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Local Euler obstructions of 1-forms on singular varieties

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This is a summary of the joint work [5] with J.-P. Brasselet and J. Seade.

There have been many works on vector fields on singular varieties and various notions of indices have been introduced and used for the study of singular varieties. Recently the attention is drawn to the corresponding notions in the case of 1-forms.

M.-H. Schwartz in [19, 20] introduced the technique of radial extension of stratified vector fields and frames on singular varieties, and used this to construct cocycles representing classes in the relative cohomology $H^*(M, M \setminus V)$, where V is a singular variety embedded in a complex manifold M. These are nowadays called the *Schwartz classes* of V. R. MacPherson in [17] introduced the notion of local Euler obstruction, an invariant defined at each point of a singular variety using an index of an appropriate radial 1-form, and used this to construct the homology Chern classes of singular varieties. Brasselet and Schwartz in [3] proved that the Alexander isomorphism $H^*(M, M \setminus V) \simeq H_*(V)$ carries the Schwartz classes into the MacPherson classes. A key ingredient for this proof is their *proportionality theorem* relating the Schwartz index and the local Euler obstruction.

These were the first indices of vector fields and 1-forms on singular spaces in the literature. Later in [10] was introduced another index for vector fields on isolated hypersurface singularities, and this definition was extended in [22] to vector fields on complete intersection germs. This is known as the GSV index. The definition of this index was recently extended in [4] for vector fields with isolated singularities on hypersurface germs with non-isolated singularities, and it was proved that this index satisfies a proportionality property analogous to the one proved in [3], the proportionality factor being now the Euler-Poincaré characteristic of a local Milnor fiber.

In [7] W. Ebeling and S. Gusein-Zade observed that when dealing with singular varieties, 1-forms have certain advantages over vector fields, as for instance the fact that for a vector field on the ambient space, the condition of being tangent to a stratified singular subvariety is very stringent, while every 1-form on the ambient space defines, by restriction, one on the singular variety. They adapted the definition of the GSV index to 1-forms on complete intersection germs with isolated singularities, and proved a very nice formula for it in the case where the form is holomorphic, generalizing the well-known formula of Lê-Greuel for the Milnor number of a function.

This article is about 1-forms on complex analytic varieties and it is particularly relevant when the variety has non-isolated singularities. We recall, in Section 2, how MacPherson's local Euler obstruction can be adapted to 1-forms in general. We show, in Section 3, how the radial extension technique of M.-H. Schwartz can be performed in the case of 1-forms. This allows us to define the Schwartz index of 1-forms with isolated singularities on singular varieties. Then we give a proportionality theorem (Theorem 3.3), analogous to the one in [3] for vector fields, relating these indices.

We then extend, in Section 4, the definition of the GSV-index to 1-forms with isolated singularities on complete intersections with non-isolated singularities that satisfy the Thom a_f-condition, which is always satisfied if the variety is a hypersurface, and we give the corresponding proportionality theorem for this index. When the form is the differential of a holomorphic function h, this index measures the number of critical points of a generic perturbation of h on a local Milnor fiber, so it is analogous to invariants studied by a number of authors (see for instance [11, 13, 21]). Section 1 includes a review of well-known facts about real and complex 1-forms.

The radial extension of 1-forms can be made global on compact varieties, and it can also be made for frames of 1-forms. We obtain in this way the dual Schwartz classes of singular varieties, which equal the usual ones up to sign. We also have the dual Chern-Mather classes of V, already envisaged in [18], and the proportionality formula in Theorem 3.3 can be used as in [3] to express the dual Chern-Mather classes as "weighted" dual Schwartz classes, the weights been given by the local Euler obstruction. Similarly, in analogy with Theorem 1.1 in [4], the corresponding GSV-index and the proportionality Theorem 4.3 extend to frames and can be used to express the dual Fulton-Johnson classes of singular hypersurfaces embedded with trivial normal bundle in compact complex manifolds, as "weighted" dual Schwartz classes, the weights been now given by the Euler-Poincaré characteristic of the local Milnor fiber. These expressions give a precise combinatorial expression of the Milnor classes of singular varieties as above (cf. [4, 1]).

1 Preliminaries

1.1 Complex and real 1-forms

Let M be a complex manifold of dimension m and TM its holomorphic tangent bundle. By the Cauchy-Riemann equation, as a real bundle, TM is canonically isomorphic to the real tangent bundle $T_{\mathbb{R}}M$ of M. In this article, a complex 1-form means a section of T^*M, the complex dual of TM, and a real 1-form a section of $T^*_{\mathbb{R}}M$, the real dual of $T_{\mathbb{R}}M$. As an oriented real bundle, T^*M is not canonically isomophic to $T^*_{\mathbb{R}}M$. However, there is a one to one correspondence between the complex and real forms as explained below. Note that, choosing a Riemannian metric, $T^*_{\mathbb{R}}M$ is isomorphic to $T_{\mathbb{R}}M$.

