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L_p - L_q maximal regularity for the Stokes equation with first order boundary condition

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1 Problem

Thoughout this paper Ω denotes a bounded or exterior domain in $\mathbb{R}^n$ $(n \geqq 2)$ with $C^{2,1}$ boundary $\partial\Omega$ and $0 < T \leqq \infty$. We consider the generalized Stokes equation:

$$u_t - \operatorname{Div} S(u, \pi) = f, \quad \operatorname{div} u = g = \operatorname{div} \tilde{g} \quad \text{in } \Omega \times (0, T) \tag{1.1}$$

which is obtained as a linealization of the Navier-Stokes equation describing the motion of a viscous incompressible Newtonian fluid in Ω. Here, f is a given exterior force field, g and $\tilde{g}$ are given functions, and $u = (u_1, \ldots, u_n)$ and π are the unknown velocity and pressure field, respectively. In (1.1) we denoted the stress tensor of the fluid by

$$\begin{aligned} &S(u, \pi) = D(u) - \pi I, \ \ D(u) = (D_{ij}(u)) \\ &D_{ij}(u) = \partial_i u_j + \partial_j u_i, \ \ I : n \times n \text{ identity matrix} \end{aligned}$$

A large number of papers have been devoted to the study of these equations. However, the overwhelming majority of those works is concerned with the Stokes equation coupled with non-slip boundary condition: $u|_{\partial\Omega} = 0$, that is, the fluid is required to be at rest at the boundary. The purpose of this paper is to study the Stokes equation with different, but nonetheless physically reasonable, boundary condition. Namely, as the boundary condition, we consider

$$B(u, \pi)|_{\partial\Omega} = T_1^\alpha(u, \pi)|_{\partial\Omega} = \begin{cases} \nu \cdot u|_{\partial\Omega} = 0 \\ (S(u, \pi)\nu - (S(u, \pi)\nu, \nu)\nu + \alpha u)|_{\partial\Omega} = h|_{\partial\Omega} \end{cases} \tag{1.2}$$

as well as

$$B(u, \pi)|_{\partial\Omega} = T_2(u, \pi)|_{\partial\Omega} = S(u, \pi)\nu|_{\partial\Omega} = h|_{\partial\Omega} \tag{1.3}$$

where ν denotes the unit outer normal to $\partial\Omega$. $T_1^\alpha(u, \pi)$ is called slip boundary condition when $\alpha = 0$ and Robin boundary condition when $\alpha > 0$ [1)], respectively. The boundary condition T_1^α admits motion of the fluid at the boundary in tangential directions. On the other hand,

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1) Throughout the paper, we assume that $\alpha \geqq 0$.

the boundary condition $T_2(u,\pi)$ is called Neumann boundary condition, which is derived as a linearization of some free boundary problem for the Navier-Stokes equations which will be discussed in section 5.

Complementing equations (1.1), (1.2) and (1.3) with initial condition for the velocity field, we arrive at the following initial boundary value problem to be studied

$$\begin{aligned} &u_t - \operatorname{Div} S(u,\pi) = f && \text{in } \Omega \times (0,T)\\ &\operatorname{div} u = g = \operatorname{div} \tilde{g} && \text{in } \Omega \times (0,T)\\ &B(u,\pi) = h && \text{on } \partial\Omega \times (0,T)\\ &u|_{t=0} = u_0 && \text{in } \Omega \end{aligned} \tag{1.4}$$

2 Analytic semigroup approach to (1.4)

In order to treat (1.4) in the analytic semigroup framework, in this section we consider the initial-boundary value problem:

$$\begin{cases} u_t - \operatorname{Div} S(u,\pi) = 0, \quad \operatorname{div} u = 0 & \text{in } \Omega \times (0,\infty)\\ B(u,\pi)|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0 \end{cases} \tag{2.1}$$

First we consider the slip or Robin boundary condition case. Namely, $B(u,\pi)$ is given by the following formula:

$$B(u,\pi) = T_1^\alpha(u,\pi) = \begin{cases} S(u,\pi)\nu - (\nu \cdot S(u,\pi)\nu)\nu + \alpha u = D(u)\nu - (\nu \cdot D(u)\nu)\nu + \alpha u\\ \nu \cdot u \end{cases}$$

To eliminate the pressure term π in the first equation of (2.1) we use the Helmholtz decomposition:

$$\begin{aligned} L_q(\Omega)^n &= J_q(\Omega) \oplus G_q(\Omega)\\ J_q(\Omega) &= \overline{\{u \in C_0^\infty(\Omega)^n \mid \operatorname{div} u = 0\}}\\ &= \{u \in L_q(\Omega)^n \mid \operatorname{div} u = 0 \text{ in } \Omega,\ \nu \cdot u|_\Gamma = 0\}\\ G_q(\Omega) &= \{\nabla\pi \mid \pi \in X_q(\Omega)\} \end{aligned}$$

The space $X_q(\Omega)$ is defined in the following way: When Ω is a bounded domain,

$$X_q(\Omega) = \{\pi \mid \pi \in W_q^1(\Omega),\ \int_\Omega \pi\, dx = 0\}, \quad \|\pi\|_{X_q(\Omega)} = \|\pi\|_{W_q^1(\Omega)}$$

When Ω is an exterior domain and $1 < q < n$,

$$\begin{aligned} X_q(\Omega) &= \{\pi \in L_{nq/(n-q)}(\Omega) \mid \nabla\pi \in L_q(\Omega),\quad \|\pi\|_{X_q(\Omega)} < \infty\}\\ \|\pi\|_{X_q(\Omega)} &= \|\pi\|_{L_{nq/(n-q)}(\Omega)} + \|\nabla\pi\|_{L_q(\Omega)} + \|\pi(d_q)^{-1}\|_{L_q(\Omega)} \end{aligned}$$

When Ω is an exterior domain and $n \leqq q < \infty$,

$$\begin{aligned} X_q(\Omega) &= \{\pi \in L_{q,\mathrm{loc}}(\overline{\Omega}) \mid \nabla\pi \in L_q(\Omega),\quad \|\pi\|_{X_q(\Omega)} < \infty\}\\ \|\pi\|_{X_q(\Omega)} &= \|\nabla\pi\|_{L_q(\Omega)} + \|\pi(d_q)^{-1}\|_{L_q(\Omega)} \end{aligned}$$

Here, $d_q(x)$ is a weight function defined by the relations:

$$d_q(x) = \begin{cases} 1 + |x| & \text{when } q \neq n,\ 1 < q < \infty \\ (1 + |x|)\log(2 + |x|) & \text{when } q = n \end{cases}$$

In fact, such decomposition was proved by Fujiwara and Morimoto [9], Farwig and Sohr [8], Galdi [11], Miyakawa [16], Simader and Sohr [24] and references therein. In this case, given $u \in L_q(\Omega)^n$, we choose $\pi \in X_q(\Omega)$ as a weak solution to the Neumann problem:

$$\Delta\pi = \nabla \cdot u \text{ in } \Omega, \quad \left.\frac{\partial\pi}{\partial\nu}\right|_{\partial\Omega} = \nu \cdot u|_{\partial\Omega} \tag{2.2}$$

And then, we define the linear continuous projection $P_q : L_q(\Omega)^n \to J_q(\Omega)$ along $G_q(\Omega)$ by the relation: $P_q u = u - \nabla\pi$. Noticing that $\operatorname{Div} S(u,\pi) = \Delta u - \nabla\pi$ when $\operatorname{div} u = 0$, we define the Stokes operator A_q^1 corresponding to (2.1) with $B(u,\pi) = T_1^\alpha(u,\pi)$ by

$$A_q^1 = P(-\Delta) \tag{2.3}$$

with domain:

$$D(A_q^1) = \{u \in J_q(\Omega) \cap W_q^2(\Omega)^n \mid (D(u)\nu - (\nu \cdot D(u)\nu)\nu + \alpha u)|_{\partial\Omega} = 0\}. \tag{SD}$$

Concerning the generation of an analytic semigroup by the Stokes operator A_q^1 with slip or Robin boundary condition, we have following theorem.

