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Continuous Dependence on Initial Data for Multidimensional, Viscous, Compressible Flows

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We prove uniqueness and continuous dependence on initial data of weak solutions of the Navier–Stokes equations of compressible flow

$$(1)_{1.1} \qquad \begin{cases} \rho_t + \operatorname{div}(\rho u) = 0 \\ (\rho u^j)_t + \operatorname{div}(\rho u^j u) + P(\rho)_{x_j} = \mu \Delta u^j + \lambda \operatorname{div} u_{x_j} + \rho f^j \end{cases}$$

for $x \in \mathbb{R}^n$, $n = 2, 3$, and $t > 0$ with initial data

$$(2)_{1.2} \qquad (\rho, u)|_{t=0} = (\rho_0, u_0).$$

Here ρ and $u = (u^1, \ldots, u^n)$ are unknown functions of x and t representing density and velocity, $P = P(\rho)$ is the pressure, f is a given external force, $\mu > 0$ and $\lambda \geq 0$ are viscosity constants, and div and Δ are the usual spatial divergence and Laplace operators.

Our purpose is to compare solutions with only minimal regularity, including the generic singular solutions of the Navier-Stokes system–solutions with codimension-one discontinuities in density, pressure, and velocity gradient (see Hoff [4]). Our result extends that of Danchin [1], which applies to solutions in certain Besov spaces of continuous functions. Danchin makes a direct comparison by subtracting the equations satisfied by different solutions and controlling the linearization errors by the assumed regularity. This regularity is absent for solutions in the class considered here, however, and a different approach is required.

Ideally, one would prove uniqueness and continuous dependence for solutions in the largest class for which existence is known, that is, solutions with finite energies and nonnegative densities (see Feireisl [2] and [3] and Lions [9], for example). Such a result may be unattainable, however, and may be in fact be false: it is known that certain anomalies can arise in solutions of (1) when densities are zero on nonnegligible sets, and there are examples of nonphysical solutions (see Hoff and Serre [7]). Moreover, it appears likely, although it is not presently known, that large energies can force positive densities to zero, so that the aforementioned anomalies may occur even when initial densities are strictly positive. It thus remains an interesting open problem to select physical solutions from the most general class of known weak solutions and to establish their uniqueness and continuous dependence on initial data. We shall discuss this issue in greater detail below. For the time being we avoid these issues and difficulties by assuming that the solutions being compared have strictly positive densities, which is known to hold for a large class of solutions with positive initial densities and small initial energies (Hoff [5]).

To be more specific, the solutions we consider are weak solutions in a regularity class which we shall call $\mathcal{C}$. Properties of solutions in $\mathcal{C}$ are rather technical and are listed in Hoff [6], (1.4)–(1.12). Rather than repeating these conditions here, we instead list the hypotheses on the system parameters, initial data, and external force under which solutions in the class $\mathcal{C}$ are known to exist (Hoff [5]): if

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$$P(\rho) = K\rho, \ K \text{ constant}, \ \mu > 0, \ \text{and } 0 < \lambda < 5\mu/4, \tag{3)$_{1.3}$ (}$$

$$C^{-1} < \rho_0 < C \text{ a.e. for some constant } C, \tag{4)$_{1.4}$ (}$$

$$\int \rho_0 |u_0|^q dx < \infty \text{ for some } q > 6 \text{ for } n = 3 \text{ and } q > 2 \text{ for } n = 2, \tag{5)$_{1.5}$ (}$$

$$\int [\rho_0 |u_0|^2 + (\rho_0 - \tilde{\rho})^2] dx \ll 1, \tag{6)$_{1.7}$ (}$$

where $\tilde{\rho}$ is a constant, positive reference density,

$$u_0 \in H^\alpha(\mathbb{R}^n) \text{ where } \alpha > 0 \text{ for } n = 2 \text{ and } \alpha > 1/2 \text{ for } n = 3, \tag{7)$_{1.8}$ (}$$

$$f \in L^\infty([0,T]; L^2) \text{ and } \int_0^T (t^7 |\nabla f|_{L^4}^2 + t^5 |f_t|_{L^2}^2) dt < \infty, \tag{8)$_{1.9}$ (}$$

then there is a weak solution (ρ, u) to the initial-value problem (1)–(2) in the class C defined on $\mathbb{R}^n \times [0,T]$, $T < \infty$. The condition in (3) on P can be relaxed considerably and the strict positivity of ρ_0 in (4) and the condition (7) can be dropped entirely for the general existence theory of [5]; all three are required here, however, for reasons which will be explained later.