Let ω be a complex 1-form and decompose it into the real and imaginary parts: $\omega = \mathrm{Re}(\omega) + \sqrt{-1}\,\mathrm{Im}(\omega)$. Both $\mathrm{Re}(\omega)$ and $\mathrm{Im}(\omega)$ are real 1-forms, and the complex linearity of ω implies that, for each tangent vector v, we have

$$\mathrm{Im}(\omega)(v) = -\mathrm{Re}(\omega)(\sqrt{-1}\,v).$$

Thus the form ω is determined by its real part and we have a one to one correspondence between the complex and real forms, assigning to each complex form its real part, and conversely, to each real 1-form η the complex form ω defined by $\omega(v) = \eta(v) - \sqrt{-1}\,\eta(\sqrt{-1}\,v)$. If $(z_1, \ldots, z_m)$ is a local coordinate system on M and if we write

$$\omega = \sum_{i=1}^{m} a_i \, dz_i \tag{1.1}$$

locally, we have

$$\mathrm{Re}(\omega) = \sum_{i=1}^{m} (u_i \, dx_i - v_i \, dy_i) \quad \text{and} \quad \mathrm{Im}(\omega) = \sum_{i=1}^{m} (v_i \, dx_i + u_i \, dy_i), \tag{1.2}$$

where u_i and v_i are the real and the imaginary parts of a_i, respectively.

1.2 Characteristic classes

The Euler classes of the real vector bundles and the top Chern classes of the complex vector bundles which appeared above are related by

$$e(T_{\mathbb{R}}^*M) = e(T_{\mathbb{R}}M) = c_m(TM) = (-1)^m c_m(T^*M).$$

1.3 Localization

For a complex 1-form ω with an isolated singularity (i.e., zero) at p, we have the Poincaré-Hopf index $\mathrm{Ind_{PH}}(\omega, p)$ as the locatization of $c_m(T^*M)$ by ω at p. In the local expression (1.1), it is the mapping degree of $(a_1, \ldots, a_m)$ on a small sphere around p. For a real 1-form η with an isolated singularity at p, we have the Poincaré-Hopf index $\mathrm{Ind_{PH}}(\eta, p)$ as the locatization of $e(T_{\mathbb{R}}^*M)$ by η at p. For a complex form ω, $\mathrm{Ind_{PH}}(\mathrm{Re}(\omega), p)$ is the mapping degree of $(u_1, -v_1, \ldots, u_m, -v_m)$, in the local expression (1.2), and thus we have

$$\mathrm{Ind_{PH}}(\omega, p) = (-1)^m \, \mathrm{Ind_{PH}}(\mathrm{Re}(\omega), p).$$

1.4 Singularities in the stratified sense

Note that, if ω is a complex 1-form, for a point p in M, we have

$$\mathrm{Ker}\, \omega(p) = \mathrm{Ker}\, \mathrm{Re}(\omega)(p) \cap \mathrm{Ker}\, \mathrm{Im}(\omega)(p).$$

Now let M be a complex manifold and V a subvariety of M of pure dimension n. We take a Whitney stratification $\{V_\alpha\}$ adapted to V, i.e., V is a union of strata. The following definition is an immediate extension for 1-forms of the corresponding (standard) definition for functions on stratified spaces in terms of its differential (c.f. [8, 9, 14]).

Definition 1.3. Let ω be a real or complex 1-form on V, i.e., a continuous section of $T_{\mathbb{R}}^*M|_V$ or of $T^*M|_V$. A point p in V is a *singular point of ω in the stratified sense*, if $\mathrm{Ker}\, \omega(p)$ contains the tangent space T_pV_α of the stratum V_α through p.

This means that the pull-back of ω to V_α vanishes at p. Note that if a stratum consisting of a point, the point is always a singular point.

2 Local Euler obstructions

Let M and V as in Section 1. We denote by $\mathrm{Sing}(V)$ the singular set of V and by $V_{\mathrm{reg}} = V \setminus \mathrm{Sing}(V)$ the regular part.