Theorem 2.1. *Let $1 < q < \infty$ and let Ω be a bounded domain or an exterior domain in $\mathbb{R}^n$ $(n \geqq 2)$ with $C^{2,1}$ boundary $\partial\Omega$. Then, A_q^1 generates an analytic semigroup $\{T_q^1(t)\}_{t\geqq 0}$ on $J_q(\Omega)$.*

Moreover, when Ω is bounded, $\{T_q^1(t)\}_{t\geqq 0}$ is exponentially stable. In particular, for $t > 0$ we have the following estimates:

$$\begin{aligned} \|T_q^1(t)u_0\|_{L_q(\Omega)} &\leqq C_q e^{-c_q t}\|u_0\|_{L_q(\Omega)} && \forall u_0 \in J_q(\Omega) \\ \|T_q^1(t)u_0\|_{W_q^2(\Omega)} &\leqq C_q t^{-1} e^{-c_q t}\|u_0\|_{L_q(\Omega)} && \forall u_0 \in J_q(\Omega) \\ \|T_q^1(t)u_0\|_{W_q^2(\Omega)} &\leqq C_q e^{-c_q t}\|u_0\|_{W_q^2(\Omega)} && \forall u_0 \in D(A_q^1) \end{aligned}$$

with some positive constants c_q and C_q.

Remark 2.2. In the bounded domain case, the generation of the analytic semigroup of the Stokes operator A_q^1 on $J_q(\Omega)$ was proved by Giga [12]. Its exponential stability was proved by Shibata-Shimada [17]. In the exterior domain case, the generation of the analytic semigroup of the Stokes operator A_q^1 on $J_q(\Omega)$ was proved by Shibata-Shimada [17]. The asymptotic behaviour has not been proved yet in the exterior domain case, but employing the similar argument to that due to Iwashita [13] in the non-slip boundary condition case, we can show the local energy decay and L_q-L_r decay estimate. We will discuss this elesewhere.

Next, we consider the Neumann boundary condition case. Namely, let us consider the initial-value problem:

$$\begin{cases} u_t - \operatorname{Div} S(u,\pi) = 0, \quad \operatorname{div} u = 0 \quad \text{in } \Omega \times (0,\infty) \\ S(u,\pi)\nu|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0 \end{cases} \tag{2.4}$$

To eliminate the pressure term π we use the second Helmholtz decomposition:

$$L_q(\Omega)^n = \hat{J}_q(\Omega) \oplus \hat{G}_q(\Omega)$$
$$\hat{J}_q(\Omega) = \{u \in L_q(\Omega)^n \mid \operatorname{div} u = 0 \text{ in } \Omega\}$$
$$\hat{G}_q(\Omega) = \{\nabla \pi \mid \pi \in X_q(\Omega),\ \pi|_\Gamma = 0\}$$

which was proved by Grubb and Solonnikov [10] and Shibata and Shimizu [21, 22]. In this case, given $u \in L_q(\Omega)^n$, we choose $\pi \in X_q(\Omega)$ as a weak solution to the Dirichet problem:

$$\Delta \pi = \operatorname{div} u \quad \text{in } \Omega,\ \ \pi|_\Gamma = 0 \tag{2.5}$$

And then, we define the continuous projection from $\hat{P}_q : L_q(\Omega)^n \to J_q(\Omega)$ along $G_q(\Omega)$ defined by the relation: $\hat{P}_q u = u - \nabla \pi$.

In order to introduce the reduced Stokes operator A_q^2 corresponding to (2.4), let us consider the resolvent problem:

$$\begin{cases} \lambda u - \operatorname{Div} S(u, \pi) = f, & \operatorname{div} u = 0 \quad \text{in } \Omega \\ S(u, \pi)\nu|_{\partial\Omega} = 0 \end{cases} \tag{2.6}$$

Let g and θ be the second Helmholtz decomposition of f:

$$f = g + \nabla\theta,\ \ \operatorname{div} g = 0,\ \ \Delta\theta = \operatorname{div} f \text{ in } \Omega,\ \ \theta|_\Gamma = 0$$

Then, we have

$$\begin{cases} \lambda u - \operatorname{Div} S(u, \pi - \theta) = g, & \operatorname{div} u = 0 \quad \text{in } \Omega \\ S(u, \pi - \theta)\nu|_{\partial\Omega} = 0 \end{cases}$$

Setting $\kappa = \pi - \theta$, we consider

$$\begin{cases} \lambda u - \operatorname{Div} S(u, \kappa) = g, & \operatorname{div} u = 0 \quad \text{in } \Omega \\ S(u, \kappa)\nu|_{\partial\Omega} = 0 \end{cases} \tag{2.7}$$

under the condition that $\operatorname{div} g = 0$ in Ω. Applying the divergence to the first equation in (2.7) and using the facts that $\operatorname{div} u = 0$ and $\operatorname{div} g = 0$ in Ω, we have

$$\Delta\kappa = 0 \quad \text{in } \Omega \tag{2.8}$$

Since $(D(u)\nu - \kappa\nu)|_\Gamma = 0$, multiplying the boundary condition by ν, using $\operatorname{div} u = 0$ and complementing the resultant boundary condition with (2.8), we arrive at the equation for π as follows:

$$\Delta\kappa = 0 \ \text{ in } \Omega,\ \ \kappa|_\Gamma = (\nu \cdot D(u)\nu - \operatorname{div} u)|_\Gamma \tag{2.9}$$

If $u \in W_q^2(\Omega)$, then we have a unique solution $\kappa \in X_q(\Omega)$ of (2.9) such that

$$\|\pi\|_{X_q(\Omega)} \leqq C\|u\|_{W_q^2(\Omega)}$$

Let K be the operator: $W_q^2(\Omega) \to X_q(\Omega)$ defined by the formula: $K(u) = \kappa$. Then, (2.7) can be written in the form:

$$\lambda u - \operatorname{Div} S(u, K(u)) = g \text{ in } \Omega,\ \ S(u, K(u))\nu|_\Gamma = 0 \tag{2.10}$$

We see that (2.10) is equivalent to (2.7) when $g \in \hat{J}_q(\Omega)$. Therefore, we set

$$A_q^2 u = -\operatorname{Div} S(u, K(u)) \tag{2.11}$$

with domain

$$D(A_q^2) = \{u \in W_q^2(\Omega) \cap \hat{J}_q(\Omega) \mid S(u, K(u))|_\Gamma = 0\}$$

Then, we have the following theorem.

