RESIDUE OF CODIMENSION 1 SINGULAR HOLOMORPHIC DISTRIBUTIONS

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1

The aim of this note is to describe the residue formula for singular holomorphic distribution in terms of the conormal sheaf $G$ in codimension 1 case.

We also prove the Baum-Bott type residue formula for singular distributions. If we define the tangent sheaf of the distribution $F$ by taking the annihilator of $G$ by the dual coupling, we will show that the residue formula for $G$ deduce the Baum-Bott type residue formula for the top Chern class of the normal sheaf $N_F$. If we assume the Frobenius integrability condition for $G$, we have the Baum-Bott residue formula

$$\int_X \varphi(N_F) = \text{Res}_\varphi(N_F, S(F))$$

for $n$-th symmetric polynomial $\varphi$. In this case, the Baum-Bott residue formula for $\varphi = c_n$ is equivalent to the formula we will prove, which means that the Bott vanishing theorem based on the involutivity of $F$ is not necessary for the top Chern class $c_n(N_F)$.

As an application of our results, we will give a residue formula for the non-transversality of a holomorphic map $F : X \rightarrow Y$ to a non-singular distribution on $Y$.

2. SINGULAR HOLOMORPHIC DISTRIBUTION

2.1. Singular holomorphic distribution. Let $X$ be a complex manifold. We define a singular holomorphic distribution $F$ on $X$ to be a coherent subsheaf of the tangent sheaf $\Theta_X$. We call $F$ the tangent sheaf of the distribution. We say $F$ is dimension $p$ if a generic stalk of $F$ is rank $p$ free $O_X$-module. We also define the normal sheaf $N_F$ of $F$ by the exact sequence

$$0 \rightarrow F \rightarrow \Theta_X \rightarrow N_F \rightarrow 0.$$  

The singular set $S(F)$ of $F$ is defined by $S(F) = \{ p \in X | N_{F,p} \text{ is not } O_p \text{-free} \}$.

We can also give a definition of a singular holomorphic distribution $G$ on $X$ to be a coherent subsheaf of the cotangent sheaf $\Omega_X$. We call $G$ the conormal sheaf of the distribution. We also say $G$ is codimension $q$ if the generic rank is $q$. We also define the cotangent sheaf $\Omega_G$ of $G$ by the exact sequence

$$0 \rightarrow G \rightarrow \Omega_X \rightarrow \Omega_G \rightarrow 0.$$  

The singular set $S(G)$ of $G$ is also defined by $S(G) = \{ p \in X | \Omega_{G,p} \text{ is not } O_p \text{-free} \}$. 

2.2. **Codimension 1 case.** We give more simple descriptions for codimension 1 singular distributions. A codimension 1 locally free singular holomorphic distribution is given by a collection of 1-forms $\omega = (\omega_\alpha, U_\alpha)$ for an open covering $\{U_\alpha\}$ of $X$ which has the transition relations $\omega_\beta = g_{\alpha\beta}\omega_\alpha$ on the intersection $U_\alpha \cap U_\beta$ with $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$. Then the cocycle $(g_{\alpha\beta})$ defines a line bundle $G$. Generically at $p$, the covector $\omega_p$ gives an embedding of the fiber $G_p$ into $T_p^*X$ by $f_p \in G_p \mapsto f_p\omega_p \in T_p^*X$. Thus $G$ is regarded as a subbundle of $T^*X$ without on the zero loci of $\omega$. Since the map of germs of sections $(f)_p \in \mathcal{O}_X(G)_p \mapsto (f\omega)_p \in \Omega_{X,p}$ are injective for all $p \in X$, the sheaf $\mathcal{G} = \mathcal{O}_X(G)$ gives the subsheaf of $\Omega_X$ in the above sense in 1.2. Since the quotient sheaf $\Omega_X$ is not $\mathcal{O}$-free on the zero loci of $\omega$ on which we can not define the quotient bundle $T^*X/G$, we see the singular set of $\mathcal{G}$ is $S(\mathcal{G}) = \{ p | \omega(p) = 0 \}$.

