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# RESIDUE OF CODIMENSION 1 SINGULAR HOLOMORPHIC DISTRIBUTIONS

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## 1

The aim of this note is to describe the residue formula for singular holomorphic distribution in terms of the conormal sheaf $\mathcal{G}$ in codimension 1 case.

We also prove the Baum-Bott type residue formula for singular distributions. If we define the tangent sheaf of the distribution $\mathcal{F}$ by taking the annihilator of $\mathcal{G}$ by the dual coupling, we will show that the residue formula for $\mathcal{G}$ deduce the Baum-Bott type residue formula for the top Chern class of the normal sheaf $\mathcal{N}_{\mathcal{F}}$. If we assume the Frobenius integrability condition for $\mathcal{G}$, we have the Baum-Bott residue formula

$$\int_X \varphi(\mathcal{N}_{\mathcal{F}}) = \mathrm{Res}_{\varphi}(\mathcal{N}_{\mathcal{F}},\, S(\mathcal{F}))$$

for $n$-th symmetric polinomial $\varphi$. In this case, the Baum-Bott residue formula for $\varphi = c_n$ is equivalent to the formula we will prove, which means that the Bott vanishing theorem based on the involutivity of $\mathcal{F}$ is not necessary for the top Chern class $c_n(\mathcal{N}_{\mathcal{F}})$.

As an application of our results, we will give a residue formula for the non-transversality of a holomorphic map $F : X \longrightarrow Y$ to a non-singular distribution on $Y$.

## 2. Singular holomorphic distribution

**2.1. Singular holomorphic distribution.** Let X be a complex manifold. We define a singular holomorphic distribution $\mathcal{F}$ on $X$ to be a coherent subsheaf of the tangent sheaf $\Theta_X$. we call $\mathcal{F}$ the tangent sheaf of the distribution. We say $\mathcal{F}$ is dimension $p$ if a generic stalk of $\mathcal{F}$ is rank $p$ free $\mathcal{O}_X$-module. We also define the normal sheaf $\mathcal{N}_{\mathcal{F}}$ of $\mathcal{F}$ by the exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \Theta_X \longrightarrow \mathcal{N}_{\mathcal{F}} \longrightarrow 0.$$

The singular set $S(\mathcal{F})$ of $\mathcal{F}$ is defined by $S(\mathcal{F}) = \{\, p \in X \,|\, \mathcal{N}_{\mathcal{F},p} \text{ is not } \mathcal{O}_p\text{-free}\}$.

We can also give a definition of a singular holomorphic distribution $\mathcal{G}$ on $X$ to be a coherent subsheaf of the cotangent sheaf $\Omega_X$. We call $\mathcal{G}$ the conormal sheaf of the distribution. We also say $\mathcal{G}$ is codimension $q$ if the generic rank is $q$. We also define the cotangent sheaf $\Omega_{\mathcal{G}}$ of $\mathcal{G}$ by the exact sequence

$$0 \longrightarrow \mathcal{G} \longrightarrow \Omega_X \longrightarrow \Omega_{\mathcal{G}} \longrightarrow 0.$$

The singular set $S(\mathcal{F})$ of $\mathcal{F}$ is also defined by $S(\mathcal{G}) = \{\, p \in X \,|\, \Omega_{\mathcal{G},p} \text{ is not } \mathcal{O}_p\text{-free}\}$.

2.2. **Codimension 1 case.** we give more simple descriptions for codimension 1 singular distributions. A codimension 1 locally free singular holomorphic distribution is given by a collection of 1-forms $\omega = (\omega_\alpha, U_\alpha)$ for an open covering $\{U_\alpha\}$ of $X$ which has the transition relations $\omega_\beta = g_{\alpha\beta}\omega_\alpha$ on the intersection $U_\alpha \cap U_\beta$ with $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$. Then the cocycle $(g_{\alpha\beta})$ defines a line bundle $G$. Generically at $p$, the covector $\omega_p$ gives an embedding of the fiber $G_p$ into $T_p^*X$ by $f_p \in G_p \longmapsto f_p\omega_p \in T_p^*X$. Thus $G$ is regarded as a subbundle of $T^*X$ without on the zero loci of $\omega$. Since the map of germs of sections $(f)_p \in \mathcal{O}_X(G)_p \longmapsto (f\omega)_p \in \Omega_{X,p}$ are injective for all $p \in X$, the sheaf $\mathcal{G} = \mathcal{O}_X(G)$ gives the subsheaf of $\Omega_X$ in the above sense in 1.2. Since the quotient sheaf $\Omega_{\mathcal{F}}$ is not $\mathcal{O}$-free on the zero loci of $\omega$ on which we can not define the quotient bundle $T^*X/G$, we see the singular set of $\mathcal{G}$ is $S(\mathcal{G}) = \{\, p \,|\, \omega(p) = 0\}$.