Theorem 2.3 (Grubb-Solonnikov [10], Shibata-Shimizu [21, 22]). *Let $1 < q < \infty$. A_q^2 generates an analytic semigroup $\{T_q^2(t)\}_{t\geqq 0}$ on $\hat{J}_q(\Omega)$.*

In order to get the exponential stability of $\{T_q^2(t)\}_{t\geqq 0}$ in the bounded domain case, we have to restrict ourself to the subspace which is orthogonal to the rigid space $\mathcal{R}$ defined by

$$\mathcal{R} = \{Ax + b \mid A\text{: anti-symmetric matrix}, \quad b \in \mathbb{R}^n\}$$

Note that $D(u) = 0$ if and only if $u \in \mathcal{R}$ and that if $u \in \mathcal{R}$ then $\operatorname{div} u = 0$. Let p_m, $m = 1, \ldots, M = n(n-1)/2 + n$, be the orthogonal basis of $\mathcal{R}$ such that $(p_m, p_\ell)_\Omega = \delta_{m\ell}$. Given closed subspace X of $L_q(\Omega)$, we set

$$X \setminus \mathcal{R} = \{u \in X \mid (u, p_\ell)_\Omega = 0, \quad \ell = 1, \ldots, M\}$$

Theorem 2.4 (Shibata-Shimizu [22]). *Let $1 < q < \infty$. Assume that Ω is a bounded domain with $C^{2,1}$ boundary. Then, A_q^2 generates an analytic semigroup $\{T_q^2(t)\}_{t\geqq 0}$ on $\hat{J}_q(\Omega) \setminus \mathcal{R}$ which enjoys the estimates:*

$$\begin{aligned}
\|T_q^2(t)u_0\|_{L_q(\Omega)} &\leqq C_q e^{-c_q t} \|u_0\|_{L_q(\Omega)} && \forall u_0 \in \hat{J}_q(\Omega) \\
\|T_q^2(t)u_0\|_{W_q^2(\Omega)} &\leqq C_q t^{-1} e^{-c_q t} \|u_0\|_{L_q(\Omega)} && \forall u_0 \in \hat{J}_q(\Omega) \\
\|T_q^2(t)u_0\|_{W_q^2(\Omega)} &\leqq C_q e^{-c_q t} \|u_0\|_{W_q^2(\Omega)} && \forall u_0 \in D(A_q^2)
\end{aligned}$$

for any $t > 0$ and $u_0 \in \hat{J}_q(\Omega) \setminus \mathcal{R}$ with some positive constants $C_{q,k} > 0$ and c_q.

Now, let us consider the asymptotic behaviour of $\{T_q^2(t)\}_{t\geqq 0}$ in the exterior domain. To state the theorems, we set

$$\begin{aligned}
&\mathbb{R}^n \setminus \Omega \subset B_{R_0}, \quad B_{R_0} = \{x \in \mathbb{R}^n \mid |x| < R_0\} \\
&L_{q,R}(\Omega) = \{f \in L_q(\Omega) \mid f(x) = 0, \ x \notin B_R\}.
\end{aligned}$$

Then, Shibata and Shimizu [21] proved the following two theorems.

Theorem 2.5 (Local energy decay estimate). *Assume that Ω is an exterior domain in $\mathbb{R}^n$ with $C^{2,1}$ boundary and that $n \geqq 3$. Let $1 < q < \infty$ and $R \geqq R_0$. Then, we have*

$$\|e^{-A_q^2 t} \hat{P}_q u_0\|_{W_q^2(\Omega \cap B_R)} \leqq C_{q,R} t^{-\frac{n}{2}} \|u_0\|_{L_q(\Omega)}$$

for all $u_0 \in L_{q,R}(\Omega)$ and $t \geqq 1$.

Theorem 2.6 (**$L_q - L_r$ estimate**). *Assume that Ω is an exterior domain in $\mathbb{R}^n$ with $C^{2,1}$ boundary and that $n \geqq 3$. Then, we have the following estimates:*

$$\|e^{-A_q t}u_0\|_{L_r(\Omega)} \leqq C_{q,r}\, t^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{r}\right)}\|u_0\|_{L_q(\Omega)} \tag{2.12}$$

$$\|\nabla e^{-A_q t}u_0\|_{L_r(\Omega)} \leqq C_{q,r}\, t^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{r}\right)-\frac{1}{2}}\|u_0\|_{L_q(\Omega)} \tag{2.13}$$

for all $u_0 \in \hat{J}_q(\Omega)$, $1 \leqq q \leqq r \leqq \infty$ ($q \neq \infty$, $r \neq 1$) and $t > 0$.

If we consider the non-slip boundary condition $u|_\Gamma = 0$, instead of the Neumann boundary condition, then we know that (2.13) holds only for $1 \leqq q \leqq r \leqq n$ ($r \neq 1$), which was proved by Iwashita [13]. Later on, several refinements of Iwashita's result were done by several authors: Maremonti-Solonnikov [15], Dan-Kobayashi-Shibata [6] and Dan-Shibata [4, 5]. Further references can be found in the references of [15, 6, 4, 5]. In particular, Maremonti-Solonnikov [15] proved that the assumption: $1 \leqq q \leqq r \leqq n$ ($r \neq 1$) is unavoidable in the non-slip condition case. Therefore the estimate (2.13) shows the eventual character of the Neumann boundary condition. Namely the null force on the boundary implies the better decay rate for the gradient of the solutions.

3 L_p-L_q maximal regularity of solutions to (1.4)

In this section, we consider the L_p-L_q maximal regularity of solutions to (1.4). We consider only the case that Ω is a bounded domain in what follows. To state our L_p-L_q maximal regularity theorem, first of all we introduce some functional spaces: Let $1 \leqq p, q \leqq \infty$, X: Banach space, $0 < T \leqq \infty$.

$$\begin{aligned}
&W^{\ell,m}_{q,p}(D \times I) = L_p(I, W^\ell_q(D)) \cap W^m_p(I, L_q(D)) \\
&\|u\|_{W^{\ell,m}_{q,p}(D\times I)} = \|u\|_{L_p(I,W^\ell_q(D))} + \|u\|_{W^m_p(I,L_q(D))} \\
&W^1_{p,0}((0,T),X) = \{u \in W^1_p((-\infty,T),X) \mid u = 0 \text{ for } t < 0\}
\end{aligned}$$

Given $\alpha \in \mathbb{R}$, we set

$$\begin{aligned}
&< D_t >^\alpha u(t) = \mathcal{F}^{-1}[(1+s^2)^{\alpha/2}\mathcal{F}u(s)](t) \\
&H^\alpha_p(\mathbb{R}, X) = \{u \in L_p(\mathbb{R}, X) \mid < D_t >^\alpha u \in L_p(\mathbb{R}, X)\} \\
&\|u\|_{H^\alpha_p(\mathbb{R},X)} = \| < D_t >^\alpha u\|_{L_p(\mathbb{R},X)} + \|u\|_{L_p(\mathbb{R},X)}
\end{aligned}$$

$\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse formula, respectively.