3. **Residue of codimension 1 distribution**

3.1. **Localization of the top Chern class.** We determine the dual homology class of $c_n(\Omega_X \otimes \mathcal{G}^\vee)$. Our main tool is the Čech-de Rham techniques. For generalities on the integration and the Chern-Weil theory on the Čech-de Rham cohomology, see [S3] or [IS]. We set for an analytic set $S$, $U_0 = X \setminus S$, $U_1$ is a regular neighborhood of $S$, and $U_{01} = U_0 \cap U_1$. For a covering $U = \{U_0, U_1\}$ of $X$, the Čech-de Rham cohomology group $H^{2\mathfrak{n}}(\mathcal{A}^*(U))$ is represented by the group of cocycles of the type $(\sigma_0, \sigma_1, \sigma_{01})$ for $\sigma_0 \in Z^{2\mathfrak{n}}(U_0)$, $\sigma_1 \in Z^{2\mathfrak{n}}(U_1)$, and $\sigma_{01} \in A^{2\mathfrak{n}-1}(U_{01})$ with $d\sigma_{01} = \sigma_1 - \sigma_0$. We note that the Čech-de Rham cohomology can be regarded as the hypercohomology of the de Rham complex $(\mathcal{A}^*, d)$. By usual spectral sequence arguments for double complexes, we see that the Čech-de Rham cohomology group is canonically isomorphic to the de Rham cohomology group. If we take the subgroup $H^{2\mathfrak{n}}(\mathcal{A}^*(U, U_0))$ of cocycles of the form $(0, \sigma_1, \sigma_{01})$, then this is also isomorphic to the relative cohomology group $H^{2\mathfrak{n}}(X, X \setminus S; \mathcal{G})$.

In the above settings, the top Chern class $c_n(E)$ of a vector bundle $E$ of rank $n$ is given by the cocycle in $H^{2\mathfrak{n}}(\mathcal{A}^*(U))$ as follows. For $i = 0, 1$, let $\nabla_i$ be a connection for $E$ on $U_i$ and $c_n(\nabla_i)$ the $n$-th Chern form of $\nabla_i$. We also write by $c_n(\nabla_0, \nabla_1)$ the transgression form of $c_n(\nabla_i)$'s on $U_{01}$. Then $c_n(E)$ is represented by

$$(c_n(\nabla_1), c_n(\nabla_1), c_n(\nabla_0, \nabla_1)).$$

If $E$ has a global section $s$ with zero loci $S$, then we take $\nabla_0$ as the $s$-trivial connection such that we have $c_n(\nabla_0) = 0$. Thus we can define the localized Chern class at $p$ in $H^{2\mathfrak{n}}(X, X \setminus S; \mathcal{G})$ by a Čech-de Rham cocycle $(0, c_n(\nabla_1), c_n(\nabla_0, \nabla_1))$.

The integration of $c_n(E) = (0, c_n(\nabla_1), c_n(\nabla_0, \nabla_1))$ is defined by

$$\int_X c_n(E) = \int_R c_n(\nabla_1) - \int_{\partial R} c_n(\nabla_0, \nabla_1)$$

for a tubular neighbourhood $R \subset U_1$ of $S$.

3.2. **Residue of codimension 1 distributions.** Now we apply the above arguments to our situations. Let $\mathcal{G}$ be a codimension 1 locally free distribution with the zero loci $S(\mathcal{G})$ and we suppose that $S(\mathcal{G})$ has connected components $S_j$. We set $U_0 = X \setminus S(\mathcal{G})$ and $U_j$ is a regular neighbourhood of $S_j$. We consider the localized class of $c_n(\Omega_X \otimes \mathcal{G}^\vee)$ in the Čech-de Rham cohomology group for the covering $U = \{U_0, U_1, \ldots, U_j\}$. Since the collection $\omega$ of 1-forms $\omega_\alpha$ defines the global section of $\Omega_X \otimes \mathcal{G}^\vee$, we can take $\nabla_0$ as the $\omega$-trivial connection such that $c_n(\nabla_0) = 0$.
as we discussed above. For all $j = 1, \ldots, k$, we can also take $\nabla_j$ as an arbitrary connection on $U_j$. So we have

$$c_n(\Omega_X \otimes G^\vee) = (0, \{c_n(\nabla_j)\}_{j=1,\ldots,k}, \{c_n(\nabla_0, \nabla_j)\}_{j=1,\ldots,k}) \in H^{2n}(X, X \setminus S(G); \mathbb{C}).$$

We denote by $R_j$ a tubular neighbourhood of $S_j$ in $U_j$. We give the following definition of residue.

**Definition 3.1.** The residue of $G$ at $S_j$ is defined by

$$\text{Res}(G, S_j) = \int_{R_j} c_n(\nabla_j) - \int_{\partial R_j} c_n(\nabla_0, \nabla_j).$$

We can describe the residue into precise form in isolated singular cases. Here we refer the result in [S3] of Theorem 5.5.