## 3. Residue of codimension 1 distribution

3.1. **Localization of the top Chern class.** We determine the dual homology class of $c_n(\Omega_X \otimes \mathcal{G}^\vee)$. Our main tool is the Čech-de Rham techniques. For generalities on the integration and the Chern-Weil theory on the Čech-de Rham cohomology, see [S3] or [IS]. We set for an analytic set $S$, $U_0 = X \setminus S$, $U_1$ is a regular neighbourhood of $S$, and $U_{01} = U_0 \cap U_1$. For a covering $\mathcal{U} = \{U_0, U_1\}$ of $X$, the Čech-de Rham cohomology group $H^{2n}(\mathcal{A}^\bullet(\mathcal{U}))$ is represented by the group of cocycles of the type $(\sigma_0, \sigma_1, \sigma_{01})$ for $\sigma_0 \in Z^{2n}(U_0)$, $\sigma_1 \in Z^{2n}(U_1)$, and $\sigma_{01} \in A^{2n-1}(U_{01})$ with $d\sigma_{01} = \sigma_1 - \sigma_0$. We note that the Čech-de Rham cohomology can be regarded as the hypercohomology of the de Rham complex $\{\mathcal{A}^\bullet, d\}$. By usual spectral sequence arguments for double complexes, we see that the Čech-de Rham cohomology group is canonically isomorphic to the de Rham cohomology group. If we take the subgroup $H^{2n}(\mathcal{A}^\bullet(\mathcal{U}, U_0))$ of cocycles of the form $(0, \sigma_1, \sigma_{01})$, then this is also isomorphic to the relative cohomology group $H^{2n}(X, X \setminus S; \mathbf{C})$.

In the above settings, the top Chern class $c_n(E)$ of a vector bundle $E$ of rank $n$ is given by the cocycle in $H^{2n}(\mathcal{A}^\bullet(\mathcal{U}))$ as follows. For $i = 0, 1$, let $\nabla_i$ be a connection for $E$ on $U_i$ and $c_n(\nabla_i)$ the $n$-th Chern form of $\nabla_i$. We also write by $c_n(\nabla_0, \nabla_1)$ the transgression form of $c_n(\nabla_i)$'s on $U_{01}$. Then $c_n(E)$ is represented by

$$(c_n(\nabla_1),\, c_n(\nabla_1),\, c_n(\nabla_0, \nabla_1)).$$

If $E$ has a global section $s$ with zero loci $S$, then we take $\nabla_0$ as the $s$-trivial connection such that we have $c_n(\nabla_0) = 0$. Thus we can define the localized Chern class at $p$ in $H^{2n}(X, X \setminus S; \mathbf{C})$ by a Čech-de Rham cocycle $(0,\, c_n(\nabla_1),\, c_n(\nabla_0, \nabla_1))$.

The integration of $c_n(E) = (0,\, c_n(\nabla_1),\, c_n(\nabla_0, \nabla_1))$ is defined by

$$\int_X c_n(E) = \int_R c_n(\nabla_1) - \int_{\partial R} c_n(\nabla_0, \nabla_1)$$

for a tubular neighbourhood $R \subset U_1$ of $S$.

