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1502 巻 2006 年 1-15

CONFORMAL GEOMETRY AND 3-PLANE FIELDS ON 6-MANIFOLDS

ROBERT L. BRYANT

ABSTRACT. The purpose of this note is to provide yet another example of the link between certain conformal geometries and ordinary differential equations, along the lines of the examples discussed by Nurowski [4].

In this particular case, I consider the equivalence problem for 3-plane fields $D \subset TM$ on a 6-manifold M satisfying the nondegeneracy condition that $D + [D, D] = TM$.

I give a solution of the equivalence problem for such D (as Tanaka has previously), showing that it defines a $\mathfrak{so}(4,3)$-valued Cartan connection on a principal right H-bundle over M where $H \subset \mathrm{SO}(4,3)$ is the subgroup that stabilizes a null 3-plane in $\mathbb{R}^{4,3}$. Along the way, I observe that there is associated to each such D a canonical conformal structure of split type on M, one that depends on two derivatives of the plane field D.

I show how the primary curvature tensor of the Cartan connection associated to the equivalence problem for D can be interpreted as the Weyl curvature of the associated conformal structure and, moreover, show that the split conformal structures in dimension 6 that arise in this fashion are exactly the ones whose $\mathfrak{so}(4,4)$-valued Cartan connection admits a reduction to a $\mathfrak{spin}(4,3)$-connection. I also discuss how this case is analogous to features of Nurowski's examples.

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Date: November 07, 2005.

1991 *Mathematics Subject Classification.* 53A55, 53C10, 58A30.

Key words and phrases. plane fields, equivalence method.

Thanks to Duke University for its support via a research grant, to the National Science Foundation for its support via grants DMS-9870164 and DMS-0103884, and to the Clay Mathematics Institute for its support in the period January-May 2002, during which a portion of this manuscript was written.

This is Version 2.0. The most recent version can be found at `arXiv:math.DG/0511110`.

1. Introduction

In [4], Nurowski considers several different equivalence problems for classes of differential equations and shows how each one leads to a natural conformal structure (of indefinite type) of an appropriate configuration space and that this conformal structure suffices to encode the original equivalence problem.

Perhaps the most striking of these examples is the one based on É. Cartan's famous 'five-variables' paper [3], in which Cartan solves the equivalence problem for 2-plane fields of maximal growth vector $(2,3,5)$ on 5-manifolds. Such 2-plane fields are now said to be 'of Cartan type' in honor of Cartan's pioneering work.

In that paper, Cartan shows that, given such a 2-plane field $D \subset TM$ where M has dimension 5, one can define what is now called a Cartan connection over M that solves the equivalence problem. Specifically, let $\mathrm{G}_2' \subset \mathrm{SO}(4,3)$ be the noncompact exceptional simple group of dimension 14. The group G_2' acts transitively on the set $Q_{3,2} \simeq S^3 \times S^2$ of null lines in $\mathbb{R}^{4,3}$. Let $H \subset \mathrm{G}_2'$ be the subgroup of codimension 5 that fixes a null line in $Q_{3,2}$. Then Cartan shows how canonically to associate to D a principal right H-bundle $\pi : P \to M$ and a $\mathfrak{g}_2'$-valued 1-form γ on P such that each (possibly locally defined) diffeomorphism $\phi : M \to M$ that preserves D lifts canonically to an H-bundle diffeomorphism $\hat{\phi} : P \to P$ that fixes γ. Cartan shows, further, that part of the curvature of γ can be interpreted as a section $\mathcal{G}$ of the bundle $S^4(D_1^*)$, where $D_1 = D + [D,D]$ is the rank 3 first derived bundle of D. He also shows that the necessary and sufficient condition for 'flatness', i.e., equivalence of D with the G_2'-invariant 2-plane field on $Q_{3,2}$, is that this section of $S^4(D_1^*)$ should vanish identically. In fact, he proves the stronger fact that $\mathcal{G}$ vanishes if and only if the 'restricted' curvature, i.e., the reduced section of $S^4(D^*)$, which Cartan denotes as $\mathcal{F}$, vanishes. (Recall that, since the inclusion $D \to D_1$ is an injection, the dual restriction map $S^4(D_1^*) \to S^4(D^*)$ is a surjection.)

Of course, G_2' preserves a conformal structure of split type on $Q_{3,2}$. What Nurowski shows is that, for general D of Cartan type, there is associated a natural conformal structure of split type on M, generalizing the case of $Q_{3,2}$. He also shows that Cartan's tensor $\mathcal{G}$ is simply the Weyl curvature of this associated conformal structure.

In this note, I point out a similar result for 3-plane fields on 6-manifolds $D \subset TM$ that satisfy the generic condition that $D + [D,D] = TM$.

In §2, I work out the equivalence problem for such 3-plane fields. Of course, following the work of Cartan, this is just a calculation. Moreover, Tanaka [5, 6] has explained how to solve this problem (and many more like it), so this aspect of the article is not at all new.[1]

One thing that is, perhaps, new, and is motivated by comparison with Nurowski's work, is the observation, made in Proposition 1, that there exists a canonical conformal structure of split type on M associated to such a 3-plane field. This conformal structure depends on two derivatives of the defining equations of the 3-plane field, as is evidenced by the fact that it is first defined in terms of the second-order frame bundle, as derived in the course of the equivalence problem.

The result of the equivalence problem calculation is that, if $H \subset \mathrm{SO}(4,3)$ is the stabilizer subgroup of a null 3-plane in $\mathbb{R}^{4,3}$, then the plane field $D \subset TM$ defines a

[1] In fact, §2 and §3 are based on calculations that I did in my 1979 thesis [1], when I was ignorant of Tanaka's work. These sections were actually written for a series of lectures that I gave in 2002 on the method of equivalence but never published.

principal right H-bundle $B_3 \to M$ and a $\mathfrak{so}(4,3)$-valued Cartan connection 1-form γ on B_3 such that every diffeomorphism $\phi : M \to M$ that preserves the plane field D induces in a canonical way a lifted H-bundle automorphism $\hat{\phi} : B_3 \to B_3$ that preserves the Cartan connection γ. Moreover, every H-bundle map $\varphi : B_3 \to B_3$ that preserves γ is of the form $\varphi = \hat{\phi}$ for a unique diffeomorphism $\phi : M \to M$ that preserves D.

I show that the fundamental curvature tensor of γ, which I denote by $\mathcal{S}$, can be regarded as a section of the rank 27 Shur-irreducible bundle

$$(S^2(D) \otimes S^2(D^*))_0 \otimes \Lambda^3(D^*). \tag{1.1}$$

This fundamental curvature tensor is the analog of Cartan's reduced curvature, i.e., in his case, the section of $S^4(D^*)$ (his 'binary quartic form' $\mathcal{F}$) rather than of $S^4(D_1^*)$ (his 'ternary quartic form' $\mathcal{G}$). Correspondingly, in this case, there is, in fact, an extended curvature tensor $\mathcal{S}^+$ that has a canonical reduction to $\mathcal{S}$, but I do not write it out explicitly here.

I show that the vanishing of $\mathcal{S}$ is the necessary and sufficient condition that D be locally equivalent to the 'flat example', i.e., the 3-plane field on $\mathrm{SO}(4,3)/H$ that is preserved by the action of $\mathrm{SO}(4,3)$. (In particular, I show that the vanishing of $\mathcal{S}$ implies that of $\mathcal{S}^+$.)