Set

$$\begin{aligned}
&H^{1,1/2}_{q,p}(D \times \mathbb{R}) = H^{1/2}_p(\mathbb{R}, L_q(D)) \cap L_p(\mathbb{R}, W^1_q(D)) \\
&H^{1,1/2}_{q,p,0}(D \times (0,\infty)) = \{u \in H^{1,1/2}_{q,p}(D \times \mathbb{R}) \mid u = 0 \text{ for } t < 0\} \\
&\|u\|_{H^{1,1/2}_{q,p}(D\times\mathbb{R})} = \|u\|_{H^{1/2}_p(\mathbb{R},L_q(D))} + \|u\|_{L_p(\mathbb{R},W^1_q(D))}
\end{aligned}$$

Finally, given $0 < T \leqq \infty$ we set

$$
\begin{aligned}
H^{1,1/2}_{q,p}(D \times (0,T)) &= \{u \mid {}^\exists v \in H^{1,1/2}_{q,p}(D \times \mathbb{R}),\ \ u = v \text{ on } D \times (0,T)\} \\
\|u\|_{H^{1,1/2}_{q,p}(D\times(0,T))} &= \inf\{\|v\|_{H^{1,1/2}_{q,p}(D\times\mathbb{R})} \mid \\
&\qquad {}^\forall v \in H^{1,1/2}_{q,p}(D \times \mathbb{R}),\ \ v = u \text{ on } D \times (0,T)\} \\
H^{1,1/2}_{q,p,0}(D \times (0,T)) &= \{u \mid {}^\exists v \in H^{1,1/2}_{q,p,0}(D \times (0,\infty)),\ \ u = v \text{ on } D \times (0,T)\} \\
\|u\|_{H^{1,1/2}_{q,p,0}(D\times(0,T))} &= \inf\{\|v\|_{H^{1,1/2}_{q,p}(D\times\mathbb{R})} \mid \\
&\qquad {}^\forall v \in H^{1,1/2}_{q,p,0}(D \times (0,\infty)),\ \ v = u \text{ on } D \times (0,T)\} \\
B^{2(1-1/p)}_{q,p}(\Omega) &= [L_q(\Omega), W^2_q(\Omega)]_{1-1/p,p} \\
\mathcal{D}_{q,p}(\Omega) &= \begin{cases} [J_q(\Omega), D(A^1_q)]_{1-1/p,p} \\ [\hat{J}_q(\Omega), D(A^2_q)]_{1-1/p,p} \end{cases}
\end{aligned}
$$

Theorem 3.1 (L_p-L_q maximal regularity). *Let $1 < p, q < \infty$, $0 < T < \infty$, u_0, f, g, $\tilde{g}$ and h satisfy the condition:*

$$
\begin{aligned}
&u_0 \in \mathcal{D}_{q,p}(\Omega),\ \ f \in L_p((0,T), L_q(\Omega))^n,\ \ g \in H^{1,1/2}_{q,p,0}(\Omega \times (0,T)) \\
&\tilde{g} \in W^1_{p,0}((0,T), L_q(\Omega))^n,\ \ h \in H^{1,1/2}_{q,p,0}(\Omega \times (0,T))^n
\end{aligned} \tag{3.1}
$$

In the slip or Robin boundary condition case, in addition we assume the compatibility condition: $\nu \cdot h|_\Gamma = 0$. Then, (1.4) *admits a unique solution*

$$
(u, \pi) \in W^{2,1}_{q,p}(\Omega \times (0,T))^n \times L_p((0,T), W^1_q(\Omega)) \tag{3.2}
$$

which enjoys the estimate:

$$
\begin{aligned}
&\|u\|_{W^{2,1}_{q,p}(\Omega\times(0,T))} + \|\pi\|_{L_p((0,T),W^1_q(\Omega))} \leqq C_T\{\|u_0\|_{\mathcal{D}_{q,p}(\Omega)} \\
&\quad + \|f\|_{L_p((0,T),L_q(\Omega))} + \|(g,h)\|_{H^{1,1/2}_{q,p,0}(\Omega\times(0,T))} + \|\tilde{g}\|_{W^1_p(0,T),L_q(\Omega))}\}
\end{aligned} \tag{3.3}
$$

where $C_T > 0$ is independent of T in the slip or Robin condition case and $C_T = C(1+T)$ in the Neumann condition case.

Theorem 3.2 (Exponential stability). *Let $T = \infty$ in* (1.4)*. Assume that* (3.1) *holds with $T = \infty$. In the Neumann condition case, in addition we assume that*

$$
\begin{aligned}
&(u_0, p_\ell)_\Omega = 0, && \ell = 1, \ldots, M \\
&(f(t), p_\ell)_\Omega + (g(t), p_\ell)_\Gamma = 0 && \ell = 1, \ldots, M,\ \ a.e.\, t > 0
\end{aligned}
$$

Then, (1.4) *admits a solution (u, π) which satisfies not only* (3.2) *and* (3.3) *with $T = \infty$ but also the following estimate:*

$$
\begin{aligned}
&\|e^{\gamma t} u\|_{W^{2,1}_{q,p}(\Omega\times(0,\infty))} + \|e^{\gamma t}\pi\|_{L_p((0,\infty),W^1_q(\Omega))} \leqq C\{\|u_0\|_{\mathcal{D}_{q,p}(\Omega)} \\
&+ \|e^{\gamma t} f\|_{L_p((0,\infty),L_q(\Omega))} + \|e^{\gamma t}(g,h)\|_{H^{1,1/2}_{q,p}(\Omega\times(0,\infty))} + \|e^{\gamma t}\tilde{g}\|_{W^1_p(0,\infty),L_q(\Omega))}\}
\end{aligned}
$$

where γ is any constant in $[0, \gamma_0]$ with some $\gamma_0 > 0$.

Moreover, in the Neumann boundary condition case, the solution $u(t)$ satisfies the orthogonal condition: $(u(t), p_\ell)_\Omega = 0$ for all $t > 0$ and $\ell = 1, \ldots, M$.

Remark 3.3. (1) In the slip or Robin boundary condition case, Theorems 3.1 and 3.2 were proved by Shibata-Shimada [18]. In the Neumann boundary condition case, Theorem 3.1 was stated by Solonnikov [25] under the assumption: $3 < p = q < \infty$, which is combined with Benedek-Calderón-Panzone theorem [2] (cf. also [1]) implies that Theorem 3.1 also holds under the assumption: $3 < q < \infty$ and $1 < p < \infty$. But, Theorem 3.2 is a quite new result compared with [25].
(2) In order to state the relationship between $\mathcal{D}_{q,p}(\Omega)$ and $B^{2(1-1/p)}_{q,p}(\Omega)$, we introduce the set $\mathcal{B}^{2(1-1/p)}_{q,p}(\Omega)$ as follows:

$$\mathcal{B}^{2(1-1/p)}_{q,p}(\Omega) = \{u \in B^{2(1-1/p)}_{q,p}(\Omega)^n \mid \operatorname{div} u = 0 \text{ in } \Omega\}$$

