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# On the Standard Expression for the Party Algebra

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## 1 Introduction

In this note we explain about the party algebra $P_{n,r}(Q)$ and its standard expression. The party algebra is originally defined as the centralizer of the unitary reflection group of type $G(r,1,k)$ which diagonally acts on the $n$ times tensor space $V^{\otimes n}$. This algebra is also defined as a diagram algebra like Temperly-Lieb's algebra, Brauer's centralizer algebra, the partition algebra, and so on. It is well known that these diagram algebras have the cellular structures.

The cellular structure of an algebra tells that how the algebra's basis is decomposed into the cells, which may turn out to be the representatives of the irreducible representations.

When we trying to find a characterization for the party algebra by generators and relations [9], we found a good expression of the basis elements of the algebra, which naturally gives the cellular structure. In this note, we present this expression as a candidate of the standard expression.

## 2 Definition of the party algebra

### 2.1 Define the party algebra as a centralizer algebra

First we quickly review the definition of the unitary reflection group of type $G(r,1,k)$. The unitary reflection group $G(r,1,k)$ in Shephard-Todd's notation [15] consists of all the monomial matrices of size $k$ whose non-zero components are powers of an $r$-th primitive root of unity and it is generated by all the permutation matrices and the identity matrix whose $(1,1)$-component was replaced by an $r$-th primitive root of unity $\zeta$. Let $V$ be a $k$-dimensional vector space on which the unitary reflection group $G(r,1,k)$ naturally acts. Consider the tensor space $V^{\otimes n}$ on which $G(r,1,k)$ acts diagonally. We assume that the dimension $k \geq n$ [resp. $k \geq 2n-1$] if $r > 1$ [resp. $r = 1$]. The party algebra $P_{n,r}(k)$ is defined as the centralizer of $G(r,1,k)$ in $V^{\otimes n}$ with respect to the above action. Namely,

$$P_{n,r}(k) := \mathrm{End}_{G(r,1,k)} V^{\otimes n}.$$

In particular $P_{n,1}(k) = \mathrm{End}_{\mathfrak{S}_k} V^{\otimes n}$ is known as the partition algebra, which has been intensively studied [5, 10, 11, 2, 17]. It is well known that in the similar setting, we have the following correspondence[1]:

$$\begin{array}{ccccc} GL_k(\mathbb{C}) & \supset & O_k(\mathbb{C}) \supset & \mathbb{C}\mathfrak{S}_k \\ \mathfrak{S}_n & \subset & B_n(k) \subset & P_{n,1}(k). \end{array}$$

## 2.2 Find the basis

Since $G(r,1,k)$ contains $G(1,1,k) = \mathfrak{S}_k$, the party algebra $P_{n,r}(k)$ must be a subalgebra of the partition algebra. To find which element is in the party algebra precisely, we observe the actions of the generators of $G(r,1,k)$.

Let $e_1, \ldots, e_k$ be the natural basis of the vector space $V$. We assume that $G(r,1,k)$ acts naturally with respect to this basis.

**Case $r = 1$**

First consider the case $r = 1$, the partition algebra case. Suppose that an endomorphism map $X$ moves one of the elements of the natural basis of the tensor space to a linear combination of the basis:

$$X(e_{f_1} \otimes \cdots \otimes e_{f_n}) = \sum_{m_1,\ldots,m_n} X_{f_1,\ldots,f_n}^{m_1,\ldots,m_n} e_{m_1} \otimes \cdots \otimes e_{m_n}. \tag{1}$$

Since $X$ commutes with the diagonal action of the symmetric group, for an arbitrary element $\sigma \in \mathfrak{S}_k$ we have

$$\sigma^{-1}X\sigma(e_{f_1} \otimes \cdots \otimes e_{f_n}) = \sum_{m_1,\ldots,m_n} X_{\sigma(f_1),\ldots,\sigma(f_n)}^{\sigma(m_1),\ldots,\sigma(m_n)} e_{m_1} \otimes \cdots \otimes e_{m_n}.$$

Hence we have

$$X_{\sigma(f_1),\ldots,\sigma(f_n)}^{\sigma(m_1),\ldots,\sigma(m_n)} = X_{f_1,\ldots,f_n}^{m_1,\ldots,m_n}.$$

From this, in the paper [5] Jones showed that the following transformations make a basis of $\mathrm{End}_{G(1,1,k)}(V^{\otimes n})$.

$$\{T^{\sim} \mid \sim \text{is an equivalent relation on } 2n, \text{ the number of classes} \leq k\}, \tag{2}$$

$$\begin{aligned} (T^{\sim})_{f_1,\ldots,f_n}^{f_{n+1},\ldots,f_{2n}} &:= \begin{cases} 1 & \text{if } (f_i = f_j \Leftrightarrow i \sim j), \\ 0 & \text{otherwise}, \end{cases} \\ f_{n+j} &:= m_j \quad j = 1, 2, \ldots, n. \end{aligned}$$

Since we assumed that the dimension $k$ of the vector space is large enough, in the following we can omit the second condition of the expression (2).

[1] A. Ram [12] called such a algebra "tantalizer". The meaning of it is "Centralizer of the tensor representation"

It is easy to see that there is a one to one correspondence between the set described in the expression (2) and the following set of the set-partitions:

$$\begin{aligned}\Sigma_n^1 \;=\; & \{\{T_1,\ldots,T_s\} \mid s=1,2,\ldots\;, \\ & T_j(\neq\emptyset)\subset F\cup M \;(j=1,2,\ldots,s), \\ & \cup T_j = F\cup M, \quad T_i\cap T_j=\emptyset \text{ if } i\neq j\}, \end{aligned} \tag{3}$$

where

$$F=\{f_1,\ldots,f_n\}, \quad M=\{m_1,\ldots,m_n\}, \quad |F\cup M|=2n.$$

**Case $r>1$**

Next consider the case $r>1$.

In this case we have to consider the action of $\xi=\mathrm{diag}(\zeta,1,\ldots,1)$. Note that $\xi$ multiplies $e_{f_1}\otimes\cdots\otimes e_{f_n}$ by $\zeta$ at each occurrence of $e_1$. Let $f_1,\ldots,f_n$ and $m_1,\ldots,m_n$ be the indices defined in the equation (1). Let $p$ be the number of 1s in the array $(m_1,\ldots,m_n)$ and $q$ the number of 1s in the array $(f_1,\ldots,f_n)$. Since

$$\begin{aligned} X\xi=\xi X \quad\Leftrightarrow\quad & \zeta^p X_{f_1,\ldots,f_n}^{m_1,\ldots,m_n} = \zeta^q X_{f_1,\ldots,f_n}^{m_1,\ldots,m_n} \\ & \text{for all possible } (f_1,\ldots,f_n) \text{ and } (m_1,\ldots,m_n), \end{aligned}$$

in order that $X$ is an element of the centralizer $\mathrm{End}_{G(r,1,k)}V^{\otimes n}$, the coefficients $X_{f_1,\ldots,f_n}^{m_1,\ldots,m_n}$ must be 0 unless that the number of 1s in $(f_1,\ldots,f_n)$ is equal to the number of 1s in $(m_1,\ldots,m_n)$ modulo $r$. Further since $\sigma\in G(1,1,k)$ runs all the permutations, the coefficients must be 0 unless that the number of $i$s in $(f_1,\ldots,f_n)$ is equal to that of $i$s in $(m_1,\ldots,m_n)$ modulo $r$ for any letter $i$,