Finally, I show that the tensor $\mathcal{S}^+$ is simply the Weyl curvature of the conformal structure on M associated to D, exactly as Nurowski shows in Cartan's case.

2. The equivalence problem

2.1. **Maximal non-integrability.** Let M be a smooth 6-manifold and let $D \subset TM$ be a smooth 3-plane field with the property that the set $D + [D, D]$ is equal to TM and has constant rank. In other words, every point $x \in M$ has a neighborhood U on which there exist vector fields X_1, X_2, X_3 that are sections of D over U, are everywhere linearly independent on U, and have the property that the six vector fields

$$X_1\,,\ X_2\,,\ X_3\,,\ [X_2,X_3]\,,\ [X_3,X_1]\,,\ [X_1,X_2] \tag{2.1}$$

are everywhere linearly indepdendent on U. Thus, D is 'maximally nonintegrable'.

A dual formulation of this maximal non-integrability condition is that there exist 1-forms θ_1, θ_2, and θ_3 on U so that each θ_i annihilates all of the vectors in D and so that $\mathrm{d}\theta_1$, $\mathrm{d}\theta_2$, and $\mathrm{d}\theta_3$ are linearly independent modulo θ_1, θ_2, and θ_3 everywhere on U.

2.2. **1-adaptation.** A coframing $\eta : TU \to \mathbb{R}^6$ on an open set $U \subset M$ of the form

$$\eta = \begin{pmatrix} \bar\theta_1 \\ \bar\theta_2 \\ \bar\theta_3 \\ \bar\omega^1 \\ \bar\omega^2 \\ \bar\omega^3 \end{pmatrix} \tag{2.2}$$

will be said to be 1-*adapted* to D if each of the $\bar\theta_i$ annihilate the vectors in D and if the equations

$$\left.\begin{aligned} d\bar\theta_1 &\equiv 2\,\bar\omega^2\wedge\bar\omega^3\\ d\bar\theta_2 &\equiv 2\,\bar\omega^3\wedge\bar\omega^1\\ d\bar\theta_3 &\equiv 2\,\bar\omega^1\wedge\bar\omega^2 \end{aligned}\right\}\quad \text{mod } \bar\theta_1,\bar\theta_2,\bar\theta_3 \tag{2.3}$$

hold on U.

The coframings 1-adapted to D are the local sections of a G_1-structure $B_1 \to M$, where $G_1 \subset \mathrm{GL}(6,\mathbb{R})$ is the group of matrices of the form

$$\begin{pmatrix} \det(A)\,{}^tA^{-1} & 0\\ AB & A\end{pmatrix} \tag{2.4}$$

where A lies in $\mathrm{GL}(3,\mathbb{R})$ and B is an arbitrary 3-by-3 matrix.

I will denote the entries of the tautological $\mathbb{R}^6$-valued 1-form on B_1 as θ_i and ω^i, as in equation (2.2). By construction, there exists on B_1 a pseudo-connection of the form

$$\begin{pmatrix} \mathrm{tr}(\alpha)\,\mathrm{I}_3 - {}^t\alpha & 0\\ \beta & \alpha\end{pmatrix} \tag{2.5}$$

where $\alpha = (\alpha^i_j)$ and $\beta = (\beta^{ij})$ take values in 3-by-3-matrices, so that equations of the form[2]

$$\begin{aligned} d\theta_i &= -\alpha^k_k\wedge\theta_i + \alpha^j_i\wedge\theta_j + \epsilon_{ijk}\,\omega^j\wedge\omega^k\\ d\omega^i &= -\beta^{ij}\wedge\theta_j - \alpha^i_j\wedge\omega^j + P^{il}\epsilon_{ljk}\,\omega^j\wedge\omega^k \end{aligned} \tag{2.6}$$

hold, where P^{il} are some functions on B_0 and where ϵ_{ijk} is totally skewsymmetric in its indices and satisfies $\epsilon_{123} = 1$.

2.3. **2-adaptation and a conformal structure.** Now, expanding out $d(d\theta_i) = 0$ and reducing the result modulo θ_1, θ_2, and θ_3 yields the relations $P^{il} = P^{li}$. One now finds that the six equations $P^{il} = 0$ define a sub-bundle $B_2 \subset B_1$ that is a G_2-structure on M, where $G_2 \subset G_1$ is the subgroup[3] consisting of those matrices of the form (2.4) in which B is skewsymmetric, i.e., ${}^tB = -B$. A coframing η that is a section of B_2 will be said to be 2-*adapted* to D.

Proposition 1. *There exists a unique pseudo-conformal structure of split type on M such that a nondegenerate quadratic form g on M represents this conformal structure if and only if its pullback to B_2 is a multiple of the quadratic form $\theta_i\circ\omega^i$.*

Proof. Note the evident fact that G_2 is a subgroup of the group $\mathrm{CO}(3,3) \subset \mathrm{GL}(6,\mathbb{R})$ consisting of the invertible matrices h that satisfy

$$ {}^th\begin{pmatrix} 0_3 & I_3\\ I_3 & 0_3\end{pmatrix} h = \left|\det(h)\right|^{1/3}\begin{pmatrix} 0_3 & I_3\\ I_3 & 0_3\end{pmatrix}. \tag{2.7}$$

The proposition now follows since B_2 is a G_2-structure on M. □

Remark 1 (Order of the conformal structure). Note that, because the bundle B_2 is constructed out of two derivatives of the plane field D, the conformal structure depends on two derivatives of the plane field D.

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[2]Here, as henceforth in this article, the summation convention is to be assumed.

[3]Of course, the reader will not confuse G_2 with G_2, the simple group of dimension 14.

Remark 2 (A weighted quadratic form). In fact, one can get a well-defined tensor on M out of this construction: Let $\eta : U \to B_2$ be a 2-adapted coframing on a domain $U \subset M$ and write η in the form (2.2). Let X_i be the sections of D over U that satisfy $\bar{\omega}^i(X_j) = \delta^i_j$. Then the tensor

$$\hat{g} = \bar{\theta}_i \circ \bar{\omega}^i \otimes (X_1 \wedge X_2 \wedge X_3) \tag{2.8}$$

is a well-defined section of $S^2(T^*M) \otimes \Lambda^3(D)$ that depends on two derivatives of D. Clearly, $\hat{g}$ determines the canonical conformal structure.

