Schwarzian Derivatives and Differential Equations

名古屋大学・多元数理科学研究科 佐藤 輯 (Hajime Sato)
名古屋大学・多元数理科学研究科 鈴木 浩志 (Hiroshi Suzuki)
Graduate School of Mathematics,
Nagoya University

0 Introduction

In this note, we write down explicitly the orbits of simple differential equations by groups of diffeomorphisms or contact diffeomorphisms using multi-dimensional Schwarzian derivatives or contact Schwarzian derivatives. We give a definition of contact Schwarzian derivatives which is equivalent to that of Fox [2]. For the lowest dimensional case, the definition is given in [7], [5].

1 System of second-order PDE's

In the section, we consider the simple system of second-order partial differential equations with $m$ independent variable and $n$ dependent variables

$$\frac{\partial^2 y_k}{\partial x_i \partial x_j} = 0 \quad \text{for all} \quad 1 \leq i \leq j \leq m, \ 1 \leq k \leq n. \quad (1)$$

The number of equations in (1) is equal to $mn(m+1)/2$. Put

$$y = (y_1, y_2, \ldots, y_n).$$

Then the system (1) is written simply as

$$\frac{\partial^2 y}{\partial x_i \partial x_j} = 0 \quad \text{for all} \quad 1 \leq i \leq j \leq m.$$
Put
\[ x = (x_1, x_2, \ldots, x_m), \quad p_{ij} = \frac{\partial y_i}{\partial x_j} \quad (1 \leq i \leq n, \ 1 \leq j \leq m), \]
and let \( p = (p_{ij}) \) be the \( n \times m \) matrix whose \((ij)\)-component is equal to \( p_{ij} \).

Let \( K \) be \( \mathbb{R} \) or \( \mathbb{C} \) and let \( \phi : K^{m+n} \to K^{m+n} \) be a nondegenerate map (diffeomorphism) given by
\[ \phi : z = (z_1, \ldots, z_{m+n}) \to (Z_1, \ldots Z_{m+n}). \]

Put \( \ell = m + n \). According to Yoshida[13], [14] or Sasaki [6], (multi dimensional) Schwarzian derivative \( S_{ij}^k(\phi) \) for \( 1 \leq i, j, k \leq \ell \) is given by
\[ S_{ij}^k(\phi) = \sum_{p=1}^{\ell} \frac{\partial^2 Z^p}{\partial z^i \partial z^j} \frac{\partial z^k}{\partial Z^p} - \frac{1}{\ell + 1} \sum_{p,q=1}^{\ell} \left( \delta^k_j \frac{\partial^2 Z^p}{\partial z^q \partial z^j} \frac{\partial z^q}{\partial Z^p} + \delta^k_i \frac{\partial^2 Z^p}{\partial z^i \partial z^q} \frac{\partial z^q}{\partial Z^p} \right), \]
(cf. also Gunning[3], Kobayashi-Ochiai[4]). Clearly \( S_{ij}^k(\phi) = S_{ji}^k(\phi) \) and further they satisfy the canonical-form relation
\[ \sum_{k=1}^{\ell} S_{ik}^k(\phi) = 0 \quad \text{for} \quad i = 1, 2, \ldots, \ell. \]
So the number of Schwarzian derivatives is \((\ell - 1)\ell(\ell + 2)/2\).

If we regard
\[ (z_1, z_2, \ldots, z_{m+n}) = (x_1, \ldots, x_m, y_1, \ldots, y_n), \]
the diffeomorphism \( \phi \) maps the system (1) to another system of partial differential equations
\[ \frac{\partial^2 y_k}{\partial x_i \partial x_j} = f^\phi_{ij}(x, y, p) \quad \text{for all} \quad 1 \leq i \leq j \leq m, \ 1 \leq k \leq n. \]
where $f^{\phi k}_{ij} = f^{\phi k}_{ij}(x, y, p)$ are smooth functions of $m + n + mn$ variables. The set of the system of equations $\frac{\partial^2 y_k}{\partial x_i \partial x_j} = f^{\phi k}_{ij}(x, y, p)$ for all diffeomorphisms $\phi$ is the orbit of the system of equations $\frac{\partial^2 y}{\partial x_i \partial x_j} = 0$ by the diffeomorphism group $\text{Diff}(\mathbb{R}^{m+n})$. This is an answer to the problem of the paper [8].

In the following Theorem 1, we write down explicitly the functions $f^{\phi k}_{ij}(x, y, p)$ by using the Schwarzian derivatives $S^k_{ij}$. The case for $m = n = 1$ is explained in [7] and higher dimensional cases are left unsolved.