If we describe this in terms of the set-partitions, the $\xi$-action adds the following restriction to the basis for the partition algebra.

$$T_j\cap F\equiv T_j\cap M \pmod{r}.$$

In fact, Tanabe [16] showed that the following set becomes a basis of the party algebra $P_{n,r}(k)$.

$$F=\{f_1,\ldots,f_n\}, \quad M=\{m_1,\ldots,m_n\}, \quad |F\cup M|=2n,$$

$$\begin{aligned}\Sigma_n^r \;=\; & \{\{T_1,\ldots,T_s\} \mid s=1,2,\ldots\;, \\ & T_j(\neq\emptyset)\subset F\cup M \;(j=1,2,\ldots,s), \\ & \cup T_j = F\cup M, \quad T_i\cap T_j=\emptyset \text{ if } i\neq j, \\ & |T_j\cap F|\equiv |T_j\cap M| \pmod{r}\}. \end{aligned} \tag{4}$$

We call an element of $\Sigma_n^r$ an *$r$-modular seat-plan*.

## 2.3 Define the party algebra as a diagram algebra

As we can see in Martin's papers [10, 11], the partition algebra is defined as a diagram algebra imposing a product on the set (3). (See also the paper [2].) In case that the partition algebra is defined as a diagram algebra on the linear span of $\Sigma_n^1$, the parameter $k$ does not have to be an integer any more.

Since the party algebra $P_{n,r}(k)$ is a subalgebra of the partition algebra (as a centralizer), it is expected that a diagram subalgebra $\widetilde{P_{n,r}}(Q)$ (which will turn out to be isomorphic to $P_{n,r}(k)$) of the partition algebra is defined on the linear span of $\Sigma_n^r$.

We explain the diagram algebra $\widetilde{P_{n,r}}(Q)$ taking an example of the case $r = 2$. Let

$$w = \{\{m_1, f_1, f_2, f_4\}, \{m_2, f_5\}, \{m_3, m_4\}, \{m_5, f_3\}\} \in \Sigma_5^2.$$

The corresponding diagram of $w$ will become the one in Fig. 1. In general, the diagram of an $r$-modular seat-plan is obtained as follows. Consider a rectangle with $n$ marked points on the bottom and the same $n$ on the top. The $n$ marked

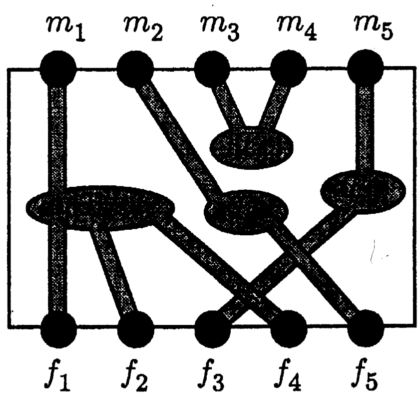



Figure 1: $w \in \Sigma_5^2$

points on the bottom are labeled by $f_1, f_2, \cdots f_n$ from left to right. Similarly, the $n$ marked points on the top is labeled by $m_1, m_2, \ldots, m_n$ from left to right. If $w \in \Sigma_n^2$ has $s$ parts, then put $s$ shaded circles in the middle of the rectangle so that they have no intersections. Each of the circles corresponds to one of the non-empty $T_j$s. Then we join the $2n$ marked points and the $s$ circles with $2n$ shaded bands so that the marked points labeled by the elements of $T_j$ are connected to the corresponding circle with $|T_j|$ bands. We call $T_j \cap M$ the *upper part* of $T_j$ and $T_j \cap F$ the *lower part* of $T_j$

Define the composition product $w_1 \circ w_2$ of diagrams $w_1$ and $w_2$ to be the new diagram obtained by placing $w_1$ above $w_2$, gluing the corresponding boundaries and shrinking half along the vertical axis as in Fig. 2. We then have a new diagram possibly containing some islands and/or lakes. If there occur islands and/or lakes in the resulting diagram, then first bury the lakes and remove each island multiplying by $Q$ (see Fig. 3). The product is the resulting diagram with

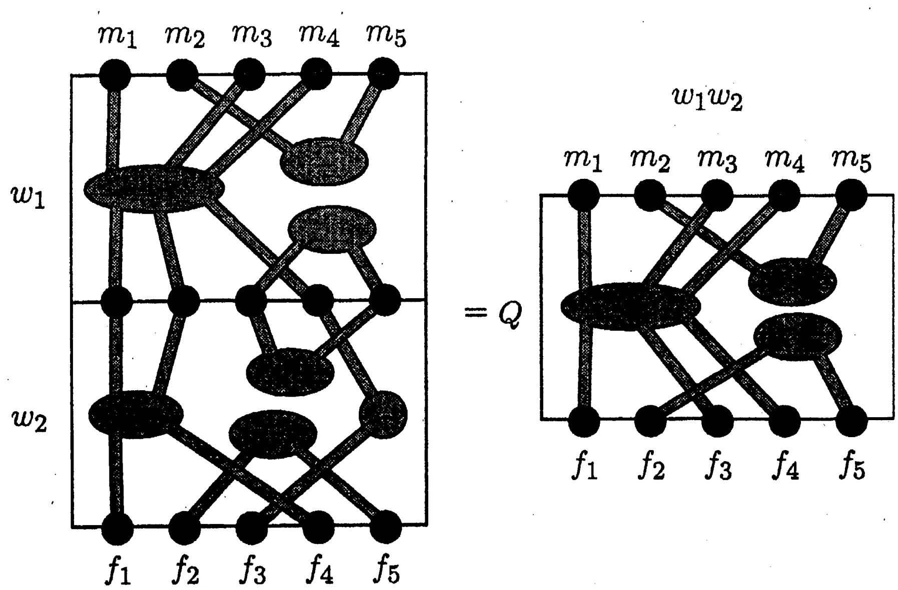



Figure 2: The product of seat-plans

the islands and lakes removed. It is easy to check that this product is again a (scalar multiple of a) set-partition defined in the expression (4). In the

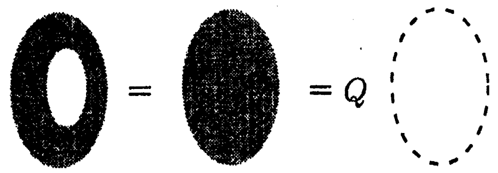



Figure 3: Remove islands multiplying by Q

following we will write $w_1w_2 = w_1 \circ w_2$ for convenience.

In the following, we use $P_{n,r}(Q)$ to denote the diagram algebra $\widetilde{P_{n,r}}(Q)$. This abuse of notation will be justified by Proposition 1 in Section 2.4.

## 2.4 Generators and special elements of $P_{n,r}(Q)$

Tanabe showed that the diagrams in Fig. 4 generate the diagram algebra $P_{n,r}(Q)$ as well as the centralizer algebra $P_{n,r}(k)$ [16]. If $Q = k \in \mathbb{Z}_{>0}$, then the correspondence between the diagram algebra and the centralizer algebra is given by the following proposition.