**Theorem 3.2.** Let $s$ be a regular section of $E$ with isolated zero \{p\} and $s$ is locally given by $(f_1, \ldots, f_n)$ near $p$. Then we have

$$\text{Res}(G, p) = \text{Res}_p[df_1 \wedge f_2 \wedge \cdots \wedge df_n]$$

where $\text{Res}_p[df_1 \wedge f_2 \wedge \cdots \wedge df_n]$ is the Grothendick residue of $(f_1, \ldots, f_n)$.

The dual correspondence in the Alexander duality

$$AL : H^{2n}(X, X \setminus S(G); \mathbb{C}) \cong \bigoplus_{j} H_0(S_j; \mathbb{C})$$

is given by

$$AL(c_n(\Omega_X \otimes G^\vee)) = \sum_j \text{Res}(G, S_j).$$

Now we have the residue formula for isolated singular cases as,

**Theorem 3.3** (The residue formula for isolated singularities). Let $\omega$ be a codimension 1 singular holomorphic distribution with the cotangent sheaf $G$ and $(f_1^{(j)}, \ldots, f_n^{(j)})$ a local expression of $\omega \in H^0(X, \Omega_X \otimes G^\vee)$ near $p_j$.

$$\int_X c_n(\Omega_X \otimes G^\vee) = \sum_{j=1}^k \text{Res}_{p_j}[df_1^{(j)} \wedge \cdots \wedge df_n^{(j)}].$$

4. **Baum-Bott type residue formula**

4.1. **Koszul resolution.** First let us remember the definition of the Koszul complex. (See [FG], Chapter 4 or [GH], Chapter 5.) Let $E$ be a locally free $O$-module of rank $n$ and $d : E \rightarrow O$ an $O$-homomorphism. Then the Koszul complex of sheaves

$$0 \rightarrow \wedge^n E \rightarrow \wedge^{n-1} E \rightarrow \cdots \rightarrow \wedge^1 E \rightarrow O \rightarrow 0$$

is defined by the boundary operator

$$d_p(\epsilon_1 \wedge \cdots \wedge \epsilon_p) = \sum_{i=1}^p (-1)^{i-1} d(\epsilon_i) \epsilon_1 \wedge \cdots \wedge \hat{\epsilon_i} \wedge \cdots \wedge \epsilon_p.$$

This complex is exact expect for the last term. If the image $I_d$ of $d$ is regular ideal, the complex

$$0 \rightarrow \wedge^n E \rightarrow \wedge^{n-1} E \rightarrow \cdots \rightarrow \wedge^1 E \rightarrow O \rightarrow I_d/O \rightarrow 0$$

is defined by the boundary operator

$$d_p(\epsilon_1 \wedge \cdots \wedge \epsilon_p) = \sum_{i=1}^p (-1)^{i-1} d(\epsilon_i) \epsilon_1 \wedge \cdots \wedge \hat{\epsilon_i} \wedge \cdots \wedge \epsilon_p.$$
is exact. We call this exact sequence the Koszul resolution of $\mathcal{O}/\mathcal{I}_d$.

Now let us consider our case. As observed in 2.1, $\omega$ can be regarded as a homomorphism $\omega : \mathcal{G} \rightarrow \Omega_X$ such that it defines a global section

$$\omega \in H^0(X, \mathcal{H}om_{\mathcal{O}}(\mathcal{G}, \mathcal{O}_X)) \simeq H^0(X, \mathcal{O}_X \otimes \mathcal{G}^{'\vee}).$$

Locally on $U_{\alpha}$, $\omega$ is given by $\omega_{\alpha} \otimes s_{\alpha}^{\vee} = \sum f_i (dx_i \otimes s_{\alpha}^{\vee})$ for some local coordinates of $X$ and a local frame $s_{\alpha}^{\vee}$ for $\mathcal{G}^{'\vee}$. In the other words, $\omega$ acts on $(\mathcal{O}_X \otimes \mathcal{G}^{'\vee})^{\alpha} \simeq \mathcal{O}_X \otimes \mathcal{G}$ as a contraction operator $\omega : \mathcal{O}_X \otimes \mathcal{G} \rightarrow \mathcal{O}$. We denote by $\mathcal{I}_\omega$ the ideal sheaf defined by $\text{Im}(\omega : \mathcal{O}_X \otimes \mathcal{G} \rightarrow \mathcal{O})$. We assume that $\mathcal{S}(\mathcal{G}) = \{ p \in X | \omega_p = 0 \}$ consists only of isolated points such that the local coefficients $(f_1, \cdots, f_n)$ of $\omega$ is regular sequence on $\mathcal{S}(\mathcal{G})$. Then the complex of sheaves