Pulling the pseudo-connection forms back to B_2 and writing $\beta^{ij} = \epsilon^{ikj}\beta_k + \tau^{ij}$ where $\tau^{ij} = \tau^{ji} \equiv 0 \mod \{\theta, \omega\}$, the structure equations take the form

$$\begin{aligned} d\theta_i &= -\alpha^k_k \wedge \theta_i + \alpha^j_i \wedge \theta_j + \epsilon_{ijk}\, \omega^j \wedge \omega^k\,, \\ d\omega^i &= -\epsilon^{ikj}\, \beta_k \wedge \theta_j - \alpha^i_j \wedge \omega^j + \tau^{ij} \wedge \theta_k\,. \end{aligned} \tag{2.9}$$

Setting

$$A^i_j = d\alpha^i_j + \alpha^i_k \wedge \alpha^k_j + 2\,\omega^i \wedge \beta_j \tag{2.10}$$

and expanding the identity $d(d\theta_i) = 0$ now yields

$$0 = -A^k_k \wedge \theta_i + A^j_i \wedge \theta_j \tag{2.11}$$

from which it follows, in particular, that $A^i_j \equiv 0 \mod \{\theta\}$. Using this congruence to expand the identity $d(d\omega^i) = 0$ and then reducing modulo $\{\theta\}$ yields

$$0 \equiv \tau^{ij} \wedge \epsilon_{jkl}\, \omega^k \wedge \omega^l \mod \{\theta\}. \tag{2.12}$$

It follows that there exist functions $T^{ij}_k = T^{ji}_k$ that satisfy $T^{ij}_i = 0$ and

$$\tau^{ij} \equiv T^{ij}_k\, \omega^k \mod \{\theta\}. \tag{2.13}$$

Now, by a replacement of the form $\alpha^i_j \mapsto \alpha^i_j + p^{ij}_k\, \theta^k$, one can retain the first equations of (2.9) (this imposes 9 linear equations on the 27 functions p^{ij}_k) while simultaneously reducing the functions T^{ij}_k to zero (this imposes 15 further linear equations on the 27 functions p^{ij}_k and these are independent from the first 9).

Thus, there exist pseudo-connection forms α^i_j and β_j on B_2 so that the equations

$$\begin{aligned} d\theta_i &= -\alpha^k_k \wedge \theta_i + \alpha^j_i \wedge \theta_j + \epsilon_{ijk}\, \omega^j \wedge \omega^k \\ d\omega^i &= -\epsilon^{ikj}\, \beta_k \wedge \theta_j - \alpha^i_j \wedge \omega^j + \epsilon^{ljk} T^i_l\, \theta_j \wedge \theta_k \end{aligned} \tag{2.14}$$

hold for some functions T^i_j on B_2. However, again, by linear algebra, there exists a unique replacement of the form $\beta_i \mapsto \beta_i + p^j_i \theta_j$ for which $T^i_j = 0$. Thus, there exist pseudo-connection forms α^i_j and β_j on B_2 so that the equations

$$\begin{aligned} d\theta_i &= -\alpha^k_k \wedge \theta_i + \alpha^j_i \wedge \theta_j + \epsilon_{ijk}\, \omega^j \wedge \omega^k \\ d\omega^i &= -\epsilon^{ikj}\, \beta_k \wedge \theta_j - \alpha^i_j \wedge \omega^j \end{aligned} \tag{2.15}$$

hold. The pseudo-connection forms are not uniquely determined by these equations; one can perform the replacements

$$\begin{aligned} \alpha^i_j &\longmapsto \alpha^i_j + \delta^i_j\, t^k\, \theta_k - t^i\, \theta_j \\ \beta_i &\longmapsto \beta_i + \epsilon_{ijk}\, t^j\, \omega^k \end{aligned} \tag{2.16}$$

for any functions t^1, t^2, t^3 without affecting (2.15). (Of course, this corresponds to the fact that the first prolongation $\mathfrak{g}_2^{(1)}$ of the algebra $\mathfrak{g}_2 \subset \mathfrak{gl}(6,\mathbb{R})$ has dimension 3.)

2.4. **Prolongation and the third-order bundle.** Let $B_3 \to B_2$ be the $\mathbb{R}^3$-bundle over B_2 whose fibers are the point pseudo-connections for which equations (2.15) hold. Then equations

$$\begin{aligned} d\theta_i &= -\alpha_k^k \wedge \theta_i + \alpha_i^j \wedge \theta_j + \epsilon_{ijk}\, \omega^j \wedge \omega^k \\ d\omega^i &= -\epsilon^{ikj}\, \beta_k \wedge \theta_j - \alpha_j^i \wedge \omega^j \end{aligned} \tag{2.17}$$

hold on B_3, where now the forms θ, ω, α, and β are tautologically defined (and, hence, canonical). Set

$$\begin{aligned} A_j^i &= d\alpha_j^i + \alpha_k^i \wedge \alpha_j^k + 2\,\omega^i \wedge \beta_j \\ B_i &= d\beta_i - \alpha_i^j \wedge \beta_j. \end{aligned} \tag{2.18}$$

The exterior derivatives of the equations (2.17) can now be expressed as

$$\begin{aligned} 0 &= -A_k^k \wedge \theta_i + A_i^j \wedge \theta_j \\ 0 &= -\epsilon^{ikj}\, B_k \wedge \theta_j - A_j^i \wedge \omega^j. \end{aligned} \tag{2.19}$$

The first equation of (2.19) implies, in particular, that $A_j^i \equiv 0 \mod \{\theta\}$, so there exist 1-forms π_j^{ik} (not unique) such that $A_j^i = \pi_j^{ik} \wedge \theta_k$. Substituting this relation into the second set of equations of (2.19) then yields

$$0 = -\epsilon^{ikj}\, B_k \wedge \theta_j - \pi_k^{ij} \wedge \theta_j \wedge \omega^k, \tag{2.20}$$

which, in turn, implies

$$0 \equiv -\epsilon^{ikj}\, B_k + \pi_k^{ij} \wedge \omega^k \mod \{\theta\}. \tag{2.21}$$

In particular, it follows that $\pi_k^{ij} + \pi_k^{ji} \equiv 0 \mod \{\theta,\omega\}$, so that one can write $\pi_k^{ij} = \epsilon^{ijl}\pi_{lk} + \sigma_k^{ij}$ where $\sigma_k^{ij} = \sigma_k^{ji} \equiv 0 \mod \{\theta,\omega\}$. One can further write $\pi_{ij} = -\epsilon_{ijk}\,\tau^k + \sigma_{ij}$ where $\sigma_{ij} = \sigma_{ji}$. This leads to the formula

$$A_j^i = \pi_j^{ik} \wedge \theta_k = \delta_j^i\, \tau^k \wedge \theta_k - \tau^i \wedge \theta_j + \epsilon^{ikl}\sigma_{jl} \wedge \theta_k + \sigma_j^{ik} \wedge \theta_k. \tag{2.22}$$

Substituting this into the first set of equations in (2.19) and using the fact that $\sigma_k^{ij} \equiv 0 \mod \{\theta,\omega\}$ shows that the 3-forms $\Sigma_j = \sigma_{jl} \wedge \epsilon^{ikl}\, \theta_k \wedge \theta_i$ are cubic expressions in the 1-forms θ_i and ω^k. In particular, it follows that $\sigma_{jl} \equiv 0 \mod \{\theta,\omega\}$. Consequently, the 2-forms A_j^i can be written in the form

$$A_j^i = \delta_j^i\, \tau^k \wedge \theta_k - \tau^i \wedge \theta_j + R_{jk}^i \epsilon^{klm}\, \theta_l \wedge \theta_m + S_{jl}^{ik}\, \theta_k \wedge \omega^l \tag{2.23}$$

for some 1-forms τ^i and functions R_{jk}^i and S_{jl}^{ik}. Comparing this with equation (2.16), one sees that the 1-forms τ^i are the components of a pseudo-connection for the bundle $B_3 \to B_2$. Of course, these τ^i are not uniquely determined by the formulae (2.23).