The following is our main result:

Theorem: *Let (ρ, u, f) and $(\overline{\rho}, \overline{u}, \overline{f})$ be weak solutions of* (1) *in the regularity class C corresponding to respective external forces f and $\overline{f}$ satisfying* (8). *Assume also that $\overline{f} \in L^1((0,T); L^{2r}(\mathbb{R}^n))$ and $\rho_0 - \overline{\rho}_0 \in L^{2s}(\mathbb{R}^n)$, where r and s are Hölder conjugates in $[1, \infty]$, and that*

$$\nabla\Gamma * (\rho - \tilde{\rho}), \ \nabla\Gamma * (\overline{\rho} - \tilde{\rho}) \in L^1((0,T); W^{1,\infty}(\mathbb{R}^n)), \tag{9)$_{1.10}$ (}$$

where Γ is the fundamental solution of the Laplace operator on $\mathbb{R}^n$. Then

$$\begin{aligned} &\left(\int_0^T \int_{\mathbb{R}^n} |u - \overline{u}|^2 dx dt \right)^{1/2} + \sup_{0 \le t \le T} \|(\rho - \overline{\rho})(\cdot, t)\|_{H^{-1}(\mathbb{R}^n)} \\ &\le C \left[\|\rho_0 - \overline{\rho}_0\|_{L^2 \cap L^{2s}} + \|\rho_0 u_0 - \overline{\rho}_0 \overline{u}_0\|_{L^2} + \left(\int_0^T \int_{\mathbb{R}^n} |f - \overline{f} \circ S|^2 dx dt \right)^{1/2} \right] \end{aligned} \tag{10)$_{1.11}$ (}$$

(the notation $\overline{f} \circ S$ is explained below in (12)). The constant C depends on the system parameters P, μ, and λ, on T, and on upper bounds for the norms of solutions and forces occurring in the above hypotheses. If $\int_0^T t\|\nabla\overline{f}(\cdot,t)\|_{L^\infty}dt < \infty$, then $\overline{f} \circ S$ may be replaced by $\overline{f}$ in (10).

Details of the proof are given in Hoff [6]. We describe here the main ideas by giving an overview of the analysis and remarking on several of the hypotheses, particular the condition in (3) that P be the pressure of an ideal isothermal fluid and the condition (9) concerning the Newtonian potentials of the densities. We first note that the smallness condition in (6), while required for the *existence* of solutions in the class C, plays no role in the proof of uniqueness and continuous dependence.

The key point of the analysis is that solutions with minimal regularity are best compared in a Lagrangean, rather than Eulerian framework. To describe this we first note that (7) and (9) insure the existence of particle trajectories $X(y,t,s)$ satisfying

(11)2.1
$$\begin{cases} \dfrac{\partial X}{\partial t}(y,t,s) = u(X(y,t,s),t) \\ X(y,s,s) = y \end{cases}$$

and similarly for $\overline{X}$. $X(y,t,s)$ is therefore the position at time t of the fluid particle whose position at time s is y. In particular, the fluid particle in (ρ,u,f) at (x,t) was at $X(x,0,t)$ initially and so corresponds to the particle in $(\overline{\rho},\overline{u},\overline{f})$ which at time t is at the point