$$0 \rightarrow \wedge^n(\mathcal{O}_X \otimes \mathcal{G}) \rightarrow \wedge^{n-1}(\mathcal{O}_X \otimes \mathcal{G}) \rightarrow \cdots \rightarrow \wedge^1(\mathcal{O}_X \otimes \mathcal{G}) \rightarrow \mathcal{O} \rightarrow \mathcal{O}/\mathcal{I}_\omega \rightarrow 0$$

is exact with the boundary operator

$$d_p(e_1 \wedge \cdots \wedge e_p) = \sum_{i=1}^{p} (-1)^{i-1} f_i e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_p$$

where we set $e_i = \frac{\partial}{\partial x_i} \otimes s$. Therefore this gives the Koszul resolution of $\mathcal{O}/\mathcal{I}_\omega$.

By using this projective resolution, we can defines the Chern character of the coherent sheaf $\mathcal{O}/\mathcal{I}_\omega$ by

**Proposition 4.1.**

$$ch(\mathcal{O}/\mathcal{I}_\omega) = c_n(\mathcal{O}_X \otimes \mathcal{G}^{'\vee}).$$

**Proof.** We use [H] of Theorem 10.1.1 and we have

$$ch(\mathcal{O}/\mathcal{I}_\omega) = ch(\sum_{i=0}^{n} (-1)^i \wedge^i(\mathcal{O}_X \otimes \mathcal{G}))$$

$$= td^{-1}(\mathcal{O}_X \otimes \mathcal{G}^{'\vee}) c_n(\mathcal{O}_X \otimes \mathcal{G}^{'\vee})$$

$$= c_n(\mathcal{O}_X \otimes \mathcal{G}^{'\vee}).$$

4.2. Baum-Bott type residue formula. Now we translate the above results in terms of differential system in the tangent sheaf $\mathcal{O}_X$. Let $\mathcal{F} = \{ v \in \mathcal{O}_X | \langle v, \omega \rangle = 0 \}$ be the annihilator of $\mathcal{G}$. Then $\mathcal{F}$ defines a $n-1$ dimensional (possibly) singular distribution. Since $\mathcal{G}$ is locally free, by applying $\otimes \mathcal{G}$ to the exact sequence

(1)$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{O}_X \rightarrow N_{\mathcal{F}} \rightarrow 0,$$

the following sequence

$$0 \rightarrow \mathcal{F} \otimes \mathcal{G} \rightarrow \mathcal{O}_X \otimes \mathcal{G} \rightarrow N_{\mathcal{F}} \otimes \mathcal{G} \rightarrow 0,$$

is also exact. Since the kernel of $\omega : \mathcal{O}_X \otimes \mathcal{G} \rightarrow \mathcal{O}_X$ is equals to $\mathcal{F} \otimes \mathcal{G}$, we have

(2)$$\mathcal{I}_\omega \simeq (\mathcal{O}_X \otimes \mathcal{G})/(\mathcal{F} \otimes \mathcal{G}) \simeq N_{\mathcal{F}} \otimes \mathcal{G}.$$ We take $\mathcal{H}om_{\mathcal{O}}(\mathcal{G}, \mathcal{O})$ of the dual exact sequence

$$0 \rightarrow \mathcal{G} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_G \rightarrow 0$$

of (1), we obtain the exact sequence

$$0 \rightarrow \mathcal{H}om_{\mathcal{O}}(\mathcal{G}, \mathcal{O}) \rightarrow \mathcal{H}om_{\mathcal{O}}(\mathcal{O}_X, \mathcal{O}) \rightarrow \mathcal{H}om_{\mathcal{O}}(\mathcal{G}, \mathcal{O}) \rightarrow \mathcal{E}xt^1_{\mathcal{O}}(\mathcal{G}, \mathcal{O}) \rightarrow 0,$$

which implies

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{O}_X \rightarrow \mathcal{G}^{'\vee} \rightarrow \mathcal{E}xt^1_{\mathcal{O}}(\mathcal{G}, \mathcal{O}) \rightarrow 0.$$
We use $\mathcal{F} = \text{Hom}_\mathcal{O}(\Omega_{\mathcal{G}}, \mathcal{O})$ and $\Theta_X = \text{Hom}_\mathcal{O}(\Omega_X, \mathcal{O})$ in the above. Thus we obtain

$$0 \rightarrow N_{\mathcal{F}} \rightarrow \mathcal{G}' \rightarrow \mathcal{E}xt^1_\mathcal{O}(\Omega_{\mathcal{G}}, \mathcal{O}) \rightarrow 0.$$  \hfill (3)