Then, it follows from Triebel [28] and Steiger [26] that there hold the following relations: When the boundary condition is a slip or Robin one and $2(1-1/p) > 1 + 1/q$,

$$\mathcal{D}_{q,p}(\Omega) = \{u \in \mathcal{B}^{2(1-1/p)}_{q,p}(\Omega) \mid \nu \cdot u|_{\partial\Omega} = 0,\ (D(u)\nu - (\nu \cdot (D(u)\nu))\nu)|_{\partial\Omega} = 0\},$$

when the boundary condition is a slip or Robin one and $1/q < 2(1-1/p) < 1 + 1/q$,

$$\mathcal{D}_{q,p}(\Omega) = \{u \in \mathcal{B}^{2(1-1/p)}_{q,p}(\Omega) \mid \nu \cdot u|_{\partial\Omega} = 0\},$$

and when the boundary condition is a slip or Robin one and $2(1-1/p) < 1/q$,

$$\mathcal{D}_{q,p}(\Omega) = \mathcal{B}^{2(1-1/p)}_{q,p}(\Omega)$$

On the other hand, when the boundary condition is a Neumann one and $2(1-1/p) > 1 + 1/q$,

$$\mathcal{D}_{q,p}(\Omega) = \{u \in \mathcal{B}^{2(1-1/p)}_{q,p}(\Omega) \mid S(u, K(u))|_{\partial\Omega} = 0\},$$

and when the boundary condition is a Neumann one and $2(1-1/p) < 1 + 1/q$,

$$\mathcal{D}_{q,p}(\Omega) = \mathcal{B}^{2(1-1/p)}_{q,p}(\Omega)$$

4 A skech of our proof of the maximal regularity results in the Neumann boundary condition case

4.1 First step

We consider the maximal regularity of the problem:

$$\begin{aligned} &u_t - \operatorname{Div} S(u, \pi) = f, \quad \operatorname{div} u = 0 \quad \text{in } \Omega \times (0, \infty) \\ &S(u, \pi)\nu|_\Gamma = 0, \quad u|_{t=0} = 0 \end{aligned} \tag{4.1}$$

where $f \in C^\infty_0(\mathbb{R}, L_q(\Omega)) \subset L_p(\mathbb{R}, L_q(\Omega))$ which satisfies the conditions:

$$(f(t), p_m)_\Omega = 0,\ t \in \mathbb{R}, m = 1, \ldots, M;\ f(t) = 0 \text{ for } t < 0.$$

By Duhamel's principle we have

$$u(t) = \int_0^t T^2_q(t-s) f(s)\, ds$$

and therefore by Theorem 2.4 we have

$$\|u(t)\|_{W_q^1(\Omega)} \leqq C \int_0^t (t-s)^{-1/2} e^{-2\gamma(t-s)} \|f(s)\|_{L_q(\Omega)}\, ds$$

with suitable $\gamma > 0$, which implis that

$$\|e^{\gamma t} u\|_{L_p((0,\infty),W_q^1(\Omega))} \leqq C_{\gamma,q} \|e^{\gamma t} f\|_{L_p((0,\infty),L_q(\Omega))} \tag{4.2}$$

The estimate (4.2) is used to estimate the perturbation terms which are obtained by the localization procedure.

4.2 2nd step

We consider L_p-L_q maximal regularity in the whole space and the half-space. We used the operator valued Fourier multiplier theorem. To state this theorem, first we start with the $\mathcal{R}$ boundedness of the operator family. Let X and Y be two Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Let $\mathcal{B}(X,Y)$ denote the set of all bounded linear operators from X into Y, and we set $\mathcal{B}(X) = \mathcal{B}(X,X)$.

Definition 4.1 ($\mathcal{R}$-boundedness). A family of operators $\mathcal{T} \subset \mathcal{B}(X,Y)$ is called $\mathcal{R}$-bounded if there exist constants $C > 0$ and $p \in (1,\infty)$ such that for each $m \in \mathbb{N}$, $\mathbb{N}$ being the set of all natural numbers, $T_j \in \mathcal{T}$, $x_j \in X$ and for all sequences $\{r_i(u)\}$ of independent, symmetric, $\{-1,1\}$-valued random variables on $[0,1]$ there holds the inequality:

$$\int_0^1 \|\sum_{j=1}^m r_j(u) T(x_j)\|_Y^p\, du \leqq C \int_0^1 \|\sum_{j=1}^m r_j(u) x_j\|_X^p\, du$$

The smallest such C is called $\mathcal{R}$-bound of $\mathcal{T}$, which is denoted by $\mathcal{R}(\mathcal{T})$.

Given $M \in L_{1,\mathrm{loc}}(\mathbb{R}, \mathcal{B}(X,Y))$ we define an operator T_M by

$$T_M \phi = \mathcal{F}^{-1}[M \mathcal{F}\phi]$$

Then the following theorem was proved by Weis ([29]).

Theorem 4.2 (Operator Valued Fourier Multiplier Theorem). *Suppose that X and Y are UMD Banach spaces and $1 < p < \infty$. Let $M \in C^1(\mathbb{R} \setminus \{0\}, \mathcal{B}(X,Y))$ be such that the following conditions are satisfied:*

$$\mathcal{R}(\{M(\rho) \mid \rho \in \mathbb{R} \setminus \{0\}\}) = \kappa_0 < \infty$$
$$\mathcal{R}(\{\rho M(\rho)' \mid \rho \in \mathbb{R} \setminus \{0\}\}) = \kappa_1 < \infty$$

Then, the operator T_M is a bounded linear operator from $L_p(\mathbb{R}, X)$ into $L_p(\mathbb{R}, Y)$ with norm:

$$\|T_M\|_{\mathcal{B}(L_p(\mathbb{R},X),L_p(\mathbb{R},Y))} \leqq C(\kappa_0 + \kappa_1)$$

where $C > 0$ depends only on p, X and Y.

To show the $\mathcal{R}$-boundedness, the following two propositions are useful (cf. Denk-Hieber-Prüss [7]) .

Proposition 4.3. *Let $1 < q < \infty$ and $\{k_s(x) \mid s \in \mathbb{R} \setminus \{0\}\}$ be a family of $L_{1,\mathrm{loc}}(\mathbb{R}^n)$ functions $k_s(x)$. Set*

$$K_s f(x) = \int_{\mathbb{R}^n} k_s(x-y) f(y)\, dy \quad s \in \mathbb{R} \setminus \{0\}$$

Suppose that there exists a constant $C > 0$ independent of $s \in \mathbb{R} \setminus \{0\}$ such that

$$\|K_s f\|_{L_2(\mathbb{R}^n)} \leqq C \|f\|_{L_2(\mathbb{R}^n)} \quad f \in L_2(\mathbb{R}^n), \; s \in \mathbb{R} \setminus \{0\}$$
$$\sum_{|\alpha|=1} |\partial_x^\alpha k_s(x)| \leqq C|x|^{-(n+1)} \quad x \in \mathbb{R}^n \setminus \{0\}, \; s \in \mathbb{R} \setminus \{0\}$$

Then, $\{K_s \mid s \in \mathbb{R} \setminus \{0\}\}$ is $\mathcal{R}$-bounded on $\mathcal{B}(L_q(\mathbb{R}^n))$ and its $\mathcal{R}$-bound is less than or equal to $C_{n,q}C$ with some constant $C_{n,q}$ depending only on n and q.

Proposition 4.4. *Let $1 < q < \infty$. Let G be a domain in $\mathbb{R}^n$ and $\mathcal{T} = \{T_\mu \mid \mu \in \mathcal{M}\} \subset \mathcal{B}(L_q(G))$ be a family of the kernel operators:*

$$[T_\mu f](x) := \int_G k_\mu(x,y)\, f(y)\, dy$$

for $x \in G$ and $f \in L_q(G)$. Suppose that there exists a $k_0(x,y)$ such that

$$|k_\mu(x,y)| \leqq k_0(x,y)$$

for almost all x, $y \in G$ and for any $\mu \in \mathcal{M}$. Set

$$[T_0 f](x) = \int_G k_0(x,y)\, f(y)\, dy$$

If $T_0 \in \mathcal{B}(L_q(G))$, then $\mathcal{T}$ is $\mathcal{R}$-bounded on $\mathcal{B}(L_q(G))$ and its $\mathcal{R}$-bound is less than or equal to $C_{q,G}\|T_0\|_{\mathcal{B}(L_q(G))}$, where $C_{q,G}$ depends only on q and G.