**Proposition 1.** *(Tanabe [16, Theorem 3.1]) Let $G(r,1,k)$ be the group of all the monomial matrices of size n whose non-zero entries are r-th roots of unity. Let V be a vector space of dimension k with the basis elements $e_1, e_2, \ldots, e_k$ on which $G(r,1,k)$ acts naturally. We note that $\{s_i \mid i = 1, \ldots, n-1\}$ in Fig. 4 generate the symmetric group $\mathfrak{S}_n$. Let $\phi$ be the representation of the symmetric*

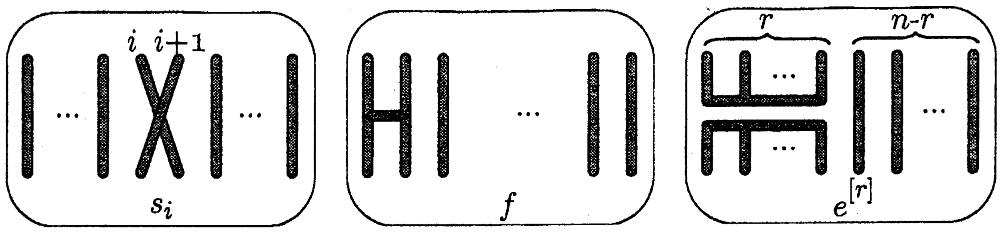



Figure 4: Generators of $P_{n,r}(Q)$

*group* $\mathfrak{S}_n$ *on* $V^{\otimes n}$ *obtained by permuting the tensor product factors, i.e., for* $v_1, v_2, \ldots, v_n \in V$ *and for* $w \in \mathfrak{S}_n$*,*

$$\phi(w)(v_1 \otimes v_2 \otimes \cdots \otimes v_n) := v_{w^{-1}(1)} \otimes v_{w^{-1}(2)} \otimes \cdots \otimes v_{w^{-1}(n)}.$$

*For* $f$ *and* $e^{[r]}$ *in Fig. 4, define* $\phi(f)$ *and* $\phi(e^{[r]})$ *as follows:*

$$\phi(f)(e_{p_1} \otimes e_{p_2} \otimes \cdots \otimes e_{p_n}) := \begin{cases} e_{p_1} \otimes e_{p_2} \otimes \cdots \otimes e_{p_n} & \text{if } p_1 = p_2, \\ 0 & \text{otherwise,} \end{cases}$$

$$\phi(e^{[r]})(e_{p_1} \otimes e_{p_2} \otimes \cdots \otimes e_{p_n}) := \begin{cases} \Sigma_{i=1}^{k}(e_i^{\otimes r}) \otimes e_{p_{r+1}} \otimes \cdots \otimes e_{p_n} & \\ & \text{if } p_1 = p_2 = \cdots = p_r, \\ 0 & \text{otherwise.} \end{cases}$$

*Then* $\mathrm{End}_{G(r,1,k)}(V^{\otimes n})$ *is generated by* $\phi(\mathfrak{S}_n)$*,* $\phi(f)$ *and* $\phi(e^{[r]})$*.*

Since $s_i$s in Fig. 4 make the symmetric group, we find that the 'conjugate' elements of $f$ and $e^{[r]}$ as in Fig. 5(a),(b) are obtained as a product of the generators. Further, the diagram (c) in Fig. 5 is also obtained from some $f$s and $s_i$s in Fig. 4. Hence if an $r$-modular seat-plan is presented as a product of the special elements in Fig. 5, then it is presented as a product of the generators in Fig. 4.

# 3 Bratteli diagram for $P_{n,r}(k)$

Thanks to the Schur-Weyl duality, we can obtain the Bratteli diagram for the party algebra observing how the tensor representation of $G(r,1,k)$ is decomposed into irreducibles in accordance with the increase of the number of tensors. The irreducible representations of the unitary reflection group $G(r,1,k)$ are indexed by the $r$-tuples of Young diagrams whose total number of the boxes is equal to $k$. (As for the irreducible representation of $G(r,1,k)$, we refer the paper [1].) We are going to explain this, taking the case $r = 3$ (Fig. 6). In this example we set $k = 5$. Since we set $r = 3$, the irreducible components are indexed by 3-tuples of Young diagrams of total size 5. First consider the case $n = 1$. In this case we have the natural representation. We know that it is irreducible and indexed by the 3-tuple of Young diagrams on the vertex

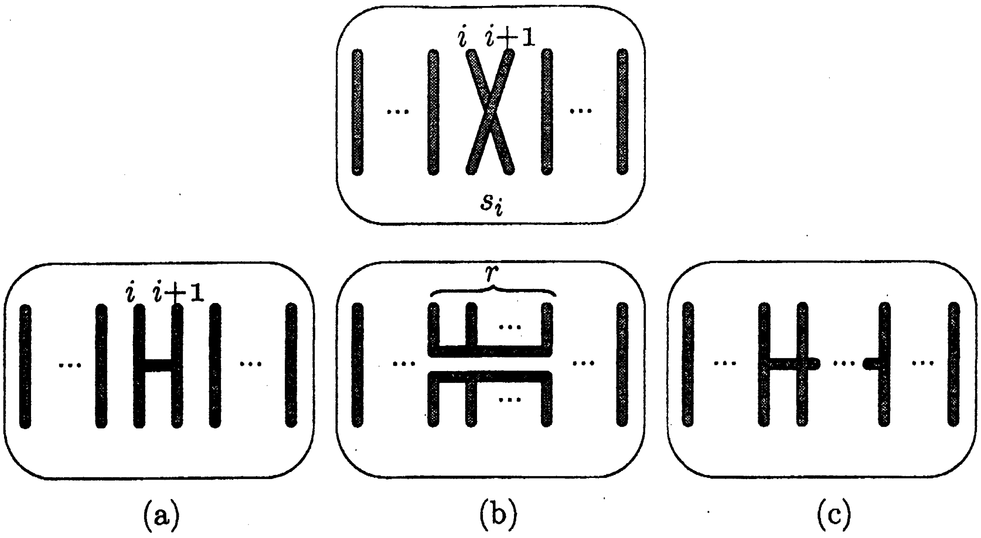



Figure 5: Special elements of $P_{n,r}(Q)$

on the 1-st floor in Fig. 6. If the number of tensors increases by one, then the irreducible components will be branched obeying the following rules:

- one of the boxes in one coordinate is moved to the next coordinate so that all the coordinates have again Young diagrams,
- in case the box to be removed is in the last ($r$-th) coordinate, this box is moved to the first coordinate.

Note that in this picture, the vertex on the 1-st floor appears on the 4-th floor.

Fig. 7 is another example of the Bratteli diagram of the party algebra in case $r = 2$, $k = 5$. In this example, we note that vertices on the 2-nd floor again appear on the 4-th floor.

In case $r = 1$, or the partition algebra case, the situation is slightly different, however, the similar argument still applies for the case $r = 1$ with slight modifications. Hereafter, we assume that $r > 1$.

Schur-Weyl's duality asserts that the multiplicity of each irreducible component becomes the degree of the corresponding irreducible representation of the centralizer. In this Bratteli diagram, each vertex on the bottom expresses an irreducible component of the party algebra as well as the corresponding irreducible representation of the unitary reflection group and the number of the paths from the top vertex to a vertex on the bottom becomes the degree of the irreducible representation of the party algebra.