By taking the Chern characters of (3), we have

$$ch(N_{\mathcal{F}}) = ch(\mathcal{G}') - ch(\mathcal{E}xt^1_\mathcal{O}(\Omega_{\mathcal{G}}, \mathcal{O})).$$  \hfill (4)

By tensoring $\mathcal{G}$ for each term of (3), we also have the exact sequence

$$0 \rightarrow \mathcal{I}_\omega \rightarrow \mathcal{O} \rightarrow \mathcal{E}xt^1_\mathcal{O}(\Omega_{\mathcal{G}}, \mathcal{O}) \otimes \mathcal{G} \rightarrow 0$$

and which gives the isomorphism $\mathcal{O}/\mathcal{I}_\omega \simeq \mathcal{E}xt^1_\mathcal{O}(\Omega_{\mathcal{G}}, \mathcal{O}) \otimes \mathcal{G}$. Thus the Chern characters of those sheaves satisfy

$$ch(\mathcal{E}xt^1_\mathcal{O}(\Omega_{\mathcal{G}}, \mathcal{O})) = ch(\mathcal{O}/\mathcal{I}_\omega)ch(\mathcal{G}').$$  \hfill (5)

Therefore by combining the two equalities (4) and (5) for the Chern characters, we obtain

**Proposition 4.2.**

$$ch(N_{\mathcal{F}}) = (1 - ch(\mathcal{O}/\mathcal{I}_\omega))ch(\mathcal{G}')$$

$$= (1 - c_n(\Omega_X \otimes \mathcal{G}'))ch(\mathcal{G}').$$

Now we find the top Chern class of $N_{\mathcal{F}}$.

**Proposition 4.3.**

$$c_n(N_{\mathcal{F}}) = (-1)^n(n - 1)!c_n(\Omega_X \otimes \mathcal{G}').$$

**Proof.** Let $\{\xi_i\}$ be the formal Chern roots of $c(N_{\mathcal{F}})$ and $ch_i$ the terms of $i$-th degree in $ch$. Then from proposition 3.1, we have

$$ch_i(N_{\mathcal{F}}) = \frac{1}{i!}c_1(\mathcal{G}')^i$$

for $i \leq n - 1$ and

$$ch_n(N_{\mathcal{F}}) = \frac{1}{n!}c_1(\mathcal{G}')^n - c_n(\Omega_X \otimes \mathcal{G}').$$

$ch_1(N_{\mathcal{F}}) = c_1(\mathcal{G}')$ is obvious. We also see that

$$\frac{1}{2!}c_1(\mathcal{G}')^2 = ch_2(N_{\mathcal{F}})$$

$$= \frac{1}{2!}(\xi_1^2 + \cdots + \xi_n^2)$$

$$= \frac{1}{2!}((\xi_1 + \cdots + \xi_n)^2 - 2 \sum \xi_i\xi_j)$$

$$= \frac{1}{2!}c_1(\mathcal{G}')^2 - c_2(N_{\mathcal{F}}),$$

which implies $c_2(N_{\mathcal{F}}) = 0$. We continue the same computations for fundamental symmetric polynomials, we have

$$c_2(N_{\mathcal{F}}) = \cdots = c_{n-1}(N_{\mathcal{F}}) = 0.$$
Thus for $n$-th term, we have
\[
\frac{1}{n!}c_{1}(G^\vee)^{n} - c_{n}(\Omega_X \otimes G^\vee) = ch_{n}(N_{F}) \]
\[
= \frac{1}{n!}(\xi_{1}^{n} + \cdots + \xi_{n}^{n})
\]
\[
= \frac{1}{n!}\{ (\xi_{1} + \cdots + \xi_{n})^{n} - (-1)^{n}n\xi_{1}\cdots\xi_{n} \}
\]
\[
= \frac{1}{n!}c_{1}(G^\vee)^{n} - \frac{(-1)^{n}}{(n-1)!}c_{n}(N_{F}),
\]
from which the result follows.