3.2. **Residue of codimension 1 distributions.** Now we apply the above arguments to our situations. Let $\mathcal{G}$ be a codimension 1 locally free distribution with the zero loci $S(\mathcal{G})$ and we suppose that $S(\mathcal{G})$ has connected components $S_j$. We set $U_0 = X \setminus S(\mathcal{G})$ and $U_j$ is a regular neighbourhood of $S_j$. We consider the localized class of $c_n(\Omega_X \otimes \mathcal{G}^\vee)$ in the Čech-de Rham cohomology group for the covering $\mathcal{U} = \{U_0, U_1, \cdots, U_j\}$. Since the collection $\omega$ of 1-forms $\omega_\alpha$ defines the global section of $\Omega_X \otimes \mathcal{G}^\vee$, we can take $\nabla_0$ as the $\omega$-trivial connection such that $c_n(\nabla_0) = 0$

as we discussed above. For all $j = 1, \cdots, k$, we can also take $\nabla_j$ as an arbitrary connection on $U_j$. So we have

$c_n(\Omega_X \otimes \mathcal{G}^\vee) = (0, \{c_n(\nabla_j)\}_{j=1,\dots,k}, \{c_n(\nabla_0, \nabla_j)\}_{j=1,\dots,k}) \in H^{2n}(X, X \setminus S(\mathcal{G}); \mathbf{C})$.

We denote by $R_j$ a tublar neighbourhood of $S_j$ in $U_j$. We give the following definition of residue.

**Definition 3.1.** *The residue of $\mathcal{G}$ at $S_j$ is defined by*

$$\mathrm{Res}(\mathcal{G}, S_j) = \int_{R_j} c_n(\nabla_j) - \int_{\partial R_j} c_n(\nabla_0, \nabla_j).$$

We can describe the residue into precise form in isolated singular cases. Here we refer the result in [S3] of Theorem 5.5.

**Theorem 3.2.** *Let $s$ be a regular section of $E$ with isolated zero $\{p\}$ and $s$ is locally given by $(f_1, \dots, f_n)$ near $p$. Then we have*

$$\mathrm{Res}(\mathcal{G}, p) = \mathrm{Res}_p \begin{bmatrix} df_1 \wedge \cdots \wedge df_n \\ f_1 \cdots f_n \end{bmatrix}$$

*where $\mathrm{Res}_p \left[\begin{smallmatrix} df_1 \wedge \cdots \wedge df_n \\ f_1 \cdots f_n \end{smallmatrix}\right]$ is the Grothendick residue of $(f_1, \dots, f_n)$.*

The dual correspondence in the Alexander duality

$$AL : H^{2n}(X\, X \setminus S(\mathcal{G}); \mathbf{C}) \xrightarrow{\sim} \bigoplus_j H_0(S_j; \mathbf{C})$$

is given by

$$AL(c_n(\Omega_X \otimes \mathcal{G}^\vee)) = \sum_j \mathrm{Res}(\mathcal{G}, S_j).$$

Now we have the residue formula for isolated singular cases as,

**Theorem 3.3** (The residue formula for isolated singularities). *Let $\omega$ be a codimension 1 singular holomorphic distribution with the cotangent sheaf $\mathcal{G}$ and $(f_1^{(j)}, \cdots, f_n^{(j)})$ a local expression of $\omega \in H^0(X, \Omega_X \otimes \mathcal{G}^\vee)$ near $p_j$.*

$$\int_X c_n(\Omega_X \otimes \mathcal{G}^\vee) = \sum_{j=1}^{k} \mathrm{Res}_{p_j} \begin{bmatrix} df_1^{(j)} \wedge \cdots \wedge df_n^{(j)} \\ f_1^{(j)} \cdots f_n^{(j)} \end{bmatrix}.$$

## 4. Baum-Bott type residue formula

4.1. **Koszul resolution.** First let us remember the definition of the Koszul complex. (See [FG], Chapter 4 or [GH], Chapter 5.) Let $\mathcal{E}$ be a locally free $\mathcal{O}$-module of rank $n$ and $d : \mathcal{E} \longrightarrow \mathcal{O}$ an $\mathcal{O}$-homomorphism. Then the Koszul complex of sheaves

$$0 \to \wedge^n \mathcal{E} \to \wedge^{n-1} \mathcal{E} \to \cdots \to \wedge^1 \mathcal{E} \to \mathcal{O} \to 0$$

is defined by the boundary operator

$$d_p(\varepsilon_1 \wedge \cdots \wedge \varepsilon_p) = \sum_{i=1}^{p} (-1)^{i-1} d(\varepsilon_i)\, \varepsilon_1 \wedge \cdots \wedge \hat{\varepsilon_i} \wedge \cdots \wedge \varepsilon_p.$$