In order to explain how to show the L_p-L_q maximal regularity, we consider the following model problem in the half-space:

$$\begin{aligned}
&u_t - \Delta u + u + \nabla \pi = 0, \; \operatorname{div} u = 0 \; \text{ in } \mathbb{R}^n_+ \times \mathbb{R}_+ \\
&\left(\frac{\partial u_j}{\partial x_n} + \frac{\partial u_n}{\partial x_j}\right)\Big|_{x_n=0} = \ell_j, \; j = 1, \ldots, n-1 \\
&\left(2\frac{\partial u_n}{\partial x_n} - \pi\right)\Big|_{x_n=0} = \ell_n
\end{aligned} \tag{4.3}$$

The solution formula for (4.3) consists of the following formulas:

$$\begin{aligned}
w_1(x,t) &= \int_0^\infty \mathcal{L}^{-1}\left[B_\lambda(\xi')^{-1} e^{-B_\lambda(\xi')(x_n+y_n)} \hat{h}(\xi', y_n, \lambda)\right](x',t)\, dy_n \\
&B_\lambda(\xi') = \sqrt{|\lambda| + |\xi'|^2 + 1}, \; \xi' = (\xi_1, \ldots, \xi_{n-1}), \lambda = \gamma + i\tau \\
w_2(x,t) &= \int_0^\infty \mathcal{L}^{-1}\left[A_\lambda(\xi')|\xi'| e^{-B_\lambda(\xi')(x_n+y_n)} \hat{h}(\xi', y_n, \lambda)\right](x',t)\, dy_n \\
w_3(x,t) &= \int_0^\infty \mathcal{L}^{-1}\left[A_\lambda(\xi')|\xi'|^2 \mathcal{M}_\lambda(\xi', x_n+y_n) \hat{h}(\xi', y_n, \lambda)\right](x',t)\, dy_n \\
&\mathcal{M}_\lambda(\xi', x_n) = \frac{e^{-B_\lambda(\xi')x_n} - e^{-|\xi'|x_n}}{B_\lambda(\xi') - |\xi'|}
\end{aligned}$$

Here, $A_\lambda(\xi')$ is a function satisfying the multiplier condition:

$$|\partial_{\xi'}^{\alpha'} A_\lambda(\xi')| \leqq C_{\alpha'}(|\lambda|^{1/2} + |\xi'| + 1)^{-2}|\xi|^{-|\alpha'|}$$

for any $\lambda = \gamma + i\tau \in [0, \infty) \times i\mathbb{R}$ and $\xi' \in \mathbb{R}^{n-1}$ where $C_{\alpha'}$ is independent of λ and ξ', $\hat{h}$ denotes the Fourier-Laplace transform of h, and $\mathcal{L}^{-1}$ denotes the inverse Fourier-Laplace transform. Namely,

$$\hat{h}(\xi', y_n, \lambda) = \int_{\mathbb{R}^n} e^{-i\xi'\cdot x' - (\gamma+i\tau)t} h(x', x_n, t)\, dx'dt = \mathcal{F}_{(x',t)}[e^{-\gamma t}h](\xi', x_n, \tau)$$

$$\mathcal{L}^{-1}[g](x', t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\xi'\cdot x' + (\gamma+i\tau)t} g(\xi', \tau)\, dx'd\tau = e^{\gamma t}\mathcal{F}^{-1}_{(\xi',\tau)}[g](x', t)$$

For example, let us explain how to estimate the following term by using Theorem 4.2:

$$e^{-\gamma t}\partial_t w_1 = \int_0^\infty \mathcal{F}^{-1}_{(\xi',\tau)}\left[\frac{\lambda e^{-B_\lambda(\xi')(x_n+y_n)}}{B_\lambda(\xi')}\hat{h}(\xi', y_n, \lambda)\right](x', t)\, dy_n$$

Set

$$k_\gamma(\tau, x) = \mathcal{F}^{-1}_{\xi'}\left[\frac{\lambda e^{-B_\lambda(\xi')x_n}}{B_\lambda(\xi')}\right](x')$$

$$K_\gamma(\tau)g = \int_{\mathbb{R}^n_+} k_\gamma(x' - y', x_n + y_n)g(y)\, dy$$

Then, we have

$$e^{-\gamma t}\partial_t w_1 = \mathcal{F}^{-1}_\tau[K_\gamma(\tau)\mathcal{F}_t[e^{-\gamma t}h](\tau)](t)$$

If we show that

$$|k_\gamma(\tau, x)| \leqq A|x|^{-n}, \quad |\tau\partial_\tau k_\gamma(\tau, x)| \leqq A|x|^{-n} \tag{4.4}$$

then by Proposition 4.4 we have

$$\mathcal{R}\{K_\gamma(\tau) \mid \tau \in \mathbb{R}\setminus\{0\}\} \leqq CA$$

with some constant $C > 0$ independent of γ and τ, because we know the $L_q(\mathbb{R}^n_+)$ boundedness of the operator:

$$T_0[g](x) = \int_{\mathbb{R}^n_+} A[|x' - y'|^2 + (x_n + y_n)^2]^{-n/2} g(y', y_n)\, dy'dy_n$$

Therefore, applying Theorem 4.2 we have

$$\|e^{-\gamma t}\partial_t w_1\|_{L_p(\mathbb{R}, L_q(\Omega))} \leqq C\|e^{-\gamma t}h\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^n_+))}$$

for any $\gamma \geqq 0$, where C is independent of $\gamma \geqq 0$.

To show (4.4), we use the following lemma.

Lemma 4.5 (Shibata-Shimizu [19]). *Let X be a Banach space and $\|\cdot\|_X$ its norm. Let α be a number $> -n$ and set $\alpha = N + \sigma - n$ where $N \geqq 0$ is an integer and $0 < \sigma \leqq 1$. Let $f(\xi)$ be a function in $C^\infty(\mathbb{R}^n \setminus \{0\}, X)$ such that*

$$\partial_\xi^\gamma f(\xi) \in L_1(\mathbb{R}^n, X) \quad \text{for } |\gamma| \leqq N$$

$$\|\partial_\xi^\gamma f(\xi)\|_X \leqq C_\gamma|\xi|^{\alpha-|\gamma|} \quad \forall\xi \neq 0, \ \forall\gamma \in \mathbb{N}_0^n$$

Set $g(x) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(\xi)\, d\xi$. *Then, we have*

$$\|g(x)\|_X \leqq C_{n,\alpha}(\max_{|\gamma|\leqq N+2} C_\gamma)|x|^{-(n+\alpha)} \quad \forall x \neq 0$$

where $C_{n,\alpha}$ *is a constant depending only on* n *and* α.