# 4 Irreducible components of $P_{n,r}(Q)$

If we define the party algebra as a diagram algebra, the parameter $Q$ does not necessarily have to be a large integer. Although the Bratteli diagrams in the

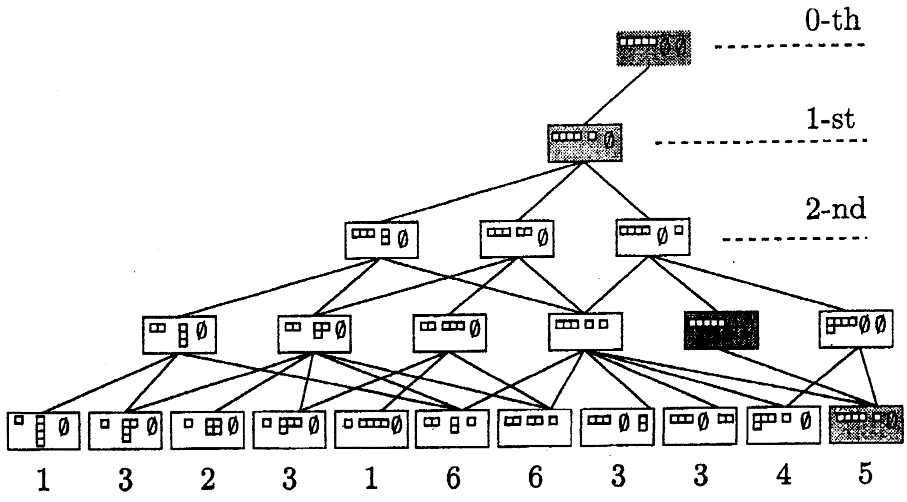



Figure 6: Bratteli diagram for $P_{4,3}(5)$

previous section seem to be made depending on the integer $k$, it is natural to guess that there must exist a description which does not depend on the choice of $k$.

Fig. 8 is such a description. We shift an $r$-tuple of Young diagrams on the $n$-th floor to the left and the left most Young diagram to the right most removing the 1-st row.

Let $\Lambda_{n,r}$ be the index set obtained from such operations. We find that $\Lambda_{n,r}$ is equal to the following set:

$$\Lambda_{n,r} = \{[\lambda^{(1)}, \ldots, \lambda^{(r)}] \ ; \ \sum_{j}^{r} j|\lambda^{(j)}| = n, n-r, n-2r, \ldots\}.$$

Let $\ell_j = |\lambda^{(j)}|$ be the size of $\lambda^j$ and $\mathbf{l} = (\ell_1, \ldots, \ell_r)$ the array of $\ell_j$s. In the following sections this array $\mathbf{l}$ will play an important role.

As for the previous examples, Fig. 6 and Fig. 7, we obtain the parametrizations Fig. 9 and Fig. 10 respectively which do not depend on the choice of $k$.

The weight sum $\|\lambda\|$ of $\lambda = [\lambda^{(1)}, \ldots, \lambda^{(r)}]$ is defined by $\sum_{j=1}^{r} j|\lambda^{(j)}|$. For example the weight sums in Fig. 9 are 4 and $4-3=1$. Those in Fig. 10 are 4 and $4-2=2$ and $4-2-2=0$.

# 5 Standard expression of $P_{n,r}(Q)$

Keeping the facts presented in the previous sections in mind, for an $r$-modular seat-plan $w \in \Sigma_n^r$, we now try to define the standard expression by the special elements in Fig. 5.

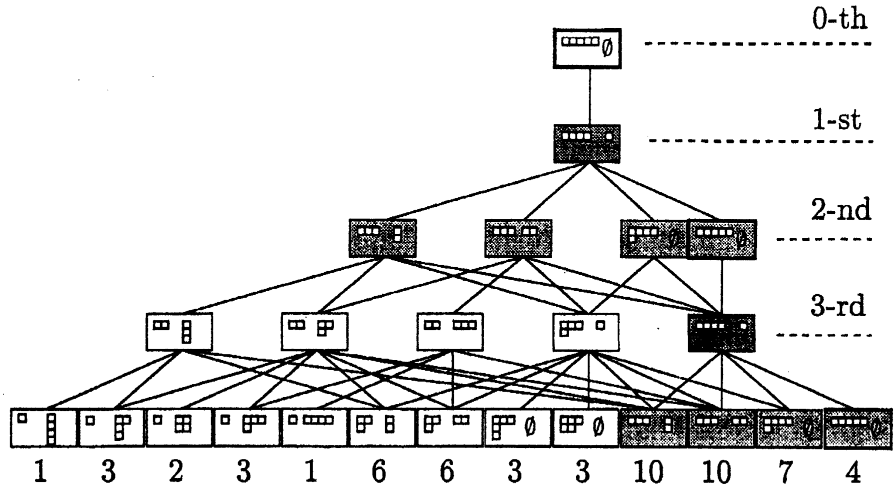



Figure 7: Bratteli diagram for $P_{4,2}(5)$

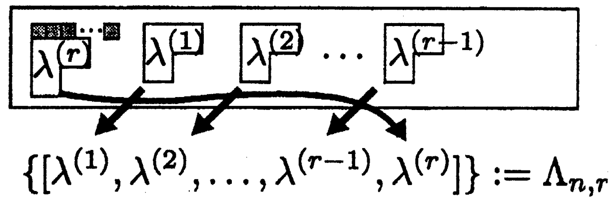


Figure 8: Irreducible representation of $P_{n,r}(Q)$

## 5.1 Propagating number

To define the standard expression, we introduce the notion of the *thickness* of the propagating parts and classify the propagating parts by the thickness. Now we quickly review the definition of propagating parts. Then we define the thickness of a part of an $r$-modular seat plan.

For a part $T$ of an $r$-modular seat-plan, if $T \cap F \neq \emptyset$ and $T \cap M \neq \emptyset$, we call $T$ *propagating*. For an $r$-modular seat-plan $w \in \Sigma_n^r$, let $\pi(w) = \{T \in w \mid T : \text{propagating}\}$ be the set of propagating parts. If $T \in w \setminus \pi(w)$, then we call $T$ *non-propagating* or *defective*. The number of the propagating parts $|\pi(w)|$ of $w$ is called the *propagating number* (of $w$).

For example, in Fig. 1, $\pi(w) = \{T_1, T_2, T_4\}$. Hence $|\pi(w)| = 3$. On the other hand $T_3$ is non-propagating. Note that the following remark holds.

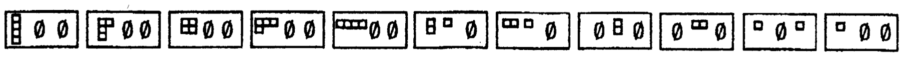


Figure 9: Irreducibles of $P_{4,3}(Q)$

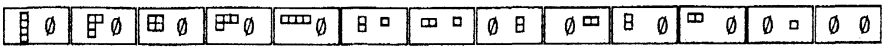


Figure 10: Irreducibles of $P_{4,2}(Q)$

**Remark 2.** *The number of elements contained in a defective part is an integer multiple of $r$. Namely, if $w \in \Sigma_n^r$ and $T_i \in w$ is non-propagating, then there exists an integer $d$ such that*

$$|T_i| = dr.$$

## 5.2 Thickness

For a propagating part of a seat-plan, we define its thickness. The notion of the thickness will also be used to define the conjugacy classes of the party algebra. As for the conjugacy classes and characters of $P_{n,r}(Q)$, it is now being studied by Naruse [13].