We combin the results in (2.3), we can derive the formula for the normal sheaf $N_{F}$, which is the Baum-Bott type residue formula.

**Theorem 4.4 (Baum-Bott type residue formula).** Let $\omega$ be a codimension 1 distribution with conormal sheaf $\mathcal{G}$, and $F$ the anihilator of $\mathcal{G}$. We suppose that $S(\mathcal{G}) = \{ p_{1}, \cdots, p_{k} \}$ and we write $\omega = \sum f_{i}^{(j)}(dx_{i} \otimes s^{\vee})$ near $p_{j}$. Then we have
\[
\int_{X}c_{n}(N_{F}) = (-1)^{n}(n-1)! \sum_{j} \text{Res} \left[ \frac{df_{1}^{(j)} \wedge \cdots \wedge df_{n}^{(j)}}{f_{1}^{(j)} \cdots f_{n}^{(j)}} \right].
\]

**proof.** This is simply given by
\[
\int_{X}c_{n}(N_{F}) = (-1)^{n}(n-1)! \int_{X}c_{n}(\Omega_X \otimes G^\vee)
\]
\[
= (-1)^{n}(n-1)! \sum_{j} \text{Res} \left[ \frac{df_{1}^{(j)} \wedge \cdots \wedge df_{n}^{(j)}}{f_{1}^{(j)} \cdots f_{n}^{(j)}} \right]
\]

**Remarks.** If we assume the integrability condition on $\mathcal{G}$, the above formula implies the Baum-Bott residue formula for singular holomorphic foliations. Since the Baum-Bott residue for $c_{n}(N_{F})$ is given by
\[
(-1)^{n}(n-1)! \dim Ext_{\mathcal{O}_{\mathrm{p}}}^{1}(\Omega_{\mathcal{G},p}, O_{\mathrm{p}}) = (-1)^{n}(n-1)! \dim O_{\mathrm{p}}/\mathcal{I}_{\omega,p},
\]
the right hand side of 3.4 coincides the Baum-Bott residue.

5. **Applications**

5.1. **Residue for the non-transversal loci of a holomorphic map.** Let $F : X^{n} \rightarrow Y^{m}$ be a holomorphic map between $n$ and $m$ dimensional compact complex manifolds. If $Y$ has a non-singular distribution $\tilde{\mathcal{G}} = O_{Y}(G)$, then the inverse image $\mathcal{G} = F^{-1}\tilde{\mathcal{G}}$ gives a distribution of $X$ which is possibly singular. In codimension 1 case, if a distribution $\tilde{\mathcal{G}}$ on $Y$ is given by a collection of 1-forms $\tilde{\omega} = (\tilde{\omega}_{\alpha})$, then the inverse image $\mathcal{G} = F^{-1}\tilde{\mathcal{G}}$ of the invertible sheaf $\tilde{\mathcal{G}}$ is given by the collection of 1-forms $\omega = (F^{*}\tilde{\omega}_{\alpha})$. If the image of the differential $DF_{p}$ does not contain the normal space $G_{p}^{*}$, we see that covector $\omega_{p}$ is zero. Thus the non-transversal loci of $F$ to $\tilde{\mathcal{G}}$ is given by
\[
S(\mathcal{G}) = \{ p \in X : F^{*}\tilde{\omega}_{\alpha}(p) = 0 \}
\]

Now we give the residue formula for the non-transversality of $F$ to $\tilde{\mathcal{G}}$. We assume that $S(\mathcal{G})$ consists of isolated points $\{ p_{1}, \cdots, p_{k} \}$. We set that, near $p_{j}$, $f_{i}^{(j)}$ are
the coefficients of $F^{*}\tilde{\omega}_{\alpha}^{(j)}$ such that we write $F^{*}\tilde{\omega}_{\alpha}^{(j)} = f_{1}^{(j)}dx_{1} + \cdots + f_{n}^{(j)}dx_{n}$. Then we have

$$
\int_{X} c_{n}(\Omega_{X} \otimes \mathcal{G}^{\vee}) = \sum_{l=0}^{n} \int_{F^{*}(c_{l}(\Theta_{X}))} c_{1}(\tilde{\mathcal{G}})^{l}
= \sum_{j=1}^{k} \text{Res}_{p_{j}} \left[ df_{1}^{(j)} \wedge \cdots \wedge df_{n}^{(j)} \right].
$$

Now we have the result.

