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Schwarzian Derivatives and Differential Equations

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0 Introduction

In this note, we write down explicitly the orbits of simple differential equations by groups of diffeomorphisms or contact diffeomorphisms using multi-dimensional Schwarzian derivatives or contact Schwarzian derivatives. We give a definition of contact Schwarzian derivatives which is equivalent to that of Fox [2]. For the lowest dimensional case, the definition is given in [7], [5].

1 System of second-order PDE's

In the section, we consider the simple system of second-order partial differential equations with m independent variable and n dependent variables

$$\frac{\partial^2 y_k}{\partial x_i \partial x_j} = 0 \qquad \text{for all} \quad 1 \leq i \leq j \leq m, \ 1 \leq k \leq n. \tag{1}$$

The number of equations in (1) is equal to $mn(m+1)/2$. Put

$$\mathbf{y} = (y_1, y_2, \ldots, y_n).$$

Then the system (1) is written simply as

$$\frac{\partial^2 \mathbf{y}}{\partial x_i \partial x_j} = 0 \qquad \text{for all} \quad 1 \leq i \leq j \leq m.$$

Put

$$\mathbf{x} = (x_1, x_2, \dots, x_m), \quad p_{ij} = \frac{\partial y_i}{\partial x_j}\ (1 \le i \le n,\ 1 \le j \le m),$$

and let $\mathbf{p} = (p_{ij})$ be the $n \times m$ matrix whose (ij)-component is equal to p_{ij}.

Let K be $\mathbb{R}$ or $\mathbb{C}$ and let $\phi : K^{m+n} \to K^{m+n}$ be a nondegenerate map (diffeomorphism) given by

$$\phi : \mathbf{z} = (z_1, \dots, z_{m+n}) \to (Z_1, \dots Z_{m+n}).$$

Put $\ell = m + n$. According to Yoshida[13], [14] or Sasaki [6], (multi dimensional) Schwarzian derivative $S^k_{ij}(\phi)$ for $1 \le i, j, k \le \ell$ is given by

$$S^k_{ij}(\phi) = \sum_{p=1}^{\ell} \frac{\partial^2 Z^p}{\partial z^i \partial z^j} \frac{\partial z^k}{\partial Z^p} - \frac{1}{\ell+1} \sum_{p,q=1}^{\ell} \left(\delta^k_i \frac{\partial^2 Z^p}{\partial z^q \partial z^j} \frac{\partial z^q}{\partial Z^p} + \delta^k_j \frac{\partial^2 Z^p}{\partial z^q \partial z^i} \frac{\partial z^q}{\partial Z^p} \right),$$

(cf. also Gunning[3], Kobayashi-Ochiai[4]). Clearly $S^k_{ij}(\phi) = S^k_{ji}(\phi)$ and further they satisfy the canonical-form relation

$$\sum_{k=1}^{\ell} S^k_{ik}(\phi) = 0 \quad \text{for} \quad i = 1, 2, \dots, \ell.$$

So the number of Schwarzian derivatives is $(\ell - 1)\ell(\ell + 2)/2$.

If we regard

$$(z_1, z_2, \dots, z_{m+n}) = (x_1, \dots, x_m, y_1, \dots, y_n),$$

the diffeomorphism ϕ maps the system (1) to another system of partial differential equations

$$\frac{\partial^2 y_k}{\partial x_i \partial x_j} = {f^{\phi}}^k_{ij}(\mathbf{x}, \mathbf{y}, \mathbf{p}) \qquad \text{for all} \quad 1 \le i \le j \le m,\ 1 \le k \le n. \tag{2}$$

where ${f^{\phi}}^k{}_{ij} = {f^{\phi}}^k{}_{ij}(\mathbf{x}, \mathbf{y}, \mathbf{p})$ are smooth functions of $m + n + mn$ variables. The set of the system of equations $\dfrac{\partial^2 y_k}{\partial x_i \partial x_j} = {f^{\phi}}^k{}_{ij}(\mathbf{x}, \mathbf{y}, \mathbf{p})$ for all diffeomorphisms ϕ is the orbit of the system of equations $\dfrac{\partial^2 \mathbf{y}}{\partial x_i \partial x_j} = 0$ by the diffeomorphism group $\mathrm{Diff}(\mathbb{R}^{m+n})$. This is an answer to the problem of the paper [8].

In the following Theorem 1, we write down explicitly the functions ${f^{\phi}}^k{}_{ij}(\mathbf{x}, \mathbf{y}, \mathbf{p})$ by using the Schwarzian derivatives $S^k_{ij}(\phi)$.

The case for $m = n = 1$ is explained in [7] and higher dimensional cases are left unsolved.