Suppose that $w \in \Sigma_n^r$ and $T_i \in \pi(w)$. We define the *thickness* $t(T_i)$ of $T_i$ as the least positive integer which is equal to the number of the elements contained in its upper part by modulo $r$:

$$\begin{aligned} t(T_i) &\in \{1, 2, \ldots, r\}, \\ t(T_i) &\equiv |T_i \cap M| \pmod{r}. \end{aligned}$$

Since $|T_i \cap F| \equiv |T_i \cap M| \pmod{r}$ for any part $T_i \in w$, we can also define the thickness using its lower part.

Put $t = t(T_i)$. Then there exist at least $t$ elements both in the upper and the lower parts of $T_i$. The number of the other elements in $T_i$ must be an integer multiple of $r$. Hence there exist permutations $w_1, w_2 \in \mathfrak{S}_n$ such that the diagram of $w_1 T_i w_2$ does not contain any crossing as in Fig. 11.

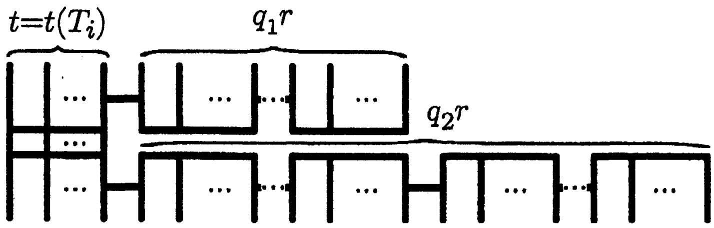



Figure 11: $w_1 T_i w_2$

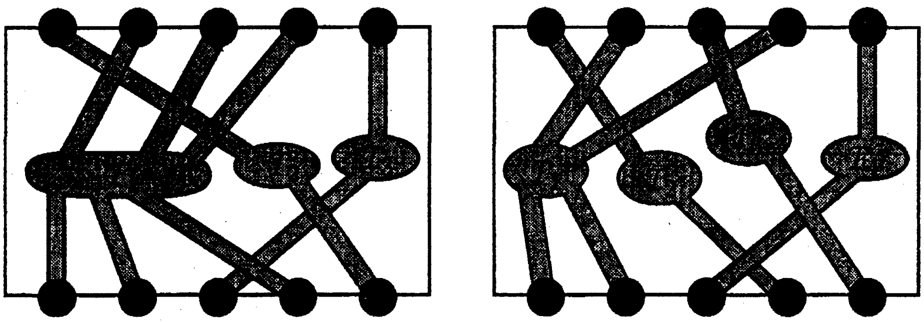
Conversely, every propagating part is obtained from such an $r$-modular seat-plan as in Fig. 11 by attaching permutations to its lower and/or upper part(s).

The *thickness array* of a seat-plan $\mathbf{t}(w) = (\ell_1, \ldots, \ell_r)$ is defined as the list of the numbers of the parts whose thickness values are $1, 2, \ldots, r$:

$$\begin{aligned} \mathbf{t}(w) &:= (\ell_1, \ldots, \ell_r) \\ &:= (\mathbf{t}(w)_1, \ldots, \mathbf{t}(w)_r) \\ &:= (\sharp\{T_i \in w \ ; \ t(T_i) = 1\}, \ldots, \sharp\{T_i \in w \ ; \ t(T_i) = r\}). \end{aligned}$$

Note that we are abusing the same notation $\ell_i$ which we have used to measure the sizes of Young diagrams for indexing the irreducible representations.

For example, in Fig. 12 if we regard $w_1, w_2$ as 3-modular seat-plans, then $\mathbf{t}(w_1) = (2, 0, 1)$ and $\mathbf{t}(w_2) = (3, 1, 0)$. On the other hand, if we regard $w_1, w_2$ as 2-modular seat-plans, then $\mathbf{t}(w_1) = (3, 0)$ and $\mathbf{t}(w_2) = (3, 1)$.



$w_1$ $w_2$

Figure 12:

Note that

$$|\mathbf{t}(w)| := \mathbf{t}(w)_1 + \cdots + \mathbf{t}(w)_r = \ell_1 + \cdots + \ell_r = \pi(w) \text{ (propageting number).}$$

## 5.3 Standard expression

To obtain the standard expression, first we rename all the propagating parts of the seat-plan so that

$$\begin{aligned} t(T_1) = t(T_2) = \cdots = t(T_{\ell_1}) &= 1, \\ t(T_{\ell_1+1}) = t(T_{\ell_1+2}) = \cdots = t(T_{\ell_1+\ell_2}) &= 2, \\ \vdots \quad \vdots \quad & \quad \vdots \\ t(T_{\ell_1+\ell_2+\cdots+\ell_{r-1}+1}) = t(T_{\ell_1+\ell_2+\cdots+\ell_{r-1}+2}) = \cdots = t(T_{\ell_1+\ell_2+\cdots+\ell_{r-1}+\ell_r}) &= r. \end{aligned}$$

Then we *twist* the parts which have the same thickness as follows. Let

$$T_{\ell_1+\ell_2+\cdots+\ell_{j-1}+1}, T_{\ell_1+\ell_2+\cdots+\ell_{j-1}+2}, \ldots, T_{\ell_1+\ell_2+\cdots+\ell_{j-1}+\ell_j}$$

be all the parts whose thickness is $j$. First we divide each of them into the upper and the lower parts. Then we sort the upper parts of them so that the minimum elements of the upper parts become increasing order. (Here we assumed that the elements of $M$ have an order, $m_1 < m_2 < \cdots < m_n$.) Next we sort the lower parts of them so that the minimum elements of the lower parts become increasing order. (Here we assumed that the elements of $F$ have an order, $f_1 < f_2 < \cdots < f_n$.) In order to restore the original parts whose thickness is $j$, join the upper and the lower parts of them. In this process, we have a permutation $v_j \in \mathfrak{S}_{\ell_j}$.

We explain this process using Fig. 13. In this picture,

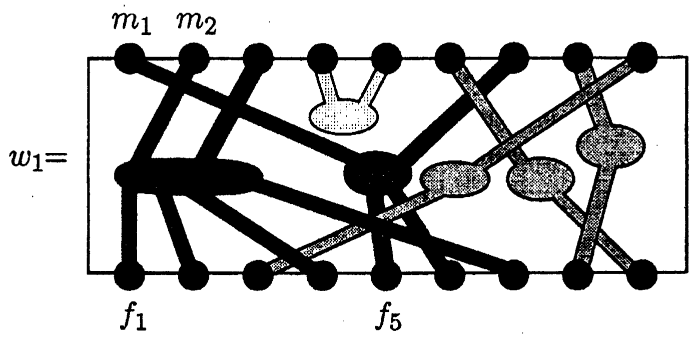



Figure 13: $w_1 \in \Sigma_9^2$

$$w_1 = \begin{array}{c} \{ \ \{m_2, m_3, f_1, f_2, f_4, f_7\}, \ \{m_1, m_7, f_5, f_6\}, \ \{m_4, m_5\}, \\ \{m_9, f_3\}, \ \{m_6, f_9\}, \ \{m_8, f_8\} \ \} \end{array} \in \Sigma_9^r.$$

The thickness of the three gray parts of $w_1$ is 1. So $\ell_1 = 3$. And the thickness of the two black pats is 2. So $\ell_2 = 2$.

