $z(0, .)$ is periodic, we have $z(0, .)$ =Const. By a similar discussion we have $w(0, .)$ =Const. This implies the result.

5 Exact solutions

In [3] F. John gave exact solutions to the equation

$$y_{tt} - (1 + y_x^2)y_{xx} = 0.$$

Along the idea of F. John we construct exact solutions of our general equation

$$y_{tt} - (G(y_x))_x = 0.$$

Let f be arbitrary function of $C_0^\infty(\mathbb{R})$. Suppose that $v = v(t, x)$ solves the equation

$$v = f'(x - \Lambda(v)t). \tag{4}$$

Put

$$y(t,x) = t\Phi(v(t,x)) + f(x - \Lambda(v(t,x))t),$$

where

$$\Phi(v) = \int_0^v \zeta\Lambda'(\zeta)d\zeta = v\Lambda(v) - \hat{G}(v).$$

Then we have

$$y_t = -\hat{G}(v), \quad y_{tt} = -\Lambda v_t, \qquad y_x = v, \quad y_{xx} = v_x.$$

On the other hand

$$v_t = -\frac{\Lambda f''}{1 + f''\Lambda' t}, \qquad v_x = \frac{f''}{1 + f''\Lambda' t},$$

as long as

$$1 + f''(x - \Lambda(v)t)\Lambda'(v)t \neq 0.$$

Thus $y(t,x)$ satisfies the equation with the initial conditions

$$y(0,x) = f(x), \qquad y_t(0,x) = -\hat{G}(f'(x)).$$

Suppose that $f \in C_0^\infty(\mathbb{R})$ satisfies that $|f'(\xi)| \leq \delta_0 (< \delta)$ for any $\xi \in \mathbb{R}$ and

$$-m = \min_\xi f''(\xi)\Lambda'(f'(\xi)) < 0 \leq \max_\xi f''(\xi)\Lambda'(f'(\xi)) \leq m.$$

Put $T = 1/m$. The equation (4), which is equivalent to the equation

$$\xi = x - \Lambda(f'(\xi))t,$$

admits a unique solution as long as $0 \leq t < T$. As $t \to T - 0$, we see that $y_x = v, y_t = -\hat{G}(v)$ remain to be bounded but $y_{xx} \to \infty$. This is a typical example of singularity which happens after a finite time for smooth solutions.

6 Estimate of life span of smooth solutions

Let us apply the theory of Lax [5] to find estimate of life span of smooth solutions.

Consider the initial value problem:

$$y_{tt} - (G(y_x))_x = 0, \qquad y(t,0) = y(t,1) = 0,$$

$$y(0,x)=\phi(x), \qquad y_t(0,x)=\psi(x),$$

where ϕ, ψ are smooth and

$$\phi(0)=\phi(1)=\psi(0)=\psi(1)=0.$$

Theorem 3. *There exist constants ε, C such that if $|\phi_x(x)|, |\psi(x)| \leq \varepsilon$ and if*

$$|\phi_{xx}(x)|, \qquad |\psi_x(x)| \leq M,$$

then there exists a solution $y(t,x)$ of class C^2 as long as $0 \leq t \leq 1/CM$.

Observe

$$w = y_t + \hat{G}(y_x), \qquad z = y_t - \hat{G}(y_x),$$

which satisfy

$$w_t - \Lambda w_x = 0, \qquad z_t + \Lambda z_x = 0.$$

Thus a priori estimates of $|w|, |z|$ are obvious.

Consider the quantities

$$A = \sqrt{\Lambda} w_x, \qquad B = \sqrt{\Lambda} z_x,$$

which satisfy

$$A_t - \Lambda A_x + \mu A^2 = 0, \qquad B_t + \Lambda B_x + \mu B^2 = 0,$$

where

$$\mu = -\frac{1}{4} G''(y_x) G'(y_x)^{-5/4}.$$

Note that $|\mu| \leq C$ a priori. As P. D. Lax said in [5]: "solution to initial-value problems exists as long as one can place an a priori limitation on the magnitude of their first derivatives." Thus this completes the proof.

7 Problem with vacuum

Originally we are interested in the equation

$$y_{tt} - \frac{1}{\rho}(PG(y_x))_x = 0,$$

where

$$G(v) = 1-(1+v)^{-\gamma}, \qquad \rho = (1-x)^{\frac{1}{\gamma-1}}, \qquad P = (1-x)^{\frac{\gamma}{\gamma-1}}.$$

This equation is derived from the motion of gas under constant gavity governed by the equation

$$\rho_t + (\rho u)_x = 0, \qquad (\rho u)_t + (\rho u^2 + P)_x = -\rho g,$$

where $P = A\rho^\gamma$ and g is positive constant. Equilibria are

$$\bar{\rho} = \begin{cases} A_1(L-x)^{\frac{1}{\gamma-1}} & (0 \leq x < L) \\ 0 \quad (L < x), \end{cases}$$

where L is positive constant determined by the total mass and

$$A_1 = \left(\frac{g(\gamma-1)}{A\gamma}\right)^{\frac{1}{\gamma-1}}.$$

The linearized problem has time periodic smooth solutions explicitely written by the Bessel function of order $1/(\gamma-1)$. Detailed discussion can be found in [1].

But we have not yet proven parallel results for this problem because of the singularity at $x = 1$.

References

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