2.4.1. *Normalizing* τ. The τ_i will be made unique by imposing the appropriate linear equations on the functions R and S as follows: First, consider the trace of (2.23):

$$A^i_i = 2\,\tau^k \wedge \theta_k + R^i_{ik}\epsilon^{klm}\,\theta_l \wedge \theta_m + S^{ik}_{il}\,\theta_k \wedge \omega^l\,. \tag{2.24}$$

By adding linear combinations of the ω^i and θ_j to the τ^k, one can arrange that

$$R^i_{ik} = S^{ik}_{il} = 0. \tag{2.25}$$

In other words, $A^i_i = 2\,\tau^k \wedge \theta_k$.

The conditions (2.25) still do not determine the τ^k completely. However, they do determine the τ^k up to a replacement of the form $\tau^k \mapsto \tau^k + p^{kl}\,\theta_l$ where $p^{kl} = p^{lk}$.

Substituting these normalized formulae into the first set of equations in (2.19) yields the relations

$$0 = \left(R^j_{ik}\epsilon^{klm}\,\theta_l \wedge \theta_m + S^{jk}_{il}\,\theta_k \wedge \omega^l\right) \wedge \theta_j\,, \tag{2.26}$$

which are equivalent to the equations

$$R^j_{ij} = S^{jk}_{il} - S^{kj}_{il} = 0. \tag{2.27}$$

This suggests a closer inspection of the functions R^i_{jk}. Consider the $\mathrm{GL}(3,\mathbb{R})$-invariant decomposition

$$R^i_{jk} = S^i_{jk} + \epsilon_{ljk}\,S^{il} + \epsilon_{ljk}\epsilon^{ilp}S_p\,, \tag{2.28}$$

where $S^i_{jk} = S^i_{kj}$ and $S^{ij} = S^{ji}$. The trace condition $R^i_{ij} = 0$ and identity $R^i_{ji} = 0$ now combine to show that $S^i_{ij} = S_j = 0$, so the decomposition of R simplifies to

$$R^i_{jk} = S^i_{jk} + \epsilon_{ljk}\,S^{il}\,, \tag{2.29}$$

where $S^i_{jk} = S^i_{kj}$ and $S^{ij} = S^{ji}$.

One can now finally complete the normalization of the τ^k by requiring, in addition to (2.25), that $S^{ij} = 0$. Thus, the τ^k are made unique by requiring them to be chosen so that

$$A^i_j = \delta^i_j\,\tau^k \wedge \theta_k - \tau^i \wedge \theta_j + S^i_{jk}\epsilon^{klm}\,\theta_l \wedge \theta_m + S^{ik}_{jl}\,\theta_k \wedge \omega^l \tag{2.30}$$

holds, where the coefficients are required to satisfy the normalizations

$$S^i_{jk} = S^i_{kj}\,,\ S^i_{ik} = 0\,,\ S^{ik}_{ij} = 0 \tag{2.31}$$

Thus, the forms θ_i, ω^j, α^i_j, β_i, and τ^j define a canonical coframing on B_3 and every diffeomorphism of M that preserves the 3-plane field D lifts to a unique diffeomorphism of B_3 that fixes the forms in this coframing. Thus, these constitute the solution of the equivalence problem in the sense of Cartan.

2.5. **Bianchi identities.** Substituting equation (2.30) into the second set of equations of (2.19), yields the relations

$$\epsilon^{ikj}\,B_k \wedge \theta_j = -\left(\delta^i_j\,\tau^k \wedge \theta_k - \tau^i \wedge \theta_j + S^i_{jk}\epsilon^{klm}\,\theta_l \wedge \theta_m + S^{ik}_{jl}\,\theta_k \wedge \omega^l\right) \wedge \omega^j. \tag{2.32}$$

It follows that $S^{ik}_{jl} = S^{ik}_{lj}$ and that one has relations of the form

$$B_i = \epsilon_{ijk}\,\tau^j \wedge \omega^k - 2S^j_{ik}\,\theta_j \wedge \omega^k + \epsilon^{jkl}S_{ij}\,\theta_k \wedge \theta_l\,, \tag{2.33}$$

where $S_{ij} = S_{ji}$.

To summarize the results so far: There are structure equations

$$\begin{aligned}
\mathrm{d}\theta_i &= -\alpha^k_k \wedge \theta_i + \alpha^j_i \wedge \theta_j + \epsilon_{ijk}\,\omega^j \wedge \omega^k \\
\mathrm{d}\omega^i &= -\epsilon^{ikj}\,\beta_k \wedge \theta_j - \alpha^i_j \wedge \omega^j \\
\mathrm{d}\alpha^i_j &= -\alpha^i_k \wedge \alpha^k_j - 2\,\omega^i \wedge \beta_j + \delta^i_j\,\tau^k \wedge \theta_k - \tau^i \wedge \theta_j \\
&\qquad + S^i_{jk}\epsilon^{klm}\,\theta_l \wedge \theta_m + S^{ik}_{jl}\,\theta_k \wedge \omega^l \\
\mathrm{d}\beta_i &= \alpha^j_i \wedge \beta_j + \epsilon_{ijk}\,\tau^j \wedge \omega^k - 2S^j_{ik}\,\theta_j \wedge \omega^k + \epsilon^{jkl}S_{ij}\,\theta_k \wedge \theta_l
\end{aligned} \tag{2.34}$$

where the functions S satisfy the trace and symmetry relations

$$S_{ij} = S_{ji}\,,\quad S^i_{jk} = S^i_{kj}\,,\quad S^i_{ik} = 0\,,\quad S^{ik}_{jl} = S^{ki}_{jl} = S^{ik}_{lj}\,,\quad S^{ik}_{ij} = 0. \tag{2.35}$$

Tracing the formula for $\mathrm{d}\alpha^i_j$ yields

$$\mathrm{d}\alpha^i_i = 2\,\tau^k \wedge \theta_k - 2\,\omega^k \wedge \beta_k \tag{2.36}$$

and, since the left-hand side is closed, taking the exterior derivative of both sides yields

$$0 = 2\,T^k \wedge \theta_k \tag{2.37}$$

where

$$T^i = \mathrm{d}\tau^i - \alpha^k_k \wedge \tau^i + \alpha^i_j \wedge \tau^j + \epsilon^{ijk}\,\beta_j \wedge \beta_k + \epsilon^{ijl}S_{lk}\,\theta_j \wedge \omega^k\,. \tag{2.38}$$

Thus, there exist 1-forms $\tau^{ij} = \tau^{ji}$ so that

$$\mathrm{d}\tau^i = \alpha^k_k \wedge \tau^i - \alpha^i_j \wedge \tau^j - \epsilon^{ijk}\,\beta_j \wedge \beta_k - \epsilon^{ijl}S_{lk}\,\theta_j \wedge \omega^k - \tau^{ij} \wedge \theta_j\,. \tag{2.39}$$

These 1-forms τ^{ij} are not unique, but are unique up to a replacement of the form $\tau^{ij} \mapsto \tau^{ij} + p^{ijk}\,\theta_k$ for some functions p^{ijk} satisfying the symmetry conditions $p^{ijk} = p^{jik} = p^{ikj}$.