Consider the gray parts first. The minimum element of the upper parts is joined to the maximum element of the lower parts. And the maximum element of the upper parts is joined to the minimum element of the lower parts. So we have a permutation $\sigma_1 = (13)(2) \in \mathfrak{S}_3$ (See the left figure of Fig. 15).

Next consider the black parts. If we sort these parts in accordance with the minimum elements of the upper parts, then the right island comes first. On the other hand, if we sort them in accordance with the minimum elements of the lower parts, then the left island comes first. Hence in order to restore the original parts, we have to join the upper parts and the lower parts with a crossing. Note that if there exist $\ell_t$ parts whose thickness is $t$, then the permutation obtained in this way is an element of the symmetric group of degree $\ell_t$. However, to present this permutation by the generators, we need $t$-parallel strings at each crossing as in Fig. 14. So the permutation is realized in the symmetric group of degree $t \times \ell_t$.

As for the black parts of $w_1$ in Fig. 13 we have a transposition $\sigma_2 = (12) \in \mathfrak{S}_2$. However, this is presented as an element of $\mathfrak{S}_{2\times 2}$ as in the right figure of Fig. 15.

The standard expression of $w_1$ in Fig. 13 is obtained from the expression in Fig. 16 by attaching the following two permutations $x$ and $y^{-1}$ on the top and

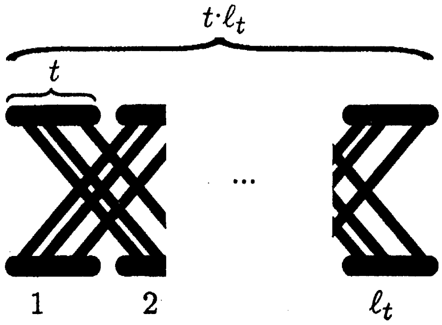



Figure 14:

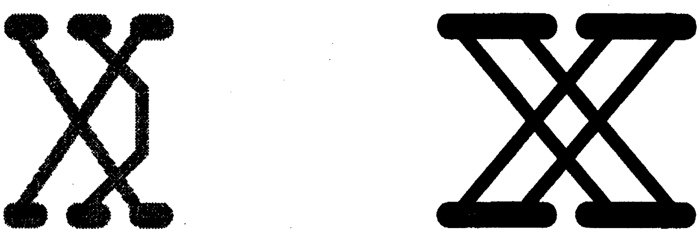


Figure 15:

the bottom respectively:

$$x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 8 & 9 & 1 & 7 & 2 & 3 & 4 & 5 \end{pmatrix},$$
$$y^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 8 & 9 & 1 & 2 & 4 & 7 & 5 & 6 \end{pmatrix}^{-1}.$$

Fig. 17 is another example of a standard expression.. Since $w_2$ is a 3-modular seat-plan, the thickness array is a 3-tuple of permutations. In this case we have a thickness array

$$\mathfrak{l} = \mathfrak{t}(w_2) = (\ell_1, \ell_2, \ell_3) = (4, 2, 2)$$

and a permutation array

$$(v_1, v_2, v_3) \in \mathfrak{S}_{\ell_1} \times \mathfrak{S}_{\ell_2} \times \mathfrak{S}_{\ell_3} \hookrightarrow \mathfrak{S}_{1\times\ell_1} \times \mathfrak{S}_{2\times\ell_2} \times \mathfrak{S}_{3\times\ell_3},$$

The permutation array is uniquely determined by the given $r$-modular seat-plan.

# 6 Application

Using the standard expression above, we can obtain the defining relation of the party algebra $P_{n,r}(Q)$. Further we can show that $P_{n,r}(Q)$ is a cellular algebra.

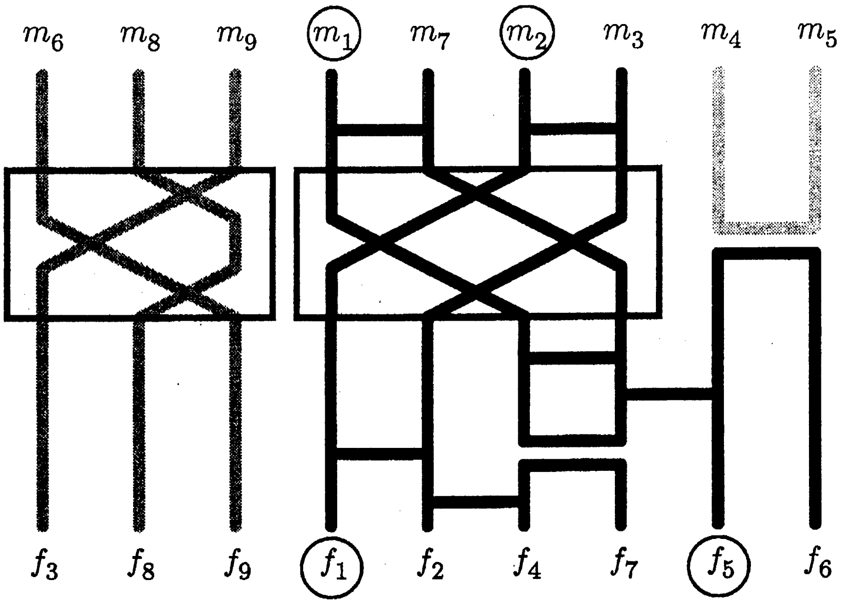



Figure 16:

## 6.1 Defining relation of $P_{n,r}(Q)$

We can find the defining relation of $P_{n,r}(Q)$ by the following try-and-error method: First guess the relations. Then try to show that multiplication of a generator and the standard expression will be transformed to a scalar multiple of another (possibly the same) word of the standard expression, only using the guessed relations.

In this manner we have shown that the following relations characterize the party algebra $P_{n,r}(Q)$[9]:

$$\begin{aligned} s_i^2 &= 1 \quad (i = 1, 2, \ldots, n-1), \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \quad (i = 1, 2, \ldots, n-2), \\ s_i s_j &= s_j s_i \quad (|i-j| \geq 2,\ i, j = 1, 2, \ldots, n-1), \end{aligned}$$

$$f^2 = f,\ fs_2fs_2 = s_2fs_2f,\ fs_2s_1s_3s_2fs_2s_1s_3s_2 = s_2s_1s_3s_2fs_2s_1s_3s_2f,$$

$$fs_1 = s_1f = f,\ fs_i = s_if\ (i = 3, 4, \ldots, n-2),$$

$$(e^{[r]})^2 = Qe,\ e^{[r]}s_i = s_ie^{[r]} = e^{[r]}\ (i = 1, 2, \ldots, r-1),$$

$$e^{[r]}f = fe^{[r]} = e^{[r]}\ e^{[r]}s_re^{[r]} = e^{[r]},$$

$$e^{[r]}Pe^{[r]}P = Pe^{[r]}Pe^{[r]},$$

$$\begin{aligned} &fs_2s_3\cdots s_{r+1}s_1s_2\cdots s_res_r\cdots s_2s_1s_{r+1}\cdots s_3s_2 \\ &\quad = s_2s_3\cdots s_{r+1}s_1s_2\cdots s_res_r\cdots s_2s_1s_{r+1}\cdots s_3s_2f, \end{aligned}$$

$$es_i = s_ie = e \quad (i = r+1, r+2, \ldots, n-1),$$

$$fs_2s_3\cdots s_res_r\cdots s_3s_2f = fs_2fs_3s_2f\cdot\cdots\cdot fs_{r-1}s_{r-2}\cdots s_2f.$$