The Nash modification of V is constructed as follows. Let $G_n(TM)$ be the Grassmann bundle over M of n-planes in TM and $\sigma : V_{\mathrm{reg}} \to G_n(TM)$ the mapping given by $p \mapsto T_p V_{\mathrm{reg}}$. The Nash modification $\tilde{V}$ of V is the closure of the image of σ. We denote by $\nu : \tilde{V} \to V$ the restriction to $\tilde{V}$ of the projection $G_n(TM) \to M$. The Nash bundle $\tilde{T} \to \tilde{V}$ is the restriction to $\tilde{V}$ of the tautological bundle over $G_n(TM)$. Note that a point in $\tilde{V}$ is an n-plane P in T_pM, $p = \nu(P)$, and the fiber of $\tilde{T}$ over the point P is P as an n-plane. We denote by $\tilde{T}^*$ and $\tilde{T}^*_{\mathbb{R}}$, respectively, the complex dual and the real dual of $\tilde{T}$. Since $\tilde{T}$ is a subbundle of ν^*TM, we have the projections

$$\nu^*T^*M \to \tilde{T}^* \quad \text{and} \quad \nu^*T^*_{\mathbb{R}}M \to \tilde{T}^*_{\mathbb{R}},$$

both of which are denoted by π. For a 1-form ω, we call $\pi\nu^*\omega$ the *lifting* of ω and denote it by $\tilde{\omega}$. It is a section of $\tilde{T}^*$ or of $\tilde{T}^*_{\mathbb{R}}$. The following lemma follows from the Whitney (a) condition.

Lemma 2.1. *Let A be a subset in V. If a 1-form ω has no singularities in A in the stratified sense, then the lifting $\tilde{\omega}$ is non-vanishing on $\nu^{-1}A$.*

Let ω be a 1-form with an isolated singularity at p in V in the stratified sense. We take a small ball B in M around p and set $S = \partial B$. By the above lemma, $\tilde{\omega}$ is non-vanishing as a section of $\tilde{T}^*$ or of $\tilde{T}^*_{\mathbb{R}}$ on $\nu^{-1}(S \cap V)$. Let $o(\tilde{\omega})$ denote the obstruction to extending $\tilde{\omega}$ as a non-vanishing section over $\nu^{-1}(B \cap V)$, which is in the relative cohomology $H^{2n}(\nu^{-1}(B \cap V), \nu^{-1}(S \cap V))$. Then we define (c.f. [2, 8]):

Definition 2.2. The *local Euler obstruction* $\mathrm{Eu}(\omega, V; p)$ of ω at p the integer obtained by evaluating $o(\omega)$ on the fundamental cycle $[\nu^{-1}(B \cap V), \nu^{-1}(S \cap V)]$.

Remark 2.3. 1. If ω is a complex 1-form, we have

$$\mathrm{Eu}(\omega, V; p) = (-1)^n \, \mathrm{Eu}(\mathrm{Re}(\omega), V; p).$$

2. If we denote by ρ the squre of the function giving the distance from p (with respect to some metric), then $\mathrm{Eu}(d\rho, V; p) = \mathrm{Eu}(V, p)$ is the local Euler obstruction of MacPherson [17].

3 Schwartz index

It is possible to make for 1-forms the classical construction of radial extension introduced by M.-H. Schwartz in [19, 20] for stratified vector fields and frames. Locally, the construction can be described as follows.

Let M, V and $\{V_\alpha\}$ be as in the previous sections. Let ω_α be a 1-form on a stratum V_α of dimension s with an isolated singularity at p in V_α. We take a small neighborhood U_α of p in V_α and a small disk D of complex dimension $m - s$ with center at p which is transverse to V_α. Let $B = U_\alpha \times D$ with two projections $\pi_1 : B \to U_\alpha$ and $\pi_2 : B \to D$. The *radial extension* ω'_α of ω_α is defined by

$$\omega'_\alpha = \begin{cases} \pi_1^*\omega_\alpha + \pi_2^* d\rho_D, & \text{if } \omega \text{ is real,} \\ \pi_1^*\omega_\alpha + \pi_2^* \widehat{d\rho_D}, & \text{if } \omega \text{ is complex,} \end{cases}$$

where ρ_D is the squre of the function on D giving the distance from p and $\widehat{d\rho_D}$ the complex form such that $\mathrm{Re}(\widehat{d\rho_D}) = d\rho_D$.