(12)1.3.4
$$\overline{X}(X(x,0,t),t,0) \equiv S(x,t)$$

(this is the S appearing in (10)). The "Lagrangean" comparison therefore consists of an estimate for $u - \overline{u} \circ S$, where we abuse notation slightly and abbreviate $\overline{u}(S(x,t),t) = (\overline{u} \circ S)(x,t)$. The advantages of the Lagrangean formulation may be understood briefly as follows. First, from its definition, $S(X(y,t,0),t) = \overline{X}(y,t,0)$, so that $S_t + \nabla S u = \overline{u} \circ S$. It then follows easily that, if $w = \overline{u} \circ S$,

$$(\overline{u}_t + \nabla\overline{u}\,\overline{u}) \circ S = w_t + \nabla w u,$$

which shows that the convective time derivative in the evolution equation for w will be the same as that for u. Given the form of the momentum equation, the comparison of u with $w = \overline{u} \circ S$ therefore appears quite natural and proceeds in the usual way: bounds for $u - \overline{u} \circ S$ are obtained by duality from estimates for solutions of the adjoint of the weak equation satisfied by $u - \overline{u} \circ S$. A bound for the Eulerian difference $u - \overline{u}$ can then be derived via the regularity assumptions and a bound for $I - S$.

We now discuss three of the estimates that go into the proof, beginning with a bound for $I - S$. Subtracting the equations (11) satisfied by X and $\bar{X}$ and integrating in time, we obtain

$$|X(y,t,0) - \bar{X}(y,t,0)| \leq \int_0^t |(u \circ X)(y,s) - (\bar{u} \circ \bar{X})(y,s)|ds$$

so that

$$|X(\cdot,t,0)-\bar{X}(\cdot,t,0)|_{L^2}^2 \le t\int_0^t |(u\circ X)(\cdot,s)-(\bar{u}\circ\bar{X})(\cdot,s)|_{L^2}^2 ds.$$

Assuming for the time being that X is Lipschitz continuous with respect to y, we can then make the change of variables $x = X(y,t,0)$ to obtain the bound

$$(13)_{2.2} \qquad |I - S(\cdot,t)|_{L^2}^2 \le Ct\int_0^t |(u-\bar{u}\circ S)(\cdot,s)|_{L^2}^2 ds\ .$$

Next we apply this estimate to obtain a bound for $\|(\rho-\bar{\rho})(\cdot,t)\|_{H^{-1}}$. First note that, as a consequence of the conservation of mass,

$$(14)_{2.25} \qquad \rho(X(y,t,s),t)\det\nabla_y X(y,t,s) = \rho(y,s)$$

a.e. (this is the precise formulation of the more intuitive statement that $\rho(\cdot,t)dX = \rho(\cdot,s)dy$.) Thus by (12) and (14),

$$\begin{aligned}\det\nabla S(x,t) =& \det(\nabla_y\bar{X}(X(x,0,t),t,0))\det(\nabla_y X(x,0,t),t,0))\\ =&\rho(x,t)/\bar{\rho}(S(x,t),t)\ .\end{aligned}$$

Then for $\varphi\in\mathcal{D}(\mathbb{R}^n)$ and t fixed,

$$\begin{aligned}\Big|\int(\rho-\bar{\rho})\varphi\,dx\Big| &= \Big|\int[\rho\varphi-(\bar{\rho}\circ S)(\varphi\circ S)\det\nabla S]dx\Big|\\ &= \Big|\int\rho[\varphi-\varphi\circ S]dx\Big|\\ &\le C\|\nabla\varphi\|_{L^2}\|I-S\|_{L^2}\ .\end{aligned}$$

It then follows from (13) that

$$(15)_{2.3} \qquad \sup_{0\le t\le T}\|(\rho-\bar{\rho})(\cdot,t)\|_{H^{-1}}^2 \le CT\int_0^T\int|u-\bar{u}\circ S|^2 dxdt\ .$$