$\partial_t w_2$, $\partial_t w_3$ and second derivatives with respect to x variables of w_1, w_2 and w_3 can be estimated in the same spirit, and therefore we arrive at the estimate:

$$\begin{aligned}&\|e^{-\gamma t}(u_t, \nabla^2 u, \nabla\pi)\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^n_+))} \\ &\leqq C\{\|(\nabla(e^{-\gamma t}\ell), < D_t >^{1/2} (e^{-\gamma t}\ell), \sqrt{\gamma}e^{-\gamma t}\ell)\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^n_+))}\} \quad \gamma \geqq 0\end{aligned}$$

where u is a solution to the equation (4.3).

4.3 3rd Step

By using the localization technique, we reduce the problem to the whole space problem and the half-space problem and by using the result in 2nd step and by representing π in terms of f and u, we can show the a priori estimate:

$$\|(u_t, \nabla^2 u)\|_{L_p(\mathbb{R}, L_q(\Omega))} \leqq C\{\|f\|_{L_p(\mathbb{R}, L_q(\Omega))} + \|u\|_{L_p(\mathbb{R}, W^1_q(\Omega))}\} \tag{4.5}$$

where u is a solution to

$$\begin{aligned}u_t - \operatorname{Div} S(u,\pi) = f, \ \ \operatorname{div} u = 0 \quad &\text{in } \Omega \times (0,\infty) \\ S(u,\pi)\nu|_\Gamma = 0, \ \ u|_{t=0} = 0&\end{aligned}$$

Finally, applying (4.2) in the 1st step to (4.5), we have

$$\|u_t\|_{L_p(\mathbb{R}, L_q(\Omega))} + \|u\|_{L_p(\mathbb{R}, W^2_q(\Omega))} + \|\pi\|_{L_p(\mathbb{R}, W^1_q(\Omega))} \leqq C\|f\|_{L_p(\mathbb{R}, L_q(\Omega))}$$

4.4 4th step

Let u be a solution to the non-homogeneous boundary value problem:

$$\begin{aligned}u_t - \operatorname{Div} S(u,\pi) = 0, \ \ \operatorname{div} u = 0 \quad &\text{in } \Omega \times (0,\infty) \\ S(u,\pi)\nu|_\Gamma = h, \ \ u|_{t=0} = 0&\end{aligned}$$

To get a priori estimate of u_t, we use the solution ψ to the adjoint problem:

$$\begin{aligned}\psi_t + \operatorname{Div} S(\psi,\theta) = \phi, \ \ \operatorname{div} \psi = 0 \quad &\text{in } \Omega \times (-\infty, T) \\ S(\psi,\theta)\nu|_\Gamma = 0, \ \ \psi|_{t=T} = 0&\end{aligned}$$

Given $T > 0$ and $\phi \in L_{p'}(\mathbb{R}, L_{q'}(\Omega))$ which vanishes for $t > T$, we know the existence of solution $(\psi, \theta) \in W^{2,1}_{q',p'}(\Omega \times \mathbb{R})^n \times L_{p'}(\mathbb{R}, W^1_{q'}(\Omega))$ such that ψ and θ vanish for $t > T$ and satisfy the estimate:

$$\|\psi_t\|_{L_{p'}(\mathbb{R}, L_{q'}(\Omega))} + \|\psi\|_{L_{p'}(\mathbb{R}, W^2_{q'}(\Omega))} + \|\theta\|_{L_{p'}(\mathbb{R}, W^1_{q'}(\Omega))} \leqq C\|\phi\|_{L_{p'}(\mathbb{R}, L_{q'}(\Omega))} \tag{4.6}$$

where $1/p + 1/p' = 1/q + 1/q' = 1$. Observing that

$$\begin{aligned}
(u_t, \phi)_{\Omega\times\mathbb{R}} &= (u_t, \psi_t + \operatorname{Div} S(\psi,\theta))_{\Omega\times\mathbb{R}} \\
&= -(u_{tt} - \operatorname{Div} S(u_t, \pi_t), \psi)_{\Omega\times\mathbb{R}} + (S(u_t,\pi_t)\nu, \psi)_{\Gamma\times\mathbb{R}} \\
&= (h_t, \psi)_{\Gamma\times\mathbb{R}} \\
&= (\nabla\cdot(\nu h_t), \psi)_{\Omega\times\mathbb{R}} + (\nu h_t, \nabla\psi)_{\Omega\times\mathbb{R}} \\
&= -(\nabla\cdot(\nu h), \psi_t)_{\Omega\times\mathbb{R}} + (\nu < D_t >^{-1/2} h_t, < D_t >^{1/2} \nabla\psi)_{\Omega\times\mathbb{R}},
\end{aligned}$$

we have

$$\begin{aligned}
|(u_t,\phi)_{\Omega\times\mathbb{R}}| \leqq C(&\|h\|_{L_p(\mathbb{R}, W_q^1(\Omega))} + \| < D_t >^{1/2} h\|_{L_p(\mathbb{R}, L_q(\Omega))})\times \\
&(\|\psi_t\|_{L_{p'}(\mathbb{R}, L_{q'}(\Omega))} + \| < D_t >^{1/2} \nabla\psi\|_{L_{p'}(\mathbb{R}, L_{q'}(\Omega))}
\end{aligned} \tag{4.7}$$

Since we know the interpolation inequality:

$$\| < D_t >^{1/2} \nabla\psi\|_{L_{p'}(\mathbb{R}, L_{q'}(\Omega))} \leqq C(\|\psi_t\|_{L_{p'}(\mathbb{R}, L_{q'}(\Omega))} + \|\psi\|_{L_{p'}(\mathbb{R}, W^2_{q'}(\Omega))})$$

by (4.6) and (4.7) we have

$$|(u_t,\phi)_{\Omega\times\mathbb{R}}| \leqq C(\|h\|_{L_p(\mathbb{R}, W_q^1(\Omega))} + \| < D_t >^{1/2} h\|_{L_p(\mathbb{R}, L_q(\Omega))})\|\phi\|_{L_{p'}(\mathbb{R}, L_{q'}(\Omega))}$$

which implies that

$$\|u_t\|_{L_p(\mathbb{R}, L_q(\Omega))} \leqq C(\|h\|_{L_p(\mathbb{R}, W_q^1(\Omega))} + \| < D_t >^{1/2} h\|_{L_p(\mathbb{R}, L_q(\Omega))})$$

In this way, we can show the L_p-L_q maximal regularity.

5 A free boundary problem for the Navier-Stokes equations

In this section, we consider a time dependent problem with free surface for the Navier-Stokes equations which describes the motion of an isolated finite volume of viscous incompressible fluid without taking surface tension into account. The region $\Omega_t \subset \mathbb{R}^n$, $n \geqq 2$, occupied by the fluid is given only on the initial time $t = 0$, while for $t > 0$ it is to be determined. The velocity vector field $v(x,t) = {}^t(v_1, \ldots, v_n)$ and the pressure $\theta(x,t)$ for $x \in \Omega_t$ satisfy the Navier-Stokes equations (cf. [25]):

$$\begin{aligned}
&v_t + (v\cdot\nabla)v - \operatorname{Div} S(v,\theta) = f(x,t) && \text{in } \Omega_t,\ t > 0 \\
&\operatorname{div} v = 0 && \text{in } \Omega_t,\ t > 0 \\
&S(v,\theta)\nu_t + \theta_0(x,t)\nu_t = 0 && \text{in } \Gamma_t,\ t > 0 \\
&v|_{t=0} = v_0 && \text{on } \Omega.
\end{aligned} \tag{5.1}$$