Define 1-forms σ_{ij}, σ^i_{jk}, and σ^{ik}_{jl} by the equations

$$\begin{aligned}
\mathrm{d}S^{ik}_{jl} &= \sigma^{ik}_{jl} + S^{ik}_{jl}\,\alpha^m_m - S^{mk}_{jl}\alpha^i_m - S^{im}_{jl}\,\alpha^k_m + S^{ik}_{ml}\alpha^m_j + S^{ik}_{jm}\alpha^m_l \\
&\qquad + \tfrac{2}{3}\left(5\delta^k_m S^i_{jl} + 5\delta^i_m S^k_{jl} - \delta^k_j S^i_{ml} - \delta^i_j S^k_{ml} - \delta^k_l S^i_{jm} - \delta^i_l S^k_{jm}\right)\omega^m \\
\mathrm{d}S^i_{jk} &= \sigma^i_{jk} + S^i_{jk}\,\alpha^m_m - S^m_{jk}\,\alpha^i_m + S^i_{mk}\,\alpha^m_j + S^i_{jm}\,\alpha^m_k + \tfrac{1}{2}S^{il}_{jk}\,\beta_l \\
&\qquad - \tfrac{1}{2}\left(4\delta^i_l S_{jk} - \delta^i_k S_{jl} - \delta^i_j S_{lk}\right)\omega^l \\
\mathrm{d}S_{ij} &= \sigma_{ij} + S_{jk}\,\alpha^m_m + S_{mj}\,\alpha^m_i + S_{im}\,\alpha^m_j - 2S^m_{ij}\,\beta_m\,.
\end{aligned} \tag{2.40}$$

Then the σs satisfy the same symmetry and trace conditions as the corresponding Ss and, moreover, the identies $\mathrm{d}(\mathrm{d}\alpha^i_j) = 0$ and $\mathrm{d}(\mathrm{d}\beta_i) = 0$ become the relations

$$\begin{aligned}
0 &= -\tau^{im} \wedge \theta_m \wedge \theta_j + \epsilon^{klm}\,\sigma^i_{jk} \wedge \theta_l \wedge \theta_m + \sigma^{ik}_{jl} \wedge \theta_k \wedge \omega^l \\
0 &= \epsilon_{ijk}\,\tau^{jl} \wedge \theta_l \wedge \omega^k - 2\sigma^j_{ik} \wedge \theta_j \wedge \omega^k + \epsilon^{jkl}\sigma_{ij} \wedge \theta_k \wedge \theta_l
\end{aligned} \tag{2.41}$$

These relations imply

$$\tau^{ij} \equiv \sigma_{ij} \equiv \sigma^i_{jk} \equiv \sigma^{ik}_{jl} \equiv 0 \mod \{\theta, \omega\}. \tag{2.42}$$

(If X is a vector field on B_3 that satisfies $\theta_i(X) = \omega^j(X) = 0$, then the above equations imply

$$\begin{aligned}
0 &= -\tau^{im}(X)\,\theta_m \wedge \theta_j + \epsilon^{klm}\,\sigma^i_{jk}(X)\,\theta_l \wedge \theta_m + \sigma^{ik}_{jl}(X)\,\theta_k \wedge \omega^l \\
0 &= \epsilon_{ijk}\,\tau^{jl}(X)\,\theta_l \wedge \omega^k - 2\sigma^j_{ik}(X)\,\theta_j \wedge \omega^k + \epsilon^{jkl}\sigma_{ij}(X)\,\theta_k \wedge \theta_l\,,
\end{aligned} \tag{2.43}$$

which implies $\tau^{ij}(X) = \sigma_{ij}(X) = \sigma^i_{jk}(X) = \sigma^{ik}_{jl}(X) = 0$. Hence the conclusion.)

It follows that there are expansions

$$\begin{aligned} \tau^{ij} &= T^{ijm}\,\theta_m + T^{ij}_m\ \ \omega^m \\ \sigma_{ij} &= T^m_{ij}\ \ \theta_m + T_{ijm}\,\omega^m \\ \sigma^i_{jk} &= T^{im}_{jk}\ \ \theta_m + T^i_{jkm}\,\omega^m \\ \sigma^{ik}_{jl} &= T^{ikm}_{jl}\,\theta_m + T^{ik}_{jlm}\,\omega^m \end{aligned} \tag{2.44}$$

and that τ^{ij} can be made unique by requiring that the full symmetrization of T^{ijm} vanish, i.e., that $T^{ijk} + T^{jki} + T^{kij} = 0$, so assume that this has been done.

The relations (2.41) can now be expressed as the following identities:

$$\begin{aligned} T^{ik}_{jlm} &= T^{ik}_{jml} \\ T^i_{jkm} &= \tfrac14\left(\epsilon_{jpq}\,T^{ipq}_{km} + \epsilon_{kpq}\,T^{ipq}_{jm}\right) \\ T^{ij}_m &= -\tfrac12\,T^{ijk}_{mk} \\ T^{ijm} &= \tfrac23\left(\epsilon^{ilm}\,T^{jk}_{lk} + \epsilon^{jlm}\,T^{ik}_{lk}\right) \\ T^m_{im} &= 0 \\ T_{ijm} &= 3\epsilon_{klm}\,T^{kl}_{ij} - 2\epsilon_{kli}\,T^{kl}_{mj} - 2\epsilon_{klj}\,T^{kl}_{mi} + \epsilon_{ilm}\,T^{lk}_{jk} + \epsilon_{jlm}\,T^{lk}_{ik}\,. \end{aligned} \tag{2.45}$$

3. The fundamental tensor and flatness

The expansions (2.44) taken with the definition of σ^{ik}_{jl} in (2.40) show that the functions S^{ij}_{kl} are constant on the fibers of $B_3 \to B_2$ and hence can be regarded as functions on B_2. In fact, because

$$\mathrm{d}S^{ik}_{jl} \equiv S^{ik}_{jl}\,\alpha^m_m - S^{mk}_{jl}\,\alpha^i_m - S^{im}_{jl}\,\alpha^k_m + S^{ik}_{ml}\,\alpha^m_j + S^{ik}_{jm}\,\alpha^m_l \mod \{\theta, \omega\}, \tag{3.1}$$

it follows that the S^{ik}_{jl} can be regarded as the components of a section of the bundle $S^2(D) \otimes S^2(D^*) \otimes \Lambda^3(D^*)$ that takes values in the (irreducible) Shur representation subbundle $\left(S^2(D) \otimes S^2(D^*)\right)_0 \otimes \Lambda^3(D^*)$, which has rank 27 (the subscript 0 denotes the kernel of the natural mapping $S^2(D) \otimes S^2(D^*) \to D \otimes D^*$ that is defined by contraction).

Specifically, if $\eta = (\bar\theta_i, \bar\omega^j)$ is a 2-adapted coframing on some domain $U \subset M$, set $\bar S^{ik}_{jl} = \eta^* S^{ik}_{jl}$ and consider the expression

$$\mathcal{S}(\eta) = \bar S^{ik}_{jl}\ \bar X_i \circ \bar X_k \otimes \bar\omega^j \circ \bar\omega^l \otimes \left(\bar\omega^1 \wedge \bar\omega^2 \wedge \bar\omega^3\right) \tag{3.2}$$

as a section of $S^2(D) \otimes S^2(D^*) \otimes \Lambda^3(D^*)$ over U, where $\bar X_i$ are the sections of D over U that are dual to $\bar\omega^i$, i.e., so that $\bar\omega^i(\bar X_j) = \delta^i_j$. Then equation (3.1) implies that $\mathcal{S}(\eta)$ is independent of the choice of 1-adapted coframing η and hence is the restriction to U of a globally defined section $\mathcal{S}$ that depends only on D.