To complete the proof we need to obtain a bound for the integral on the right side of (15) in terms of the left side and differences in the initial data and forces. This is the most difficult and technical part of the analysis, and, as indicated earlier, proceeds by duality by subtracting the weak forms of the equations satisfied by the different solutions and choosing the test function to satisfy a suitably mollified adjoint equation. Complete details are given in Hoff [6]. We shall content ourselves here with the observation that the momentum equation, which may be written

$$\Big[\rho(\frac{\partial}{\partial t}+u\cdot\nabla)-\mu\Delta-\lambda\text{div}\Big]u = -\nabla P(\rho)+\rho f\ ,$$

shows that L^2 differences in velocities u are driven by H^{-1} differences in pressures $P(\rho)$ and L^2 differences in forces f. We therefore anticipate an estimate of the form

$$\int_0^T \int |u - \bar{u} \circ S|^2 dxdt \leq \ldots + CT \sup_{0 \leq t \leq T} \|P(\rho(\cdot,t)) - P(\bar{\rho}(\cdot,t)\|^2_{H^{-1}} ,$$

where the ellipsis indicates norms of differences in forces and initial data. This together with (15) then completes the proof provided that $P = K\rho$, which is the assumption required to insure that H^{-1} differences in pressures are equivalent to H^{-1} differences in densities. We remark that this is the only point in the analysis where this condition is required. More general pressures could be included in this analysis, but only at the expense of having to impose unreasonably strong conditions on the regularity of the solutions being compared.

We now discuss briefly the requirement (9) concerning the Newtonian potentials of the densities. As indicated above, we require at several points of the analysis that the particle trajectories $X(y,t,s)$ be Lipschitz continuous functions of their initial positions y. This follows from (11) when the Lipschitz constants with respect to space of u and $\bar{u}$ are integrable are time, that is, when $u, \bar{u} \in L^1((0,T); W^{1,\infty}(\mathbb{R}^n))$. Now by adding and subtracting terms we obtain that

$$\begin{aligned}(\mu+\lambda)\Delta u^j =& ((\mu+\lambda)\operatorname{div} u - P(\rho))_{x_j} + (\mu+\lambda)(u^j_{x_k} - u^k_{x_j})_{x_k} + (P(\rho) - P(\tilde{\rho}))_{x_j} \\ \equiv& (\mu+\lambda)\Delta u^j_{F,\omega} + (\mu+\lambda)\Delta u^j_P\end{aligned}$$

where $(\mu+\lambda)u^j_P = \Gamma_{x_j} * (P(\rho) - P(\tilde{\rho}))$ and summation over repeated indices is understood. (The standard notation is that $F = (\mu+\lambda)\operatorname{div} u - P(\rho)$ and $\omega^{j,k} = u^j_{x_k} - u^k_{x_j}$.) For the solutions constructed in [5], F and ω are more regular than the summands which define them, this fact being a consequence of the cancellation of singularities predicted by the Rankine-Hugoniot conditions applied to (1) (see the introduction to [4], for example). A straightforward argument based on standard elliptic theory and Sobolev estimates ([6], section 1) then shows that $\nabla u_{F,\omega} \in L^1((0,T); L^\infty(\mathbb{R}^n))$, as required. (The hypotheses (7) and (8) are needed here to insure that $\|\nabla u_{F,\omega}(\cdot,t)\|_{L^\infty}$ is integrable in the initial layer near $t = 0$.) On the other hand, the relation $(\mu+\lambda)\Delta u^j_P = (P(\rho) - P(\tilde{\rho}))_{x_j}$ implies that $\|\nabla u(\cdot,t)\|_{L^p} \leq C_p \|P(\rho) - P(\tilde{\rho})\|_{L^p}$ only for $1 < p < \infty$, not for $p = 1$ or $p = \infty$. We must therefore impose (9) as an posteriori condition on the solutions being compared. We do note, however, that a large class of piecewise smooth solutions exhibiting codimension-one singularities and satisfying the condition (9) is constructed in Hoff [4].

We now turn to the question of uniqueness and continuous dependence for more general, large-energy solutions (again see Feireisl [2] and [3] and Lions [9] for the corresponding existence theory.) These solutions are much less regular than those in the class C and consequently are not amenable to the analysis outlined above. Additionally, it is known that certain anomalies can arise in solutions in which densities vanish on nonnegligible sets, and it appears likely, although it is not presently known, that large initial energies can cause spontaneous cavitation even when the initial density is strictly positive. This raises the possibility that the known set of large-energy weak solutions includes nonphysical solutions, even when initial densities are strictly positive.