Here, ${}^tM$ denotes the transposed M, Γ_t denotes the boundary of Ω_t and $\nu_t(x)$ is the unit outer normal to Γ_t at the point $x \in \Gamma_t$. $\nabla = (\partial_1, \ldots, \partial_n)$ with $\partial_i = \partial/\partial x_i$. $S(v,\theta)$ is the stress tensor defined by the formula:

$$S(v,\theta) = D(v) - \theta I$$

where $D(v)$ is the deformation tensor of the velocities with elements $D_{ij}(v) = \partial_i v_j + \partial_j v_i$ and I is the $n \times n$ identity matrix. For the $n \times n$ matrix of functions $S = (S_{ij})$

$$\operatorname{Div} S = {}^t(\sum_{j=1}^{n} \partial_j S_{1j}, \ldots, \sum_{j=1}^{n} \partial_j S_{nj})$$

The external force $f(x,t)$ and the pressure $\theta_0(x,t)$ are functions defined on the whole space. Below, we shall always assume that $\theta_0(x,t) = 0$, since we can arrive at this case by replacing $\theta(x,t)$ by $\theta + \theta_0$.

Aside from the dynamical boundary condition, a further kinematic condition for Γ_t is satisfied which gives Γ_t as a set of points $x = x(\xi,t)$, $\xi \in \Gamma_0$, where $x(\xi,t)$ is the solution of the Cauchy problem:

$$\frac{dx}{dt} = v(x,t), \quad x|_{t=0} = \xi. \tag{5.2}$$

This expresses the fact that the free surface Γ_t consists for all $t > 0$ of the same fluid particles, which do not leave it and are not incident on it from inside Ω_t. It is clear that $\Omega_t = \{x = x(\xi,t) \mid \xi \in \Omega_0\}$ and $\Gamma_t = \{x = x(\xi,t) \mid \xi \in \Gamma_0\}$.

The problem (5.1) can therefore be written as an initial boundary value problem in the given region Ω_0 if we go over the Euler coordinates $x \in \Omega_t$ to Lagrange coordinates $\xi \in \Omega$ connected with x by (5.2). If a velocity vector field $u(\xi,t) = (u_1, \ldots, u_n)$ is known as a function of the Lagrange coordinates ξ, then this connection can be written in the form:

$$x = \xi + \int_0^t u(\xi,\tau)\, d\tau \equiv X_u(\xi,t) \tag{5.3}$$

Passing to Lagrange coordinate in (5.1) and setting $\theta(X_u(\xi,t),t) = \pi(\xi,t)$, we obtain

$$\begin{aligned}
&u_t - \operatorname{Div}[S(u,\pi) + U(u,\pi)] = f(X_u(\xi,t),t) && \text{in } \Omega \times (0,T)\\
&\operatorname{div} u + E(u) = \operatorname{div}[u + \tilde{E}(u)] = 0 && \text{in } \Omega \times (0,T)\\
&[S(u,\pi) + U(u,\pi)]\nu = 0 && \text{on } \Gamma \times (0,T)\\
&u|_{t=0} = v_0 && \text{in } \Omega.
\end{aligned} \tag{5.4}$$

Here and hereafter, Ω is a bounded domain in $\mathbb{R}^n$, $n \geqq 2$, whose boundary Γ is assumed to be a $C^{2,1}$ compact hypersurface, ν is the unit outer normal to Γ, $U(u,\pi)$, $E(u)$ and $\tilde{E}(u)$ are nonlinear terms of the following forms:

$$U(u,\pi) = V_1(\int_0^t \nabla u\, d\tau)\nabla u + V_2(\int_0^t \nabla u\, d\tau)\pi$$

$$E(u) = V_3(\int_0^t \nabla u\, d\tau)\nabla u \quad \tilde{E}(u) = V_4(\int_0^t \nabla u\, d\tau)u$$

with some polynomials $V_j(\cdot)$ of $\int_0^t \nabla u\, d\tau$, $j = 1,2,3,4$, such as $V_j(0) = 0$.

As a linearized problem of (5.4), we have the Stokes equation with Neumann boundary condition:

$$\begin{aligned}
&u_t - \operatorname{Div} S(u,\pi) = f && \text{in } \Omega \times (0,T)\\
&\operatorname{div} u = g = \operatorname{div} \tilde{g} && \text{in } \Omega \times (0,T)\\
&S(u,\pi)\nu|_\Gamma = h, \quad u|_{t=0} = u_0 &&
\end{aligned} \tag{5.5}$$

By using Theorems 3.1 and 3.2 and the contraction mapping principle, for (5.4) we have the following two theorems ([23]).

Theorem 5.1. *Let $2 < p < \infty$ and $n < q < \infty$. Then, given $v_0 \in \mathcal{D}_{q,p}(\Omega)$ and $f \in L_p(\mathbb{R}, L_q(\mathbb{R}^n))$ which has bounded derivatives with respect to x for each t, there exists a $T = T(\|v_0\|_{\mathcal{D}_{q,p}(\Omega)}, \|f\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^n))}, \sup_{t \geqq 0} \|\nabla f(\cdot, t)\|_{L_\infty(\mathbb{R}^n)}) > 0$ such that (5.4) admits a unique solution*

$$(u, \pi) \in W^{2,1}_{q,p}(\Omega \times (0,T))^n \times L_p((0,T), W^1_q(\Omega))$$

which satisfies the estimate:

$$\|u\|_{W^{2,1}_{q,p}(\Omega\times(0,T))} + \|\pi\|_{L_p((0,T),W^1_q(\Omega))} \leqq C\{\|v_0\|_{\mathcal{D}_{q,p}(\Omega)} + \|f\|_{L_p((0,T),L_q(\mathbb{R}^n))}\}$$

Theorem 5.2. *Let $2 < p < \infty$ and $n < q < \infty$. Then, there exist positive numbers ϵ and γ such that if $v_0 \in \mathcal{D}_{q,p}(\Omega)$, $\|v_0\|_{\mathcal{D}_{q,p}(\Omega)} \leqq \epsilon$ and $(v_0, p_\ell)_\Omega = 0$ for $\ell = 1, \ldots, M$, then (5.4) with $T = \infty$ and $f = 0$ admits a unique solution*

$$(u, \pi) \in W^{2,1}_{q,p}(\Omega \times (0,\infty))^n \times L_p((0,\infty), W^1_q(\Omega))$$

which satisfies the estimate:

$$\|e^{\gamma t} u\|_{W^{2,1}_{q,p}(\Omega\times(0,\infty))} + \|e^{\gamma t}\pi\|_{L_p((0,\infty),W^1_q(\Omega))} \leqq C\|v_0\|_{\mathcal{D}_{q,p}(\Omega)}$$

for some $\gamma > 0$ and the condition:

$$(u(\cdot,t), p_\ell)_\Omega = 0 \quad \text{for } \ell = 1, \ldots, M \text{ and } t \geqq 0$$

Remark 5.3. Theorems 5.1 and 5.2 have been proved by Solonnikov [25] when $p = q$, but our approach is completely different from [25]. Moreover, since we show the maximal regularity result with exponential stability on the whole time axis $(0, \infty)$ for (1.7) with $T = \infty$ (cf. Theorem 3.2), our proof of Theorem 5.2 is very simple compared with the corresponding proof in §4 of Solonnikov [25].

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