Definition 1 (The fundamental tensor). The tensor $\mathcal{S}$ will be referred to as the *fundamental tensor* of D.

The following vanishing result is the analog for nondegenerate 3-plane fields in dimension 6 of Cartan's characterization in [3, §VII] of the 'flat' 2-plane fields of Cartan type in dimension 5.

Proposition 2. *Suppose that S vanishes identically. Then the following hold: First, S^i_{jk} and S_{ij} vanish identically. Second, the structure equations simplify to*

$$\begin{aligned}
d\theta_i &= -\alpha^k_k \wedge \theta_i + \alpha^j_i \wedge \theta_j + \epsilon_{ijk}\,\omega^j \wedge \omega^k \\
d\omega^i &= -\epsilon^{ikj}\,\beta_k \wedge \theta_j - \alpha^i_j \wedge \omega^j \\
d\alpha^i_j &= -\alpha^i_k \wedge \alpha^k_j - 2\,\omega^i \wedge \beta_j + \delta^i_j\,\tau^k \wedge \theta_k - \tau^i \wedge \theta_j \\
d\beta_i &= \alpha^j_i \wedge \beta_j + \epsilon_{ijk}\,\tau^j \wedge \omega^k \\
d\tau^i &= \alpha^k_k \wedge \tau^i - \alpha^i_j \wedge \tau^j - \epsilon^{ijk}\,\beta_j \wedge \beta_k\,.
\end{aligned} \tag{3.3}$$

Third, for any 1-connected open $U \subset M$, the Lie algebra of vector fields on U whose (local) flows preserve D is isomorphic to the Lie algebra of SO(4, 3)*. Fourth, any point of M is the center of a coordinate system $\big(U, (x^j, y_i)\big)$ in which the plane field D is annihilated by the three 1-forms $\bar\theta_i = dy_i + \epsilon_{ijk}\,x^j\,dx^k$.*

Proof. First, note that, by the first equation of (2.40), the vanishing of the functions S^{ik}_{jl} implies that

$$\begin{aligned}
\sigma^{ik}_{jl} &= -\tfrac{2}{3}\left(5\delta^k_m S^i_{jl} + 5\delta^i_m S^k_{jl} - \delta^k_j S^i_{ml} - \delta^i_j S^k_{ml} - \delta^k_l S^i_{jm} - \delta^i_l S^k_{jm}\right)\omega^m \\
&= T^{ikm}_{jl}\,\theta_m + T^{ik}_{jlm}\,\omega^m
\end{aligned} \tag{3.4}$$

which, in turn, implies both $T^{ikm}_{jl} = 0$ and

$$T^{ik}_{jlm} = -\tfrac{2}{3}\left(5\delta^k_m S^i_{jl} + 5\delta^i_m S^k_{jl} - \delta^k_j S^i_{ml} - \delta^i_j S^k_{ml} - \delta^k_l S^i_{jm} - \delta^i_l S^k_{jm}\right). \tag{3.5}$$

However, by the first equation of (2.45), T^{ik}_{jlm} is fully symmetric in its lower indices, which implies

$$S^i_{jk} = 0. \tag{3.6}$$

Using this, by the second equation of (2.40) and by (2.44), one has that

$$\sigma^i_{jk} = \tfrac{1}{2}\left(4\delta^i_l S_{jk} - \delta^i_k S_{jl} - \delta^i_j S_{lk}\right)\omega^l = T^{im}_{jk}\ \theta_m + T^i_{jkl}\,\omega^l \tag{3.7}$$

which implies that $T^{im}_{jk}\ = 0$ and

$$T^i_{jkl} = \tfrac{1}{2}\left(4\delta^i_l S_{jk} - \delta^i_k S_{jl} - \delta^i_j S_{lk}\right). \tag{3.8}$$

Now, however, the second equation of (2.45) coupled with $T^{ikm}_{jl} = 0$ (which was derived above) show that $T^i_{jkl} = 0$ which, in turn, now implies $S_{ij} = 0$.

Next, since (2.40) now implies that $\sigma_{ij} = 0$, it follows from (2.44), that

$$T^{m}_{ij}\ = T_{ijm} = 0. \tag{3.9}$$

A final appeal to (2.45) then shows that

$$T^{ijm} = T^{ij}_{m}\ = 0, \tag{3.10}$$

i.e., that $\tau^{ij} = 0$. Consequently, the structure equations simplify to (3.3), as claimed.

Now, the exterior derivatives of the equations (3.3) are identities, so it follows that these are the left-invariant forms on a Lie group of dimension 21. An examination of the weights associated to the (maximal) torus dual to the diagonal αs shows that the Lie algebra is $\mathfrak{so}(4,3)$, thus verifying the third claim.

One can also see this directly by noting that the equations (3.3) are equivalent to $d\gamma = -\gamma\wedge\gamma$, where

$$\gamma = \begin{pmatrix} -\alpha^1_1 & -\alpha^2_1 & -\alpha^3_1 & 2\beta_1 & 0 & -\tau_3 & \tau_2 \\ -\alpha^1_2 & -\alpha^2_2 & -\alpha^3_2 & 2\beta_2 & \tau_3 & 0 & -\tau_1 \\ -\alpha^1_3 & -\alpha^2_3 & -\alpha^3_3 & 2\beta_3 & -\tau_2 & \tau_1 & 0 \\ \omega^1 & \omega^2 & \omega^3 & 0 & -\beta_1 & -\beta_2 & -\beta_3 \\ 0 & \theta_3 & -\theta_2 & -2\omega^1 & \alpha^1_1 & \alpha^1_2 & \alpha^1_3 \\ -\theta_3 & 0 & \theta_1 & -2\omega^2 & \alpha^2_1 & \alpha^2_2 & \alpha^2_3 \\ \theta_2 & -\theta_1 & 0 & -2\omega^3 & \alpha^3_1 & \alpha^3_2 & \alpha^3_3 \end{pmatrix}. \tag{3.11}$$

Obviously, γ takes values in the Lie algebra $\mathfrak{so}(4,3) \subset \mathfrak{gl}(7,\mathbb{R})$, which is the space of matrices a that satisfy $Qa + {}^t aQ = 0$, where

$$Q = \begin{pmatrix} 0_{3\times3} & 0_{3\times1} & I_{3\times3} \\ 0_{1\times3} & 2 & 0_{1\times3} \\ I_{3\times3} & 0_{3\times1} & 0_{3\times3} \end{pmatrix} \tag{3.12}$$

is a symmetric matrix of type $(4,3)$.