To be more specific, we say that a solution (ρ, u) is *locally momentum conserving* if, whenever E and V are bounded open sets in x-space with $\bar{E} \subset V$ and with $\rho = 0$ a.e. in $(V - E) \times [t_1, t_2]$, then the change in the momentum of the fluid in E from time t_1 to time t_2 should be the impulse $\int_{t_1}^{t_2} \int_E \rho f \, dxdt$ applied by the external force to the fluid in E. Weak solutions violating this condition do in fact exist, one such being constructed in Hoff and Serre [7]: initial data (ρ_0, u_0) is given corresponding to two fluids initially separated by a vacuum; it is shown that, if ρ_0 is replaced by $\rho_0 + \delta$, then the limit as $\delta \downarrow 0$ of the corresponding perturbed solutions exists and is a weak solution in the entire physical space, but fails to be locally momentum conserving. This failure is a consequence of the fact that the momentum equation in (1) determines a velocity u even in open sets in which $\rho = 0$, or, stated differently, the model posits nonzero viscous forces generated from the motion of nonexistent fluid particles. It is also shown in [8] for the above example that there is a different solution corresponding to the same initial data which *is* locally momentum conserving, but this solution is defined only on the support of the density and cannot be extended as a weak solution to the entire physical space.

These observations suggest a remedy: the viscous force terms in (1) should be made to vanish when the density vanishes and the momentum equation should be imposed only on the support of the density. We recall that kinetic theory predicts that, to first order, μ and λ are functions of temperature θ. If it is assumed that the entropy of each fluid particle is fixed, that is, that the flow is isentropic, then θ will be a function of ρ, and under reasonable assumptions on the functional relationships, μ and λ will be zero when ρ is zero. Whether or not the isentropic assumption is consistent with the presence of viscosity is in some doubt, however. If we instead regard μ and λ as functions of θ but assume that θ is constant, that is, that the flow is isothermal, then μ and λ are effectively constant. This is the case we consider, and for this case we propose that the second equation in (1) should be modified by replacing the terms $\mu u^j_{x_k x_k} + \lambda u^k_{x_j x_k}$ on the right side by $(\chi \mu u^j_{x_k})_{x_k} + (\chi \lambda u^k_{x_j})_{x_k}$, where $\chi(\rho)$ is one or zero according as ρ is positive or zero.

To be more specific, we propose that, for viscous, isothermal flow governed by the Navier-Stokes system (1), there should be established an existence, uniqueness, continuous dependence theory as follows: given initial density ρ_0 and momentum density m_0 which vanishes when $\rho_0 = 0$ (m is a surrogate for ρu), both defined, let us say, on $\mathbb{R}^3$, there should exist a unique pair (ρ, m) with $\rho, m \in C([0, \infty); L^1_{loc}(\mathbb{R}^n))$ and $m^2/\rho \in L^1_{loc}(\mathbb{R}^n \times [0, \infty))$ satisfying the weak form of the mass equation

$$\int \rho \varphi dx \Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} \int_{\mathbb{R}^n} (\rho \varphi_t + m \cdot \nabla \varphi) dxdt$$

for Lipschitz test functions φ having compact support; for each $t > 0$ there should be an open set G_t such that $\rho(\cdot, t) > 0$ a.e. on G_t and $\rho(\cdot, t) = 0$ a.e. on the complement of G_t; the set $G = \{(x, t) : x \in G_t, \, t > 0\}$ should be open and $u \equiv m/\rho$ should be in $L^1_{loc}(G)$, so that $\nabla u \in \mathcal{D}'(G)$; finally, the weak form of the momentum equation should hold on G in

the sense that
(16)2.4

$$\int_{\mathbb{R}^n} m\cdot\psi dx\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2}\int_{\mathbb{R}^n} [m\cdot(\psi_t+\nabla\psi u) + (P(\rho)-\widetilde{P})\operatorname{div}\psi$$
$$- \chi_G\mu\nabla u^j\cdot\nabla\psi^j - \chi_G\lambda(\operatorname{div}u)(\operatorname{div}\psi) + \rho f\cdot\psi]dxdt$$

where χ_G is the characteristic function of G and $\psi \in \mathcal{D}(\mathbb{R}^n \times (-\infty,\infty))^n$. It is easy to check that solutions in this sense are locally momentum conserving.