Theorem 1 *By the inverse diffeomorphism ϕ^{-1}, the system of 2O-PDE*

$$\left\{\frac{\partial^2 Y_k}{\partial X_i \partial X_j} = 0 \quad (1 \le i \le j \le m,\ 1 \le k \le n)\right\}$$

is mapped to a system of 2O-PDE

$$\left\{\frac{\partial^2 y_k}{\partial x_i \partial x_j} = {f^{\phi}}^k{}_{ij}(\mathbf{x}, \mathbf{y}, \mathbf{p}) \quad (1 \le i \le j \le m,\ 1 \le k \le n)\right\}$$

where

$$\begin{aligned} {f^{\phi}}^k{}_{ij}(\mathbf{x}, \mathbf{y}, \mathbf{p}) = -S^{m+k}_{ij} + \sum_{s=1}^{m} S^s_{ij}\, p_{ks} - \sum_{t=1}^{n} (S^{m+k}_{m+t\, j}\, p_{ti} + S^{m+k}_{i\, m+t}\, p_{tj}) \\ + \sum_{s=1}^{m} \sum_{t=1}^{n} (S^s_{m+t\, j}\, p_{ks}\, p_{ti} + S^s_{i\, m+t}\, p_{ks}\, p_{tj}) - \sum_{s,t=1}^{n} S^{m+k}_{m+t\, m+s}\, p_{ti}\, p_{sj} \\ + \sum_{s=1}^{m} \sum_{u,t=1}^{n} S^s_{m+u\, m+t}\, p_{ks}\, p_{ui}\, p_{tj}. \end{aligned}$$

This is an extension of the following result (cf. [7]).

Example 1 *Let $\phi : K^2 \to K^2$ given by $\phi(x,y) = (X(x,y), Y(x,y))$ be a diffeomorphism. By the inverse diffeomorphism ϕ^{-1},*

$$Y''(X) = 0$$

is mapped to

$$y''(x) = -S^2_{11} + 3S^1_{11}y' - 3S^2_{22}(y')^2 + S^1_{22}(y')^3.$$

2 System of third-order PDE's

In this section, we consider the simple system of third order partial differential equations of n independent variables and one dependent variable

$$\frac{\partial^3 y}{\partial x_i \partial x_j \partial x_k} = 0 \qquad \text{for all} \quad 1 \leq i \leq j \leq k \leq m,$$

where $y = y(x_1, \ldots, x_m)$. The space $K^{2m+1} = \{\mathbf{x}, y, \mathbf{p}\}$ has the natural contact form $\eta = dy - \sum_{i=1}^m p_i dx_i$. For $1 \leq i \leq j \leq m$, put $q_{ij} = \dfrac{\partial^2 y}{\partial x_i \partial x_j}$ and put $\mathbf{q} = \{q_{11}, q_{12}, \ldots, q_{mm}\}$. Then $\mathbf{q}$ consists of $\frac{1}{2}m(m+1)$ functions.

Let $\varphi : K^{2m+1} \to K^{2m+1}$ be a contact diffeomorphism with a nonvanishing function f such that $\varphi^*\eta = f\eta$. The contact diffeomorphism φ maps the equations $\dfrac{\partial^3 y}{\partial x_i \partial x_j \partial x_k} = 0$ to another system of partial differential equations

$$\frac{\partial^3 y}{\partial x_i \partial x_j \partial x_k} = f^{\varphi}{}_{ijk}(\mathbf{x}, y, \mathbf{p}, \mathbf{q}) \quad \text{for} \quad 1 \leq i \leq j \leq k \leq m.$$

Here $f^{\varphi}{}_{ijk}$ are smooth functions of $2m+1+\frac{1}{2}m(m+1) = \frac{m^2+5m+2}{2}$ variables.

We define (multi-dimensional) contact Schwarzian derivatives $C^k_{ij}(\varphi)$ (Definition 2.1) which is equivalent to that of Fox [2]. For $m = 1$, the definition is given in [7], [5].

In the following Theorem 2, we write down explicitly the function $f^{\varphi}{}_{ij}(\mathbf{x}, y, \mathbf{p}, \mathbf{q})$ by using the contact Schwarzian derivatives $C^k_{ij}(\varphi)$. This extends a result of [7] for the case of $m = 1$.