This complex is exact expect for the last term. If the image $\mathcal{I}_d$ of $d$ is regular ideal, the complex

$$0 \to \wedge^n \mathcal{E} \to \wedge^{n-1} \mathcal{E} \to \cdots \to \wedge^1 \mathcal{E} \to \mathcal{O} \to \mathcal{O}/\mathcal{I}_d \to 0$$

is exact. We call this exact sequence the Koszul resolution of $\mathcal{O}/\mathcal{I}_d$.

Now let us consider our case. As observed in 2.1, $\omega$ can be regarded as a homomorphism $\omega : \mathcal{G} \longrightarrow \Omega_X$ such that it defines a global section

$$\omega \in H^0(X, \mathcal{H}om_{\mathcal{O}}(\mathcal{G}, \ \Omega_X)) \simeq H^0(X, \ \Omega_X \otimes \mathcal{G}^\vee).$$

Locally on $U_\alpha$, $\omega$ is given by $\omega_\alpha \otimes s_\alpha^\vee = \sum f_i(dx_i \otimes s_\alpha^\vee)$ for some local coordinates of $X$ and a local frame $s_\alpha^\vee$ for $\mathcal{G}^\vee$. In the other words, $\omega$ acts on $(\Omega_X \otimes \mathcal{G}^\vee)^\vee \simeq \Theta_X \otimes \mathcal{G}$ as a contraction operator $\omega : \Theta_X \otimes \mathcal{G} \longrightarrow \mathcal{O}$. We denote by $\mathcal{I}_w$ the ideal sheaf defined by $\mathrm{Im}(\omega : \Theta_X \otimes \mathcal{G} \longrightarrow \mathcal{O})$. We assume that $S(\mathcal{G}) = \{ p \in X \,|\, \omega_p = 0 \}$ consists only of isolated points such that the local coefficients $(f_1, \cdots, f_n)$ of $\omega$ is regular sequence on $S(\mathcal{G})$. Then the complex of sheaves

$$0 \to \wedge^n(\Theta_X \otimes \mathcal{G}) \to \wedge^{n-1}(\Theta_X \otimes \mathcal{G}) \to \cdots \to \wedge^1(\Theta_X \otimes \mathcal{G}) \to \mathcal{O} \to \mathcal{O}/\mathcal{I}_\omega \to 0$$

is exact with the boundary operator

$$d_p(e_1 \wedge \cdots \wedge e_p) = \sum_{i=1}^{p} (-1)^{i-1} f_i \, e_1 \wedge \cdots \wedge \hat{e_i} \wedge \cdots \wedge e_p$$

where we set $e_i = \frac{\partial}{\partial x_i} \otimes s$. Therefore this gives the Koszul resolution of $\mathcal{O}/\mathcal{I}_\omega$.

By using this projective resolution, we can defines the Chern character of the coherent sheaf $\mathcal{O}/\mathcal{I}_\omega$ by

**Proposition 4.1.**

$$ch(\mathcal{O}/\mathcal{I}_\omega) = c_n(\Omega_X \otimes \mathcal{G}^\vee).$$

**Proof.** We use [H] of Theorem 10.1.1 and we have

$$\begin{aligned} ch(\mathcal{O}/\mathcal{I}_\omega) &= ch(\sum_{i=0}^{n} (-1)^i \wedge^i (\Theta_X \otimes \mathcal{G})) \\ &= td^{-1}(\Omega_X \otimes \mathcal{G}^\vee) c_n(\Omega_X \otimes \mathcal{G}^\vee) \\ &= c_n(\Omega_X \otimes \mathcal{G}^\vee). \end{aligned}$$