A version of this program has been carried out in the simplest case that $\Omega = (0,1) \subset \mathbb{R}^1$ with the obvious modifications required to accommodate no-slip boundary conditions. The hypotheses on the initial data are that ρ_0 is nonnegative and bounded, that there is an open set $G_0 \subset (0,1)$ with $\rho_0 > 0$ a.e. on G_0 and $\rho_0 = 0$ a.e. on the complement of G_0, that the initial energy $\int_{G_0} \rho_0 u_0^2 dx$ is finite, and that there exists a constant $\bar{u}^2$ such that

$$\int_E [\tfrac{1}{2}\rho_0 u_0^2 + (\rho_0 - \tilde{\rho})^2]dx \le \tfrac{1}{2}\bar{u}^2 \int_E \rho_0 dx$$

for all measureable sets $E \subset G_0$. The latter condition insures that the there are no fluid particles of fixed energy and vanishingly small mass, and the second condition disallows data with mass concentrated on Cantor-like sets.

Complete details of this work will be presented elsewhere. We do note, however, that locally momentum conserving solutions in this sense are in fact not unique: an additional condition, which might be called "maximal Lagrangean structure" is required. Rather than giving precise technical details here, we instead describe an example which illustrates the issue in a fairly simple way. First recall the result of Serre [10] showing that, given a fluid initially occupying a one-dimensional interval $(a(0), b(0))$ surrounded entirely by vacuum, there is a weak solution (ρ, u) of (1) defined on the set $\{(x,t) : a(t) < x < b(t),\, t > 0\}$, where $a(t)$ and $b(t)$ are fluid particle paths satisfying $\dot{a} = u(a,t)$ and $\dot{b} = u(b,t)$ defining the free boundaries of the fluid, subject to the weak form of momentum conservation boundary conditions $(\mu+\lambda)u_x - P = 0$ at $x = a(t)$ and $b(t)$. It is easily checked that this solution is locally momentum conserving in the sense described above. We can apply this result to demonstrate the nonuniqueness of locally momentum conserving solutions as follows. First apply Serre's result to a fluid with strictly positive density initially occupying $(a(0), b(0)) = (0,1)$, perform a Galilean translation of the x-axis to transform the resulting solution to one for which $a(t) > 0$ for all $t > 0$, then reflect about the t-axis to obtain a locally momentum conserving solution (ρ_1, u_1) with ρ_1 even in x, u_1 odd in x, and $G_t = (-b(t), -a(t)) \cup (a(t), b(t))$ and $G_0 = (-1,1)$. A second solution (ρ_2, u_2) can be obtained more simply by applying Serre's result directly to the same initial data $(\rho_1(0), u_1(0))$ on $(-1,1)$. For this second solution, $G_t = (-c(t), c(t))$ for a free boundary function c, and since $\rho_2(\cdot, t)$ is strictly positive on G_t, $(\rho_1, u_1) \neq (\rho_2, u_2)$. Our theory picks out (ρ_2, u_2) as the unique momentum conserving solution having the property that there is a unique particle trajectory emanating from each point of $\bar{G}_0$. Observe that this condition fails for (ρ_1, u_1) since there are *two* distinct particle trajectories for this solution emanating from $x = 0$.

We conclude by remarking that the extension of this program to three space dimensions promises to be an especially interesting and challenging mathematical project, owing to the increased complexity of the geometry of the set G and to the fact that spontaneous cavitation may be possible after the initial time, a phenomenon which cannot occur for one-dimensional flows (Hoff and Smoller [8]).

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