**Theorem 1** By the inverse diffeomorphism $\phi^{-1}$, the system of 2O-PDE

\[
\left\{ \frac{\partial^2 Y_k}{\partial X_i \partial X_j} = 0 \quad (1 \leq i \leq j \leq m, \ 1 \leq k \leq n) \right\}
\]

is mapped to a system of 2O-PDE

\[
\left\{ \frac{\partial^2 y_k}{\partial x_i \partial x_j} = f^{\phi k}_{ij}(x, y, p) \quad (1 \leq i \leq j \leq m, \ 1 \leq k \leq n) \right\}
\]

where

\[
f^{\phi k}_{ij}(x, y, p) = -S^m_{ij} + \sum_{s=1}^{m} S^{s}_{ij} p_{ks} - \sum_{t=1}^{n} (S^{m+k}_{m+tj} p_{ti} + S^{m+k}_{im+t} P_{tj})
\]

\[
+ \sum_{s=1}^{m} \sum_{t=1}^{n} (S^{s}_{m+tj} p_{ks} p_{ti} + S^{s}_{i m+t} P_{ks} P_{tj}) - \sum_{s,t=1}^{n} S^{m+k}_{m+t m+s} P_{ti} P_{sj}
\]

\[
+ \sum_{s=1}^{m} \sum_{u,t=1}^{n} S^{s}_{m+u+m+t} P_{ks} P_{ui} P_{tj}.
\]

This is an extension of the following result (cf. [7]).
Example 1 Let $\phi : K^2 \rightarrow K^2$ given by $\phi(x, y) = (X(x, y), Y(x, y))$ be a diffeomorphism. By the inverse diffeomorphism $\phi^{-1}$,

$$Y''(X) = 0$$

is mapped to

$$y''(x) = -S_{11}^2 + 3S_{11}^1 y' - 3S_{22}^2 (y')^2 + S_{22}^1 (y')^3.$$  

2 System of third-order PDE's

In this section, we consider the simple system of third order partial differential equations of $n$ independent variables and one dependent variable

$$\frac{\partial^3 y}{\partial x_i \partial x_j \partial x_k} = 0 \quad \text{for all} \quad 1 \leq i \leq j \leq k \leq m,$$

where $y = y(x_1, \ldots, x_m)$. The space $K^{2m+1} = \{x, y, p\}$ has the natural contact form $\eta = dy - \sum_{i=1}^{m} p_i dx_i$. For $1 \leq i \leq j \leq m$, put $q_{ij} = \frac{\partial^2 y}{\partial x_i \partial x_j}$ and put $q = \{q_{11}, q_{12}, \ldots, q_{mm}\}$. Then $q$ consists of $\frac{1}{2} m(m+1)$ functions.

Let $\varphi : K^{2m+1} \rightarrow K^{2m+1}$ be a contact diffeomorphism with a nonvanishing function $f$ such that $\varphi^* \eta = f \eta$. The contact diffeomorphism $\varphi$ maps the equations $
abla^3 y \nabla x_i \partial x_j \partial x_k = 0$ to another system of partial differential equations

$$\frac{\partial^3 y}{\partial x_i \partial x_j \partial x_k} = f^\varphi_{ijk}(x, y, p, q) \quad \text{for} \quad 1 \leq i \leq j \leq k \leq m.$$  

Here $f^\varphi_{ijk}$ are smooth functions of $2m+1+\frac{1}{2} m(m+1) = \frac{m^2+5m+2}{2}$ variables.
We define (multi-dimensional) contact Schwarzian derivatives $C_{ij}^{k}(\varphi)$ (Definition 2.1) which is equivalent to that of Fox [2]. For $m = 1$, the definition is given in [7], [5].

In the following Theorem 2, we write down explicitly the function $f^\varphi_{ij}(x, y, p, q)$ by using the contact Schwarzian derivatives $C_{ij}^{k}(\varphi)$. This extends a result of [7] for the case of $m = 1$.