4.2. **Baum-Bott type residue formula.** Now we translate the above results in terms of differential system in the tangent sheaf $\Theta_X$: Let $\mathcal{F} = \{ v \in \Theta_X \mid < v, \omega >= 0\}$ be the annihilator of $\mathcal{G}$. Then $\mathcal{F}$ defines a $n-1$ dimensional (possibly) singular distribution. Since $\mathcal{G}$ is locally free, by applying $\otimes \mathcal{G}$ to the exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \Theta_X \longrightarrow \mathcal{N}_{\mathcal{F}} \longrightarrow 0, \tag{1}$$

the following sequence

$$0 \longrightarrow \mathcal{F} \otimes \mathcal{G} \longrightarrow \Theta_X \otimes \mathcal{G} \longrightarrow \mathcal{N}_{\mathcal{F}} \otimes \mathcal{G} \longrightarrow 0.$$

is also exact. Since the kernel of $\omega : \Theta_X \otimes \mathcal{G} \longrightarrow \mathcal{O}_X$ is equals to $\mathcal{F} \otimes \mathcal{G}$, we have

$$\mathcal{I}_\omega \simeq (\Theta_X \otimes \mathcal{G})/(\mathcal{F} \otimes \mathcal{G}) \simeq \mathcal{N}_{\mathcal{F}} \otimes \mathcal{G}. \tag{2}$$

We take $\mathcal{H}om_{\mathcal{O}}( \ \ , \mathcal{O})$ of the dual exact sequence

$$0 \longrightarrow \mathcal{G} \longrightarrow \Omega_X \longrightarrow \Omega_{\mathcal{G}} \longrightarrow 0$$

of (1), we obtain the exact sequence

$$0 \longrightarrow \mathcal{H}om_{\mathcal{O}}(\Omega_{\mathcal{G}}, \mathcal{O}) \longrightarrow \mathcal{H}om_{\mathcal{O}}(\Omega_X, \mathcal{O}) \longrightarrow \mathcal{H}om_{\mathcal{O}}(\mathcal{G}, \mathcal{O}) \longrightarrow \mathcal{E}xt^1_{\mathcal{O}}(\Omega_{\mathcal{G}}, \mathcal{O}) \longrightarrow 0,$$

which implies

$$0 \longrightarrow \mathcal{F} \longrightarrow \Theta_X \longrightarrow \mathcal{G}^\vee \longrightarrow \mathcal{E}xt^1_{\mathcal{O}}(\Omega_{\mathcal{G}}, \mathcal{O}) \longrightarrow 0.$$

We use $\mathcal{F} = \mathcal{H}om_{\mathcal{O}}(\Omega_{\mathcal{G}}, \mathcal{O})$ and $\Theta_X = \mathcal{H}om_{\mathcal{O}}(\Omega_X, \mathcal{O})$ in the above. Thus we obtain

$$0 \longrightarrow \mathcal{N}_{\mathcal{F}} \longrightarrow \mathcal{G}^{\vee} \longrightarrow \mathcal{E}xt^1_{\mathcal{O}}(\Omega_{\mathcal{G}}, \mathcal{O}) \longrightarrow 0. \tag{3}$$

By taking the Chern characters of (3), we have

$$ch(\mathcal{N}_F) = ch(\mathcal{G}^{\vee}) - ch(\mathcal{E}xt^1_{\mathcal{O}}(\Omega_{\mathcal{G}}, \mathcal{O})). \tag{4}$$

By tensoring $\mathcal{G}$ for each term of (3), we also have the exact sequence

$$0 \longrightarrow \mathcal{I}_{\omega} \longrightarrow \mathcal{O} \longrightarrow \mathcal{E}xt^1_{\mathcal{O}}(\Omega_{\mathcal{G}}, \mathcal{O}) \otimes \mathcal{G} \longrightarrow 0$$

and which gives the isomorphism $\mathcal{O}/\mathcal{I}_{\omega} \simeq \mathcal{E}xt^1_{\mathcal{O}}(\Omega_{\mathcal{G}}, \mathcal{O}) \otimes \mathcal{G}$. Thus the Chern characters of those sheaves satisfy

$$ch(\mathcal{E}xt^1_{\mathcal{O}}(\Omega_{\mathcal{G}}, \mathcal{O})) = ch(\mathcal{O}/\mathcal{I}_{\omega})ch(\mathcal{G}^{\vee}). \tag{5}$$