**Theorem 5.1** (Residue formula for non-transversality). Let $F : X^{n} \rightarrow Y^{m}$ be a holomorphic map of generic rank $r$ and $\tilde{\mathcal{G}}$ a codimension 1 non-singular distribution of $Y$. We assume that the non-transversal points of $F$ to $\tilde{\mathcal{G}}$ are $\{p_{1}, \cdots, p_{k}\}$, then we have

$$
\chi(X) + \sum_{l=1}^{r} \int_{F^{*}(c_{l}(\Theta_{X}))} c_{1}(\tilde{\mathcal{G}})^{l} = \sum_{j=1}^{k} \text{Res}_{p_{j}} \left[ df_{1}^{(j)} \wedge \cdots \wedge df_{n}^{(j)} \right].
$$

**Proof.** We denote by $X^{*}$ the set of generic points where $F$ has rank $k$. By using projection formula,

$$
\int_{X} c_{n-l}(\Theta_{X})c_{1}(\tilde{\mathcal{G}})^{l} = \int_{X^{*}} c_{n-l}(\Theta_{X})F^{*}(c_{1}(\tilde{\mathcal{G}})^{l})
= \int_{F^{*}(c_{n-l}(\Theta_{X})-\{X\})} c_{1}(\tilde{\mathcal{G}})^{l}.
$$

It is obvious that the above terms are zero for $k \leq l$.

Here let $F : X^{2} \rightarrow Y^{m}$ be a map from compact complex surface. In this case we write down the above general form of the formula into geometric forms. We set that $y_{m} = F_{m}^{(j)}(x_{1}, x_{2})$ is the $m$-th entry of a local representation of $F$ near $p_{j}$ and also write $dF_{m}^{(j)} = f_{1}^{(j)}dx_{1} + f_{2}^{(j)}dx_{2}$. Then the above formula is

$$
\chi(X) + \int_{F^{*}(c_{1}(\Theta_{X})-\{X\})} c_{1}(\tilde{\mathcal{G}}) + \int_{F^{*}[X]} c_{1}(\tilde{\mathcal{G}})^{2} = \sum_{j=1}^{k} \text{Res}_{p_{j}} \left[ df_{1}^{(j)} \wedge f_{2}^{(j)} \right].
$$

We remark that if the generic rank of $F$ is 1, the last term in the left-hand side of the above vanishes and we have

$$
\chi(X) + \chi(M_{F}) \int_{F^{*}[X]} c_{1}(\tilde{\mathcal{G}}) = \sum_{j=1}^{k} \text{Res}_{p_{j}} \left[ df_{1}^{(j)} \wedge f_{2}^{(j)} \right].
$$

In the above, $M_{F}$ is the generic fiber of $F$.

As an other example let us consider the case that $F : X^{n} \rightarrow C$ is a map for a curve $C$ and $\tilde{\mathcal{G}} = \Omega_{C}$ is the point distribution. Then the above formula implies the multiplicity formula. (See [IS], [F].)

**Theorem 5.2** (The multiplicity formula). Let $F : X^{n} \rightarrow C$ be a holomorphic function for a compact complex curve $C$ with the generic fiber $M_{F}$. If $F$ has finite
number of isolated critical points \(\{p_1, \cdots, p_k\}\), then we have

\[
\chi(X) - \chi(M_F)\chi(C) = (-1)^n \sum_{j=1}^{k} \mu(F, p_j)
\]

where \(\mu(F, p_j)\) is the Milnor number of \(F\) at \(p_j\).

Remarks. The one dimensional cases of theorem 4.1 is the classical Riemann-Hurwitz formula for a morphism of Riemann surfaces \(F : C \to \overline{C}\). We note that it cannot be deduced from the Baum-Bott type formula for \(c_1(N_F)\) in the above settings, however we can still apply the residue formula for \(G\) in theorem 2.4. By taking the anihilator of the inverse image \(G\) of \(\Omega_{\overline{C}}\), the given tangent sheaf \(F\) of the lifted foliation turn out to be reduced. Since 1 dimensional manifolds only admits point foliations, the zero schemes of singularities are the points with multiplicities. Thus those kinds of singularities become non-singular by taking reduction. Therefore in our pull-back situation, the normal sheaf \(N_F\) is always locally free and only \(G\) itself keeps the informations of singularities of \(F\).

References