Finally, the system $\alpha^i_j = \beta_j = \tau^i = 0$ is a Frobenius system, and a leaf of this system in B_3 defines a (local) 2-adapted coframing η on an open set $U \subset M$ that satisfies

$$\begin{aligned} d\bar\omega^j &= 0\,, \\ d\bar\theta_i &= \epsilon_{ijk}\,\bar\omega^j\wedge\bar\omega^k\,. \end{aligned} \tag{3.13}$$

Consequently, assuming that U is simply connected, there exist functions x^j on U such that $\bar\omega^j = dx^j$ and there exist functions y_i on U such that

$$dy_i = \bar\theta_i - \epsilon_{ijk}\,x^j\,dx^k. \tag{3.14}$$

These provide the desired local coordinates. □

Corollary 1 (Maximal symmetry). *The Lie group* Aut(M,D) *has dimension at most* 21 *and this upper limit is reached only when D is locally equivalent to the 3-plane field on $\mathbb{R}^6$ defined by the equations*

$$dy_i + \epsilon_{ijk}\,x^j\,dx^k = 0. \tag{3.15}$$

Proof. The Lie group Aut(M,D) is embedded into the group of diffeomorphisms of B_3 that preserve the coframing defined by θ_i, ω^j, α^j_i, β_j, and τ^i. This group can only have dimension 21 if all of the functions S^{ik}_{jl} are constant. However, these functions cannot be constant on the fibers of $B_3 \to M$ unless they vanish. Now apply Proposition 2 □

Remark 3 (The homogeneous model). Note that the proof of Proposition 2 identifies the homogeneous model for the 'flat' case: Let $M^6 \subset \mathrm{Gr}(3,\mathbb{R}^{4,3})$ be the space of isotropic (i.e., null) 3-planes in the split signature inner product space $\mathbb{R}^{4,3}$. The group $\mathrm{O}(4,3)$ acts transitively on this 6-manifold and preserves a nondegenerate 3-plane field on it. By Proposition 2, the identity component of $\mathrm{O}(4,3)$ is the identity component of the group of automorphisms of this 3-plane field.

Remark 4 (Irregular D-curves). Note that the Cartan system of the 1-form θ_3 on B_3 is the Pfaffian system J spanned by θ_1, θ_2, θ_3, ω^1, ω^2, α^1_3, and α^2_3. Consequently, this Pfaffian system is Frobenius (as can be directly verified by a glance at the structure equations) and hence there is a submersion $\nu : B_3 \to N^7$ for some (not necessarily Hausdorff) 7-manifold N^7 such that the fibers of ν are the leaves of J.

The points of N^7 represent the *irregular D-curves* in M^6, as defined in [2].[4] Specifically, a leaf of the system J projects to M as a submersion onto a curve in M and, in this way, one sees that each J-leaf represents a curve in M.

This 7-parameter family of curves has the property that exactly one curve of the family passes through a given point in M with a given tangent direction in D.

Note that, in the homogeneous model, N^7 is simply the space of null (i.e., isotropic) 2-planes in $\mathbb{R}^{4,3}$. Each such 2-plane lies in a 1-parameter family of null 3-planes and this gives the interpretation of such 2-planes as curves in M. In fact, given a null 2-plane $E \subset \mathbb{R}^{4,3}$, the restriction of the quadratic form to the 5-plane $E^\perp \subset \mathbb{R}^{4,3}$ has kernel equal to E and hence descends to a nondegenerate form (of type $(2,1)$ on $E^\perp/E \simeq \mathbb{R}^{2,1}$. The null 3-planes that contain E are in 1-to-1 correspondence with the null lines in $E^\perp/E$, a space which is known to be 1-dimensional and, in fact, naturally isomorphic to $\mathbb{RP}^1$.

Similarly, in the general case, each of the irregular D-curves inherits a natural projective structure. In fact, on a leaf of J, one has $d\omega^3 = -\alpha^3_3{\wedge}\omega^3$, $d\alpha^3_3 = 2\,\beta_3{\wedge}\omega^3$, and $d\beta_3 = \alpha^3_3{\wedge}\beta_3$, so that ω^3 is a differential on the corresponding D-curve that is well-defined up to a projective change of parameter.

Remark 5 (An extended tensor). The reader cannot have helped but notice that equations (2.40) actually imply that S is the reduction of an extended tensor S^+ of rank $48 = 27{+}15{+}6$ that uses all of the components S^{ij}_{kl}, S^i_{kl}, and S_{kl}. This extended tensor will play a role in the next section, but it is not worthwhile to write it out explicitly here. Instead, I will just note that S^+ takes values in a certain rank 48 subbundle of the bundle $S^2(\mathfrak{g}_2) \otimes \Lambda^3(D^*)$, where $\mathfrak{g}_2 \subset \mathfrak{gl}(6,\mathbb{R})$ is the Lie algebra of the subgroup G_2 defined at the beginning of §2.3.

For a comparison with Nurowski's examples, see Remark 7.

4. The fundamental tensor and Weyl curvature

Consider the 1-form $\hat\gamma$ with values in $\mathfrak{so}(4,4) \subset \mathfrak{gl}(8,\mathbb{R})$ defined on B_3 by the formula

$$(4.1)\qquad \hat\gamma = \begin{pmatrix} -\phi & \beta_1 & \beta_2 & \beta_3 & \tau^1 & \tau^2 & \tau^3 & 0 \\ \omega^1 & \alpha^1_1-\phi & \alpha^1_2 & \alpha^1_3 & 0 & -\beta_3 & \beta_2 & \tau^1 \\ \omega^2 & \alpha^2_1 & \alpha^2_2-\phi & \alpha^2_3 & \beta_3 & 0 & -\beta_1 & \tau^2 \\ \omega^3 & \alpha^3_1 & \alpha^3_2 & \alpha^3_3-\phi & -\beta_2 & \beta_1 & 0 & \tau^3 \\ \theta_1 & 0 & \omega^3 & -\omega^2 & \phi-\alpha^1_1 & -\alpha^2_1 & -\alpha^3_1 & \beta_1 \\ \theta_2 & -\omega^3 & 0 & \omega^1 & -\alpha^1_2 & \phi-\alpha^2_2 & -\alpha^3_2 & \beta_2 \\ \theta_3 & \omega^2 & -\omega^1 & 0 & -\alpha^1_3 & -\alpha^2_3 & \phi-\alpha^3_3 & \beta_3 \\ 0 & \theta_1 & \theta_2 & \theta_3 & \omega^1 & \omega^2 & \omega^3 & \phi \end{pmatrix}$$

where $\phi = \frac{1}{2}(\alpha^i_i)$.

[4] Actually, in [2], these curves are called 'non-regular', but I now prefer the more standard English term 'irregular'.

In the flat case, this 1-form satisfies $d\hat{\gamma} = -\hat{\gamma}\wedge\hat{\gamma}$. In particular, it follows that $\hat{\gamma}$ takes values in a Lie algebra $\mathfrak{g} \subset \mathfrak{so}(4,4)$ that is isomorphic to $\mathfrak{so}(4,3)$. The corresponding subgroup of $\mathrm{SO}(4,4)$ is isomorphic to $\mathrm{Spin}(4,3)$. Thus, I will denote the algebra $\mathfrak{g}$ by $\mathfrak{spin}(4,3)$ and call the corresponding subgroup $\mathrm{Spin}(4,3)$.