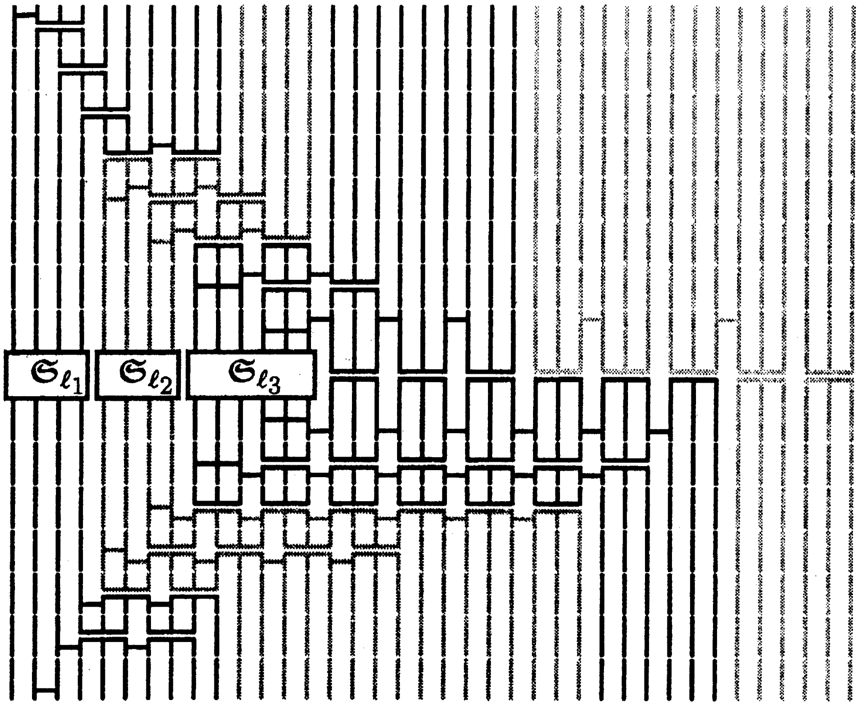



Figure 17: $w_2 \in \Sigma_{38}^3$

Here $P$ denotes a product of $s_i$s which corresponds to the following permutation:

$$\begin{pmatrix} 1 & 2 & \cdots & r & r+1 & r+2 & \cdots & 2r \\ r+1 & r+2 & \cdots & 2r & 1 & 2 & \cdots & r \end{pmatrix}.$$

## 6.2 $P_{n,r}(Q)$ is a cellular algebra

As another application of the standard expression, we can show that $P_{n,r}(Q)$ is a cellular algebra. For the precise description for the cellular algebras, we refer the papers [3, 17]. Here to show that $P_{n,r}(Q)$ is cellular, we use the following lemma which is a version of Lemma 3.3 in [17].

**Lemma 3.** *(Xi [17, Lemma 3.3]) Let $A$ be an algebra with an involution $i$. Suppose there is a decomposition*

$$A = \bigoplus_{\mathbf{j} \le \mathbf{l}} V_{\mathbf{j}} \otimes_k V_{\mathbf{j}} \otimes_k B_{\mathbf{j}} \quad \textit{direct sum of vector space},$$

*where $\mathbf{l}$ is an $r$-tuple of non-negative integers, the partial order $<$ among the indices is introduced by saying that $(j_1', \ldots, j_r') < (j_1, \ldots, j_r)$ if and only if the partition $(1^{j_1}, \ldots, r^{j_r})$ is a refinement of $(1^{j_1'}, \ldots, r^{j_r'})$, $V_{\mathbf{j}}$ is a vector space, and $B_{\mathbf{j}}$ is a cellular algebra with respect to an involution $\sigma_{\mathbf{j}}$ and a cell chain $J_1^{\mathbf{j}} \subset \cdots \subset J_{s_{\mathbf{j}}}^{\mathbf{j}} = B_{\mathbf{j}}$ for each $\mathbf{j}$. Define*

$$J_{\mathbf{t}} = \bigoplus_{\mathbf{j} \le \mathbf{t}} V_{\mathbf{j}} \otimes_k V_{\mathbf{j}} \otimes_k B_{\mathbf{j}}$$

*and*

$$J_{<\mathbf{t}} = \bigoplus_{\mathbf{j}<\mathbf{t}} V_{\mathbf{j}} \otimes_k V_{\mathbf{j}} \otimes_k B_{\mathbf{j}}.$$

*Assume that the restriction of $i$ on $V_{\mathbf{j}} \otimes_k V_{\mathbf{j}} \otimes_k B_{\mathbf{j}}$ is given by $w \otimes v \otimes b \mapsto v \otimes w \otimes \sigma_{\mathbf{j}}(b)$. If for each $\mathbf{j}$ there is a bilinear form $\phi_{\mathbf{j}} : V_{\mathbf{j}} \otimes V_{\mathbf{j}} \to B_{\mathbf{j}}$ such that $\sigma_{\mathbf{j}}(\phi_{\mathbf{j}}(w,v)) = \phi_{\mathbf{j}}(v,w)$ for all $w, v \in V_{\mathbf{j}}$ and that the multiplication of two elements in $V_{\mathbf{j}} \otimes V_{\mathbf{j}} \otimes B_{\mathbf{j}}$ is governed by $\phi_{\mathbf{j}}$ modulo $J_{<\mathbf{j}}$, that is, for $x, y, u, v \in V_{\mathbf{j}}$ and $b, c \in B_{\mathbf{j}}$, we have*

$$(x \otimes y \otimes b)(u \otimes v \otimes c) = x \otimes v \otimes b\phi_{\mathbf{j}}(y,u)c$$

*modulo the ideal $J_{<\mathbf{j}}$, and if $V_{\mathbf{j}} \otimes V_{\mathbf{j}} \otimes J_t^{\mathbf{j}} + J_{<\mathbf{j}}$ is an ideal in $A$ for all $\mathbf{j}$ and $t$ ($1 \le t \le s_{\mathbf{j}}$), then $A$ is a cellular algebra.*

For a finite set $E$ of size $n$, let $\Sigma_E$ be the set of all set-partitions of $E$:

$$\begin{aligned}\Sigma_E &= \{v = \{E_1, \dots, E_s\} \mid s = 1, 2, \dots, \\ &\qquad E_j (\neq \emptyset) \subset E \ (j = 1, 2, \dots, s); \\ &\qquad \cup E_j = E, \quad E_i \cap E_j = \emptyset \text{ if } i \neq j\}.\end{aligned}$$

Suppose that $v \in \Sigma_E$ and $E_j \in v$. We define the thickness $t(E_j)$ of $E_j$ as the least positive integer which is equal to $|E_j|$ modulo $r$:

$$t(E_j) \in \{1, 2, \dots, r\}, \quad t(E_j) \equiv |E_j| \pmod r.$$

The thickness array $\mathbf{t}(v) = (\ell_1, \dots, \ell_r)$ of $v$ is defined as the list of the numbers of the sets whose thicknesses are $1, 2, \dots, r$:

$$\begin{aligned}\mathbf{t}(v) &:= (\ell_1, \dots, \ell_r) \\ &:= (\mathbf{t}(v)_1, \dots, \mathbf{t}(v)_r) \\ &:= (\sharp\{E_j \in v \ ; \ t(E_j) = 1\}, \dots, \sharp\{E_j \in v \ ; \ t(E_j) = r\}).\end{aligned}$$