Therefore by combining the two equalities (4) and (5) for the Chern characters, we obtain

**Proposition 4.2.**

$$\begin{aligned} ch(\mathcal{N}_{\mathcal{F}}) &= (1 - ch(\mathcal{O}/\mathcal{I}_{\omega}))ch(\mathcal{G}^{\vee}) \\ &= (1 - c_n(\Omega_X \otimes \mathcal{G}^{\vee}))ch(\mathcal{G}^{\vee}). \end{aligned}$$

Now we find the top Chern class of $\mathcal{N}_{\mathcal{F}}$.

**Proposition 4.3.**

$$c_n(\mathcal{N}_{\mathcal{F}}) = (-1)^n (n-1)!\, c_n(\Omega_X \otimes \mathcal{G}^{\vee}).$$

**Proof.** Let $\{\xi_i\}$ be the formal Chern roots of $c(\mathcal{N}_{\mathcal{F}})$ and $ch_i$ the terms of $i$-th degree in $ch$. Then from proposition 3.1, we have

$$ch_i(\mathcal{N}_{\mathcal{F}}) = \frac{1}{i!} c_1(\mathcal{G}^{\vee})^i$$

for $i \leq n-1$ and

$$ch_n(\mathcal{N}_{\mathcal{F}}) = \frac{1}{n!} c_1(\mathcal{G}^{\vee})^n - c_n(\Omega_X \otimes \mathcal{G}^{\vee}).$$

$ch_1(\mathcal{N}_{\mathcal{F}}) = c_1(\mathcal{G}^{\vee})$ is obvious. we also see that

$$\begin{aligned} \frac{1}{2!} c_1(\mathcal{G}^{\vee})^2 &= ch_2(\mathcal{N}_{\mathcal{F}}) \\ &= \frac{1}{2!}(\xi_1^2 + \cdots + \xi_n^2) \\ &= \frac{1}{2!}\{(\xi_1 + \cdots + \xi_n)^2 - 2\sum \xi_i \xi_j\} \\ &= \frac{1}{2!} c_1(\mathcal{G}^{\vee})^2 - c_2(\mathcal{N}_{\mathcal{F}}), \end{aligned}$$

which implies $c_2(\mathcal{N}_{\mathcal{F}}) = 0$. We continue the same computations for fundamental symmetric polynomials, we have

$$c_2(\mathcal{N}_{\mathcal{F}}) = \cdots = c_{n-1}(\mathcal{N}_{\mathcal{F}}) = 0.$$

Thus for $n$-th term, we have

$$\begin{aligned}\frac{1}{n!}c_1(\mathcal{G}^\vee)^n - c_n(\Omega_X \otimes \mathcal{G}^\vee) &= ch_n(\mathcal{N}_\mathcal{F}) \\ &= \frac{1}{n!}(\xi_1^n + \cdots + \xi_n^n) \\ &= \frac{1}{n!}\{ (\xi_1 + \cdots + \xi_n)^n - (-1)^n n\, \xi_1 \cdots \xi_n \} \\ &= \frac{1}{n!}c_1(\mathcal{G}^\vee)^n - \frac{(-1)^n}{(n-1)!}c_n(\mathcal{N}_\mathcal{F}),\end{aligned}$$

from which the result follows.

We combin the results in (2.3), we can derive the formula for the normal sheaf $\mathcal{N}_\mathcal{F}$, which is the Baum-Bott type residue formula.