It now follows from (2.34) that, if $P \to M$ is the Cartan structure bundle associated to the canonical conformal structure with $\mathfrak{so}(4,4)$-valued connection form Γ and fiber isomorphic to the parabolic subgroup $H \subset \mathrm{SO}(4,4)$ that is the stabilizer of a null line in $\mathbb{R}^{4,4}$, then there exists a bundle embedding $\iota : B_3 \to P$ such that

$$\hat{\gamma} = \iota^*(\Gamma). \tag{4.2}$$

In particular, the structure equations (2.34) show that the functions S^{ik}_{jl}, S^{i}_{jl}, S_{jl} are the components of the Weyl curvature of the conformal structure in this reduction. Thus, one has the following result:

Proposition 3. *The Weyl tensor of the conformal structure associated to D is the extended tensor S^+. In particular, the associated conformal structure is conformally flat if and only if the plane field D is locally equivalent to the flat example.* □

Remark 6 (An algebraic characterization of the Weyl tensor). Recall that, for a split-conformal manifold of dimension 6, the Weyl tensor takes values in a bundle associated to an irreducible, 84-dimensional representation space W of $\mathfrak{co}(3,3) = \mathbb{R} \oplus \mathfrak{so}(3,3)$ that can be described as follows: Use the 'exceptional isomorphism' $A_3 = D_3$ to regard $\mathfrak{co}(3,3)$ as $\mathfrak{gl}(4,\mathbb{R})$ and let V be the standard representation of dimension 4 of $\mathfrak{gl}(4,\mathbb{R})$. Then it is not difficult to establish the isomorphism of representations

$$W = \left(S^2(V)\otimes S^2(V^*)\right)_0 \otimes \left(\Lambda^4(V)\right)^{-1/2} \tag{4.3}$$

where $\left(S^2(V)\otimes S^2(V^*)\right)_0 \subset S^2(V)\otimes S^2(V^*)$ is the kernel of the natural (and surjective) contraction mapping

$$S^2(V) \otimes S^2(V^*) \longrightarrow V \otimes V^*. \tag{4.4}$$

Now, if $\xi \subset V^*$ is a hyperplane, one can define the subspace

$$\left(S^2(V) \otimes S^2(\xi)\right)_0 \subset \left(S^2(V) \otimes S^2(V^*)\right)_0 \tag{4.5}$$

to be the kernel of the natural (and surjective) contraction mapping

$$S^2(V) \otimes S^2(\xi) \longrightarrow V \otimes \xi. \tag{4.6}$$

The dimension of this space is 48 and it is a representation space of the 12-dimensional subgroup $G_\xi \subset \mathrm{GL}(V)$ that preserves the hyperplane ξ. Under the isomorphism (actually, a double cover) $\mathrm{GL}(V) \to \mathrm{CO}(3,3)$, the subgroup G_ξ goes to the subgroup $G_2 \subset \mathrm{CO}(3,3)$.

Now, the Weyl curvature function of the conformal structure pulls back to B_3 to take values in the 48-dimensional subspace

$$W_\xi = \left(S^2(V)\otimes S^2(\xi)\right)_0 \otimes \left(\Lambda^4(V)\right)^{-1/2}, \tag{4.7}$$

This subspace is characterized as the kernel of the contraction

$$C_e : W \to (S^2(V) \otimes V^*)_0 \otimes \left(\Lambda^4(V)\right)^{-1/2} \tag{4.8}$$

where $e \subset V$ is a nonzero vector annihilated by ξ.

The group G_ξ preserves the filtration

$$S^2(\xi^\perp)\otimes S^2(\xi)\subset\left(\xi^\perp\circ V\otimes S^2(\xi)\right)_0\subset\left(S^2(V)\otimes S^2(\xi)\right)_0 \tag{4.9}$$

whose graded pieces have dimensions 6, 15, and 27. This filtration corresponds to the representation of the Weyl curvature by the components S_{jk}, S^i_{jk} and S^{il}_{jk}, which are the components of the tensor S^+. In particular, the top associated graded piece

$$\frac{\left(S^2(V)\otimes S^2(\xi)\right)_0}{\left(\xi^\perp\circ V\otimes S^2(\xi)\right)_0}\simeq\left(S^2(V/\xi^\perp)\otimes S^2(\xi)\right)_0 \tag{4.10}$$

of dimension 27 gives the associated bundle in which the tensor $\mathcal{S}$ takes values.

Thus, the algebraic characterization of the Weyl tensors that arise from conformal structures associated to nondegenerate 3-plane fields on 6-manifolds is that there should exist a nonzero conformal half-spinor s whose contraction with the Weyl tensor should vanish. This nonvanishing half-spinor field then defines the structure reduction that locates B_3 as a subbundle of the conformal Cartan connection bundle.

Remark 7 (Analogy with the 5-dimensional case). By Nurowski's calculations in [4], Proposition 3 has a direct analog in the case of Cartan-type 2-plane fields in dimension 5.

In that case, Nurowski shows that Cartan's ternary quartic form $\mathcal{G}$ can be interpreted as the Weyl curvature of the conformal structure associated to the 2-plane field.

In fact, the analogy is even more striking when one looks at the algebraic characterization of the Weyl curvature in that case. There, Cartan's structure bundle $\pi : P \to M^5$ and $\mathfrak{g}_2'$-valued connection form γ are embedded via an equivariant inclusion $\iota : P \to P^+$ into the conformal structure bundle $\pi : P^+ \to M^5$ with $\mathfrak{so}(4,3)$-valued connection form Γ. This corresponds to the inclusion $\mathrm{G}_2'\subset\mathrm{SO}(4,3)$ and the fiber group $H\subset\mathrm{G}_2'$ of the bundle P is the intersection of G_2' with the subgroup $H^+\subset\mathrm{SO}(4,3)$ that consists of those elements that fix a given null line in $\mathbb{R}^{4,3}$.

The group H^+ has a natural homomorphism onto $\mathrm{CO}(3,2)$ and the image of H under this natural homomorphism is the 7-dimensional subgroup $K\subset\mathrm{CO}(3,2)$ that fixes a null 2-plane.

Now, the Weyl curvature of a conformal structure of split type in dimension 5 is an irreducible, 35-dimensional representation W of $\mathrm{CO}(3,2)$. Using the exceptional isomorphism $\mathfrak{co}(3,2)\simeq\mathbb{R}\oplus\mathfrak{sp}(2,\mathbb{R})$ and letting V denote the irreducible 4-dimensional representation of $\mathbb{R}^*\cdot\mathrm{Sp}(2,\mathbb{R})$, then W is isomorphic to $S^4(V^*)$.

Also, the subgroup of $\mathrm{SO}(3,2)$ that fixes a null 2-plane corresponds, in $\mathrm{Sp}(2,\mathbb{R})$ to the 7-dimensional subgroup that fixes a line in V, or, equivalently, a 3-plane $\xi\subset V^*$.

Tracing through this isomorphism and comparing it with the calculations of Nurowski, one sees that the Weyl curvature of the conformal structure associated to a Cartan-type 2-plane field in dimension 5 takes values in a subspace of the form $S^4(\xi)\subset \S^4(V^*)\simeq W$ for an appropriately chosen ξ. Moreover, since the subgroup of $\mathrm{Sp}(2,\mathbb{R})$ that fixes ξ preserves the subspace $\xi^\perp\subset\xi$, it follows that this subgroup preserves a filtration of $S^4(\xi)$ based on the number of factors of $\xi^\perp$ that appear in the quartic. This filtration has graded pieces of degrees 1, 2, 3, 4, and 5, which corresponds exactly to the filtration of Cartan's tensor $\mathcal{G}$ into the components that he labels E, D_i, C_i, B_i and A_i (with 1, 2, 3, 4, and 5 components, respectively.)

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Duke University Mathematics Department, P.O. Box 90320, Durham, NC 27708-0320

E-mail address: `bryant@math.duke.edu`

URL: `http://www.math.duke.edu/~bryant`