Note that the definitions of the thickness and the thickness array above are slightly different from the one defined in Section 5.2: we do not have the notion of 'propagating' or 'defective' for $v \in \Sigma_E$. For an $r$-tuple of non-negative integers $\mathbf{l} = (\ell_1, \dots, \ell_r)$ such that $\|\mathbf{l}\| = \sum_{j=1}^r j\ell_j \le n$, we define a vector space $V_{\mathbf{l}}$ whose basis is indexed by the set

$$\begin{aligned}\mathcal{S}_{\mathbf{l}} &= \{(v, S) \mid v \in \Sigma_M, \\ &\qquad (\mathbf{t}(v)_1, \dots, \mathbf{t}(v)_{r-1}) = (\ell_1, \dots, \ell_{r-1}), \ \mathbf{t}(v)_r \ge \ell_r, \\ &\qquad S \text{ is a subset of the set of all parts of } v \text{ with } \mathbf{t}(v)_r = \ell_r\}.\end{aligned}$$

If $v \in \Sigma_M$, we may assume

$$v = \{M_j = \{a_1^{(j)}, a_2^{(j)}, \dots, a_{t_j}^{(j)}\} \mid j = 1, 2, \dots, s\} \tag{5}$$

is written *in standard from.* That is to say the expression (5) satisfies

$$\begin{array}{rcl} t(M_1)=t(M_2)=\cdots=t(M_{\ell_1}) & = & 1,\\ t(M_{\ell_1+1})=t(M_{\ell_1+2})=\cdots=t(M_{\ell_1+\ell_2}) & = & 2,\\ \vdots \quad \vdots & & \vdots\\ t(M_{\ell_1+\ell_2+\cdots+\ell_{r-1}+1})=t(M_{\ell_1+\ell_2+\cdots+\ell_{r-1}+2})=\cdots=t(M_{\ell_1+\ell_2+\cdots+\ell_{r-1}+\ell_r}) & = & r, \end{array}$$

$$a_1^{(j)}<a_2^{(j)}<\cdots<a_{t_j}^{(j)} \text{ for } j=1,2,\ldots,s$$

and

$$a_1^{(\ell_{j-1}+1)}<a_1^{(\ell_{j-1}+2)}<\cdots<a_1^{(\ell_{j-1}+\ell_j)} \text{ for } j=1,2,\ldots,s,$$

where we put $\ell_0=0$. It is clear that there is only one standard form for each $v$. We may also introduce an order on the set of all parts of $v$ by saying that $M_j<M_k$ if and only if $a_1^{(j)}<a_1^{(k)}$. Suppose that $D\subset E$ and $v\in\Sigma_E$. Let $r_D(v)$ denote the partition of $E\setminus D$ obtained from $v$ by deleting all elements in $D$ from the parts of $v$. Let $\mathfrak{S}_{\mathbf{l}}$ be the direct product of the symmetric groups $\mathfrak{S}_{\ell_j}$ $(j=1,2,\ldots,r)$ and $k\mathfrak{S}_{\mathbf{l}}$ the tensor product of the group algebras of them over the field $k$.

Recall that from the standard expression of an $r$-modular seat-plan $w$, we obtain the thickness array $\mathbf{t}(w)=(\ell_1,\ell_2,\cdots,\ell_r)$. More precisely, we have the following lemma.

**Lemma 4.** *For each $r$-modular seat-plan $w\in\Sigma_n^r$, there exists uniquely an $r$-tuple of non-negative integers $\mathbf{l}=(\ell_1,\ldots,\ell_r)$ such that $\sum_{j=1}^r j\ell_j=n$ and $w$ can be written uniquely as an element of $V_{\mathbf{l}}\otimes V_{\mathbf{l}}\otimes_k k\mathfrak{S}_{\mathbf{l}}$.*

*Proof.* Take an $r$-modular seat-plan $w\in\Sigma_n^r$, we define $x:=r_F(w)\in\Sigma_M$ and $y:=r_M(w)\in\Sigma_F$. For $t\in\{1,\ldots,r\}$, let $S^{(t)}$ [resp. $T^{(t)}$] be a subset of $x$ [resp. $y$] obtained from $\{T_j\in\pi(w)\mid t(T_j)=t\}$ by deleting the elements contained in $F$ [resp. $M$]. It is clear that both $|S^{(t)}|=|T^{(t)}|=\mathbf{t}(w)_t$. Now if we write $S^{(t)}=\{S_1^{(t)},\ldots,S_{\ell_t}^{(t)}\}$ and $T^{(t)}=\{T_1^{(t)},\ldots,T_{\ell_t}^{(t)}\}$ in standard form, there exists a permutation $b^{(t)}\in\mathfrak{S}_{\ell_t}$ such that $T_{b^{(t)}(j)}^{(t)}\cup S_j^{(t)}\in\pi(w)$ for $j=1,2,\ldots,\ell_t$. Put $S=\cup_{t=1}^r S^{(t)}$ and $T=\cup_{t=1}^r T^{(t)}$. Note that $x$, $y$ and $b^{(t)}$ $(t=1,\ldots,r)$ are uniquely determined by $w$ in standard form. Note also that if we identify the set $F$ with $M$ by sending $f_j$ to $m_j$, then $T\subset y\in\Sigma_M$. Thus, we can associate with the given $w$ a unique elements

$$(x,S)\otimes(y,T)\otimes b^{(1)}\otimes\cdots\otimes b^{(r)}.$$

Obviously, $(x,S)$ and $(y,T)$ belong to $V_{\mathbf{l}}$ and $b\in\mathfrak{S}_{\ell_1}\times\cdots\times\mathfrak{S}_{\ell_r}$. Conversely, each element $(x,S)\otimes(y,T)\otimes b$ with $(x,S)$, $(y,T)\in S_{\mathbf{l}}$ and $b\in\mathfrak{S}_{\mathbf{l}}$ corresponds to a unique $r$-modular party algebra $w\in\Sigma_n^r$. □

Observing the actions of the generators of $P_{n,r}(Q)$ on the set of $r$-modular seat-plans (presented in the standard expressions), we can define a bilinear form $\phi_{\mathrm{j}} : V_{\mathrm{j}} \otimes V_{\mathrm{j}} \mapsto B_{\mathrm{j}}$ so that it satisfies the condition in Lemma 3. Moreover

$$J_{\mathrm{l}} = \bigoplus_{\mathrm{j} \leq \mathrm{l}} V_{\mathrm{j}} \otimes_k V_{\mathrm{j}} \otimes_k B_{\mathrm{j}}.$$

and

$$J_{<\mathrm{l}} = \bigoplus_{\mathrm{j} < \mathrm{l}} V_{\mathrm{j}} \otimes_k V_{\mathrm{j}} \otimes_k B_{\mathrm{j}}.$$

satisfy the condition in Lemma 3. Hence we finally obtain the following Theorem.

**Theorem 5.** *The party algebra $P_{n,r}(Q)$ is a cellular algebra.*

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