**Theorem 4.4** (Baum-Bott type residue formula). *Let $\omega$ be a codimension 1 distribution with conormal sheaf $\mathcal{G}$, and $\mathcal{F}$ the anihilator of $\mathcal{G}$. We suppose that $S(\mathcal{G}) = \{p_1, \cdots, p_k\}$ and we write $\omega = \sum f_i^{(j)}(dx_i \otimes s^\vee)$ near $p_j$. Then we have*

$$\int_X c_n(\mathcal{N}_\mathcal{F}) = (-1)^n(n-1)! \sum_j \mathrm{Res}\begin{bmatrix} df_1^{(j)} \wedge \cdots \wedge df_n^{(j)} \\ f_1^{(j)} \cdots f_n^{(j)} \end{bmatrix}.$$

**proof.** This is simply given by

$$\begin{aligned}\int_X c_n(\mathcal{N}_\mathcal{F}) &= (-1)^n(n-1)! \int_X c_n(\Omega_X \otimes \mathcal{G}^\vee) \\ &= (-1)^n(n-1)! \sum_j \mathrm{Res}\begin{bmatrix} df_1^{(j)} \wedge \cdots \wedge df_n^{(j)} \\ f_1^{(j)} \cdots f_n^{(j)} \end{bmatrix}\end{aligned}$$

**Remarks**. If we assume the integrability condition on $\mathcal{G}$, the above formula implies the Baum-Bott residue formula for singular holomorphic foliations. Since the Baum-Bott residue for $c_n(\mathcal{N}_\mathcal{F})$ is given by

$$(-1)^n(n-1)! \dim Ext^1_{\mathcal{O}_p}(\Omega_{\mathcal{G},p}, \mathcal{O}_p) = (-1)^n(n-1)! \dim \mathcal{O}_p/\mathcal{I}_{\omega,p},$$

the right hand side of 3.4 coincides the Baum-Bott residue.

## 5. Applications

5.1. **Residue for the non-transversal loci of a holomorphic map.** Let $F : X^n \longrightarrow Y^m$ be a holomorphic map between $n$ and $m$ dimensional compact complex manifolds. If $Y$ has a non-singular distribution $\tilde{\mathcal{G}} = \mathcal{O}_Y(G)$, then the inverse image $\mathcal{G} = F^{-1}\tilde{\mathcal{G}}$ gives a distribution of $X$ which is possibly singular. In codimension 1 case, if a distribution $\tilde{\mathcal{G}}$ on $Y$ is given by a collection of 1-forms $\tilde{\omega} = (\tilde{\omega}_\alpha)$, then the inverse image $\mathcal{G} = F^{-1}\tilde{\mathcal{G}}$ of the invertible sheaf $\tilde{\mathcal{G}}$ is given by the collection of 1-forms $\omega = (F^*\tilde{\omega}_\alpha)$. If the image of the differential $DF_p$ dose not contain the normal space $G_p^*$, we see that covector $\omega_p$ is zero. Thus the non-transversal loci of $F$ to $\tilde{\mathcal{G}}$ is given by

$$S(\mathcal{G}) = \{p \in X \ : \ F^*\tilde{\omega}_\alpha(p) = 0\}$$

Now we give the residue formula for the non-transversality of $F$ to $\tilde{\mathcal{G}}$. We assume that $S(\mathcal{G})$ consists of isolated points $\{p_1, \cdots, p_k\}$. We set that, near $p_j$, $f_i^{(j)}$ are the coefficients of $F^*\tilde{\omega}_\alpha^{(j)}$ such that we write $F^*\omega_\alpha^{(j)} = f_1^{(j)}dx_1 + \cdots + f_n^{(j)}dx_n$. Then we have

$$\begin{aligned}\int_X c_n(\Omega_X \otimes \mathcal{G}^\vee) &= \sum_{l=0}^{n} \int_X c_{n-l}(\Theta_X)c_1(\mathcal{G})^l \\ &= \sum_{j=1}^{k} \mathrm{Res}_{p_j}\begin{bmatrix} df_1^{(j)} \wedge \cdots \wedge df_n^{(j)} \\ f_1^{(j)} \cdots f_n^{(j)} \end{bmatrix}.\end{aligned}$$

Now we have the result.