Contact Schwarzian Derivatives

On the contact space $K^{2n+1} = \{(\mathbf{x}, y, \mathbf{p})\}$, the total differential $\frac{d}{dx_k}$ is defined by

$$\frac{d}{dx_k} = \frac{\partial}{\partial x_k} + p_k \frac{\partial}{\partial y}.$$

For a function A, the total differential of A is expresses as

$$A_{;k} = \frac{dA}{dx_k} = A_{x_k} + A_y p_k.$$

Let $\varphi : K^{2n+1} \to K^{2n+1}$ be a contact transformation given by

$$\varphi(x_1, \dots, x_n, y, p_1, \dots, p_n) = (X_1, \dots, X_n, Y, P_1, \dots, P_n).$$

Corresponding to φ, define $(n \times n)$ matrices $X_;$, X_p , $P_;$, P_p by

$$(X_;)_{ij} = X_{i;j}, \quad (X_p)_{ij} = \frac{\partial X_i}{\partial p_j}, \quad (P_;)_{ij} = P_{i;j}, \quad (P_p)_{ij} = \frac{\partial P_i}{\partial p_j},$$

and define $(2n) \times (2n)$-matrix $\mathcal{X}_\varphi$ by

$$\mathcal{X}_\varphi = \begin{pmatrix} X_; & X_p \\ P_; & P_p \end{pmatrix}.$$

Then $(\mathcal{X}_\varphi)^{-1} = \mathcal{X}_{\varphi^{-1}} = \begin{pmatrix} x_; & x_P \\ p_; & p_P \end{pmatrix}$. For $1 \le r, s \le 2n$, let α_{rs} be the (r, s)-component of the matrix $\mathcal{X}_\varphi$; $\alpha_{rs} = (\mathcal{X}_\varphi)_{rs}$. Let β_{rs} be the (r, s)-component of the matrix $\mathcal{X}_{\varphi^{-1}}$; $\beta_{rs} = (\mathcal{X}_{\varphi^{-1}})_{rs}$. Then β_{st} is equal to α_{st} with replacing small letters x, p with capital letters X, P.

For $1 \le r, s, t \le 2n$, put

$$\alpha_{rst} = \begin{cases} \alpha_{rs;t} & if\ 1 \le t \le n \\ \frac{\partial \alpha_{st}}{\partial p_{t-n}} & if\ n \le t \le 2n \end{cases}, \qquad \mu^t_{r\,s} = \sum_{u=1}^{2n} \beta_{tu}\alpha_{urs}$$

Then, for $1 \le i, j, k \le 2n$, the contact Schwarzian derivatives $C^k_{i\,j} = C^k_{i\,j}(\varphi)$ is defined by

$$C^k_{ij} = \frac{1}{2}\left(\mu^k_{i\,j} + \mu^k_{j\,i}\right) - \frac{1}{2(2n+1)}\sum_{s=1}^{2n}(\delta_{ik}\mu^s_{j\,s} + \delta_{ik}\mu^s_{s\,j} + \delta_{jk}\mu^s_{i\,s} + \delta_{jk}\mu^s_{s\,i}).$$

Now we have the following theorem;

Theorem 2 *By the inverse contact diffeomorphism φ^{-1}, the system of 3O-PDE*

$$\left\{\frac{\partial^3 Y}{\partial X_i \partial X_j \partial X_k} = 0 \quad (1 \le i \le j \le k \le m)\right\}$$

is mapped to a system of 3O-PDE

$$\left\{\frac{\partial^3 y}{\partial x_i \partial x_j \partial x_k} = f^{\varphi}{}_{ijk}(\mathbf{x}, y, \mathbf{p}, \mathbf{q}) \quad (1 \le i \le j \le k \le m)\right\}$$

where

$$f^{\varphi}{}_{ijk}(\mathbf{x}, y, \mathbf{p}, \mathbf{q})$$

$$= -C^{m+i}_{j\,k} + \sum_{\ell=1}^{m} C^{\ell}_{j\,k} q_{i\ell} - \sum_{\ell=1}^{m} C^{m+i}_{m+\ell\,k} q_{j\ell} - \sum_{\ell=1}^{m} C^{m+i}_{j\,m+\ell} q_{\ell k}$$

$$+ \sum_{\ell,r=1}^{m} C^{\ell}_{j\,m+r} q_{i\ell} q_{rk} + \sum_{\ell,r=1}^{m} C^{\ell}_{m+r\,k} q_{i\ell} q_{rj} - \sum_{\ell,r=1}^{m} C^{m+i}_{m+\ell\,m+r} q_{\ell j} q_{rk} + \sum_{\ell,r,s=1}^{m} C^{s}_{m+\ell\,m+r} q_{is} q_{\ell j} q_{rk}.$$

This is an extension of the following result ([7],[5]).

Example 2 *Let* $\varphi : K^3 \to K^3$ *given by* $\varphi(x,y,p) = (X(x,y,p), Y(x,y,p), P(x,y,p))$ *be a contact diffeomorphism. By the inverse contact diffeomorphism* φ^{-1},

$$Y'''(X) = 0$$

is mapped to

$$y'''(x) = -C^2_{11} + 3C^1_{11}y'' - 3C^2_{22}(y'')^2 + C^1_{22}(y'')^3.$$

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