Lemma 3.1. *Let ω_α be as above and $\omega = \omega'_\alpha$ its radial extension. Then we have*

$$\begin{aligned} \mathrm{Ind}_{\mathrm{PH}}(\omega, M; p) &= \mathrm{Ind}_{\mathrm{PH}}(\omega, M; p), && \text{if } \omega \text{ is real,} \\ (-1)^m \mathrm{Ind}_{\mathrm{PH}}(\omega, M; p) &= (-1)^s \mathrm{Ind}_{\mathrm{PH}}(\omega, M; p), && \text{if } \omega \text{ is complex.} \end{aligned}$$

Definition 3.2. Let ω_α and $\omega = \omega'_\alpha$ be as in Lemma 3.1. The Schwartz index of ω at p in V is defined by

$$\begin{aligned} \mathrm{Ind}_{\mathrm{Sch}}(\omega, V; p) &= \mathrm{Ind}_{\mathrm{PH}}(\omega, M; p), && \text{if } \omega \text{ is real,} \\ (-1)^n \mathrm{Ind}_{\mathrm{Sch}}(\omega, V; p) &= (-1)^m \mathrm{Ind}_{\mathrm{PH}}(\omega, M; p), && \text{if } \omega \text{ is complex.} \end{aligned}$$

Theorem 3.3 (Proportionality Theorem). *For a radial extension ω as above, we have*

$$\mathrm{Eu}(\omega, V; p) = \mathrm{Eu}(V; p) \cdot \mathrm{Ind}_{\mathrm{Sch}}(\omega, V; p).$$

For the proof (as well as that of Theorem 4.3 below), we refer to [5]. A similar line of proof works for the case of vector fields and frames of vector fields and it significantly simplifies the original proof of the proportionality theorem in [3] (see [6]).

4 GSV-index

We recall ([10, 22]) that the GSV index of a vector field v on a complete intersection V with an isolated singularity can be defined to be the Poincaré-Hopf index of an extension of v to a Milnor fiber F. Similarly, the GSV index of a 1-form ω on V can be defined to be the Poincaré-Hopf index of the form on F, i.e., the number of singularities of ω in F counted with multiplicities [7]. If V has non-isolated singularities we may not have a Milnor fibration in general, but we do if V admits a Whitney stratification with the Thom a_f-condition, $f = (f_1, \cdots, f_k)$ being the functions defining V (c.f. [15, 16, 4]).

Let (V, p) be a complete intersection of dimension n in a neighborhood U of the origin 0 in $\mathbb{C}^m$, defined by functions $f = (f_1, \cdots, f_k)$, $k = m - n$, and assume p $(= 0)$ is a singular point of V (not necessarily an isolated singularity). As before, we endow U with a Whitney stratification adapted to V, and we assume that it satisfies the a_f-condition of Thom (see, e.g., [16]). In particular we always have such a stratification if $k = 1$, by [12].

Let ω be as before, a real or complex 1-form on U, and assume that its restriction to V has an isolated singularity at p in the stratified sense. Now let F be a Milnor fiber of

V, i.e., $F = f^{-1}(t) \cap B$, where B is a small ball in U around p and $t \in \mathbb{C}^k$ is a regular value of f with $\|t\|$ sufficiently small. From the Thom a_f-condition, we have

Lemma 4.1. *The pull-back of ω to a Milnor fiber F is a section of $T^*_{\mathbb{R}}F$ or of T^*F which is non-vanishing near the boundary $\partial F = f^{-1}(t) \cap S$, $S = \partial B$.*

Let $o(\omega)$ denote the obstruction to extending ω as a non-vanishing section of $T^*_{\mathbb{R}}F$ or of T^*F over F, which is in the relative cohomology $H^{2n}(F, \partial F)$.

Definition 4.2. The *GSV index* $\mathrm{Ind}_{\mathrm{GSV}}(\omega, p)$ of ω at p the integer obtained by evaluating $o(\omega)$ on the fundamental cycle $[F, \partial F]$.

For a complex 1-form ω, we have

$$\mathrm{Ind}_{\mathrm{GSV}}(\omega, p) = (-1)^n \mathrm{Ind}_{\mathrm{GSV}}(\mathrm{Re}(\omega), p).$$

We remark that if V has an isolated singularity at p, this coincides the index introduced in [7], i.e., the degree of the map from the link of V into the Stiefel manifold of complex $(k+1)$-frames in $\mathbb{C}^m$ given by the map $(\omega, df_1, \cdots, df_k)$. Also notice that this index is somehow dual to the index defined in [4] for vector fields, which is related to the top Fulton-Johnson class of singular hypersurfaces.

Theorem 4.3 (Proportionality Theorem). *Let ω_α be a real or complex 1-form on the stratum V_α through p as above. Suppose that p is an isolated singularity of ω_α in the stratified sense and let $\omega = \omega'_\alpha$ be its radial extension. Then we have*

$$\mathrm{Ind}_{\mathrm{GSV}}(\omega, p) = \chi(F) \cdot \mathrm{Ind}_{\mathrm{Sch}}(\omega, V; p),$$

where $\chi(F)$ denotes the Euler-Poincaré characteristic of a Milnor fiber F.

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