**Contact Schwarzian Derivatives**

On the contact space $K^{2n+1} = \{(x, y, p)\}$, the total differential $\frac{d}{dx_k}$ is defined by

$$\frac{d}{dx_k} = \frac{\partial}{\partial x_k} + p_k \frac{\partial}{\partial y}.$$

For a function $A$, the total differential of $A$ is expresses as

$$A_{;k} = \frac{dA}{dx_k} = A_{x_k} + A_y p_k.$$

Let $\varphi : K^{2n+1} \rightarrow K^{2n+1}$ be a contact transformation given by

$$\varphi(x_1, \ldots, x_n, y, p_1, \ldots, p_n) = (X_1, \ldots, X_n, Y, P_1, \ldots, P_n).$$

Corresponding to $\varphi$, define $(n \times n)$ matrices $X_\cdot$, $X_p$, $P_\cdot$, $P_p$ by

$$(X_\cdot)_{ij} = X_{i;j}, \quad (X_p)_{ij} = \frac{\partial X_i}{\partial p_j}, \quad (P_\cdot)_{ij} = P_{i;j}, \quad (P_p)_{ij} = \frac{\partial P_i}{\partial p_j},$$

and define $$(2n) \times (2n)$$-matrix $\mathcal{X}_\varphi$ by

$$\mathcal{X}_\varphi = \begin{pmatrix} X_\cdot & X_p \\ P_\cdot & P_p \end{pmatrix}.$$ Then $(\mathcal{X}_\varphi)^{-1} = \mathcal{X}_{\varphi^{-1}} = \begin{pmatrix} x_\cdot & x_p \\ p_\cdot & p_p \end{pmatrix}$. For $1 \leq r, s \leq 2n$, let $\alpha_{rs}$ be the $(r, s)$-component of the matrix $\mathcal{X}_\varphi$; $\alpha_{rs} = (\mathcal{X}_\varphi)_{rs}$. Let $\beta_{rs}$ be the $(r, s)$-component of the matrix $\mathcal{X}_{\varphi^{-1}}$; $\beta_{rs} = (\mathcal{X}_{\varphi^{-1}})_{rs}$. Then $\beta_{st}$ is equal to $\alpha_{st}$ with replacing small letters $x, p$ with capital letters $X, P$.
For $1 \leq r, s, t \leq 2n$, put
\[ \alpha_{rst} = \begin{cases} \frac{\alpha_{rs} \partial \alpha}{\partial p_{t-n}} & \text{if } 1 \leq t \leq n \\ \beta_{tu} \alpha_{urs} & \text{if } n \leq t \leq 2n \end{cases}, \quad \mu_{r,s}^{t} = \sum_{u=1}^{2n} \sqrt{t-u} \alpha_{u,rs} \]

Then, for $1 \leq i, j, k \leq 2n$, the contact Schwarzian derivatives $C_{ij}^{k} = C_{ij}^{k}(\varphi)$ is defined by
\[ C_{ij}^{k} = \frac{1}{2} (\mu_{ij}^{k} + \mu_{ji}^{k}) - \frac{1}{2(2n+1)} \sum_{s=1}^{2n} (\delta_{ik} \mu_{j,s}^{s} + \delta_{ik} \mu_{s,j}^{s} + \delta_{jk} \mu_{i,s}^{s} + \delta_{jk} \mu_{s,i}^{s}). \]

Now we have the following theorem;

**Theorem 2** By the inverse contact diffeomorphism $\varphi^{-1}$, the system of 3O-PDE
\[ \left\{ \frac{\partial^{3}Y}{\partial X_{i} \partial X_{j} \partial X_{k}} = 0 \quad (1 \leq i \leq j \leq k \leq m) \right\} \]

is mapped to a system of 30-PDE
\[ \left\{ \frac{\partial^{3}y}{\partial x_{i} \partial x_{j} \partial x_{k}} = f_{ijk}^{\varphi}(x, y, p, q) \quad (1 \leq i \leq j \leq k \leq m) \right\} \]

where
\[ f_{ijk}^{\varphi}(x, y, p, q) = -C_{jk}^{m+i} + \sum_{\ell=1}^{m} C_{j,\ell}^{m+i} q_{\ell k} - \sum_{\ell=1}^{m} C_{m+j,\ell}^{m+i} q_{\ell k} \]
\[ + \sum_{\ell,r=1}^{m} C_{j,m+r,\ell} q_{\ell k} q_{rk} + \sum_{\ell,r=1}^{m} C_{m+r,\ell} q_{ij} q_{rk} + \sum_{\ell,r,s=1}^{m} C_{m+\ell m+r}^{s} q_{is} q_{\ell j} q_{rk}. \]

This is an extension of the following result ([7],[5]).
Example 2 Let $\varphi : K^3 \rightarrow K^3$ given by $\varphi(x, y, p) = (X(x, y, p), Y(x, y, p), P(x, y, p))$ be a contact diffeomorphism. By the inverse contact diffeomorphism $\varphi^{-1}$,

$$Y'''(X) = 0$$

is mapped to

$$y'''(x) = -C_{11}^2 + 3C_{11}^1 y'' - 3C_{22}^2 (y'')^2 + C_{22}^1 (y'')^3.$$ 

References