**Theorem 5.1** (Residue formula for non-transversality). *Let $F : X^n \longrightarrow Y^m$ be a holomorphic map of generic rank $r$ and $\tilde{\mathcal{G}}$ a codimension 1 non-singular distribution of $Y$. We assume that the non-transversal points of $F$ to $\tilde{\mathcal{G}}$ are $\{p_1, \cdots, p_k\}$, then we have*

$$\chi(X) + \sum_{l=1}^{r} \int_{F_*(c_{n-l}(X)\frown[X])} c_1(\tilde{\mathcal{G}})^l = \sum_{j=l}^{k} \mathrm{Res}_{p_j}\begin{bmatrix} df_1^{(j)} \wedge \cdots \wedge df_n^{(j)} \\ f_1^{(j)} \cdots f_n^{(j)} \end{bmatrix}.$$

**Proof.** We denote by $X^*$ the set of generic points where $F$ has rank $k$. By using projection formula,

$$\begin{aligned}\int_X c_{n-l}(\Theta_X)c_1(\mathcal{G})^l &= \int_{X^*} c_{n-l}(\Theta_X)F^*(c_1(\tilde{\mathcal{G}})^l) \\ &= \int_{F_*(c_{n-l}(X)\frown[X])} c_1(\tilde{\mathcal{G}})^l.\end{aligned}$$

It is obvious that the above terms are zero for $k \leq l$.

Here let $F : X^2 \longrightarrow Y^m$ be a map from compact complex surface. In this case we write down the above general form of the formula into geometric forms. We set that $y_m = F_m^{(j)}(x_1, x_2)$ is the $m$-th entry of a local representation of $F$ near $p_j$ and also write $dF_m^{(j)} = f_1^{(j)}dx_1 + f_2^{(j)}dx_2$. Then the above formula is.

$$\chi(X) + \int_{F_*(c_1(X)\frown[X])} c_1(\tilde{\mathcal{G}}) + \int_{F_*[X]} c_1(\tilde{\mathcal{G}})^2 = \sum_{j=l}^{k} \mathrm{Res}_{p_j}\begin{bmatrix} df_1^{(j)} \wedge f_2^{(j)} \\ f_1^{(j)}, f_2^{(j)} \end{bmatrix}.$$

We remark that if the generic rank of $F$ is 1, the last term in the left-hand side of the above vanishes and we have

$$\chi(X) + \chi(M_F)\int_{F_*[X]} c_1(\tilde{\mathcal{G}}) = \sum_{j=l}^{k} \mathrm{Res}_{p_j}\begin{bmatrix} df_1^{(j)} \wedge f_2^{(j)} \\ f_1^{(j)}, f_2^{(j)} \end{bmatrix}.$$

In the above, $M_F$ is the generic fiber of $F$.

As an other example let us consider the case that $F : X^n \longrightarrow C$ is a map for a curve $C$ and $\tilde{\mathcal{G}} = \Omega_C$ is the point distribution. Then the above formula implies the multiplicity formula. (See [IS], [F].)

**Theorem 5.2** (The multiplicity formula). *Let $F : X^n \longrightarrow C$ be a holomorphic function for a compact complex curve $C$ with the generic fiber $M_F$. If $F$ has finite* *number of isolated critical points* $\{p_1, \cdots, p_k\}$, *then we have*

$$\chi(X) - \chi(M_F)\chi(C) = (-1)^n \sum_{j=1}^{k} \mu(F, p_j)$$

*where* $\mu(F, p_j)$ *is the Milnor number of* $F$ *at* $p_j$.

**Remarks.** The one dimensional cases of theorem 4.1 is the classical Riemann-Hurwitz formula for a morphism of Riemann surfaces $F : C \longrightarrow \tilde{C}$. We note that it cannot be deduced from the Baum-Bott type formula for $c_1(\mathcal{N}_{\mathcal{F}})$ in the above settings, however we can still apply the residue formula for $\mathcal{G}$ in theorem 2.4. By taking the anihilator of the inverse image $\mathcal{G}$ of $\Omega_{\tilde{C}}$, the given tangent sheaf $\mathcal{F}$ of the lifted foliation turn out to be reduced. Since 1 dimensional manifolds only admits point foliations, the zero schemes of singularities are the points with multiplicities. Thus those kinds of singularities become non-singular by taking reduction. Therefore in our pull-back situation, the normal sheaf $\mathcal{N}_{\mathcal{F}}$ is always locally free and only $\mathcal{G}$ itself keeps the informations of singularities of $F$.

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