数理解析研究所講究録
1502 巻 2006 年 125-139

Goursat equations and twistor theory —two dualities—

待田 芳徳 (沼津高専)
Yoshinori MACHIDA (Numazu College of Technology)

Introduction.

1. Motivation.

We consider a second order partial differential equation

$$F(x, y, z, p, q, r, s, t) = 0$$

with two independent variables x, y of $\mathbb{R}^2$ (or $\mathbb{C}^2$), a dependent variable z of x, y, and $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}$.

Goursat ([G]) considered the so-called Gousat equations (see I.1.1.). And then Cartan ([C], cf.[T],[Y]) showed the followings: for a Goursat equation

$$\text{(i) } 9r^2 + 12t^2(rt - s^2) + 32s^3 - 36rst = 0$$

and an involutive system

$$\text{(ii) } r = \frac{1}{3}t^3, \quad s = \frac{1}{2}t^2,$$

1: the surface (i) is the tangent developable of the space curve (ii) in (r, s, t)-space.

2: the symmetry of infinitesimal contact transformations of (i) and (ii) is the Lie algebra of the type exceptional Lie group G_2' of non-compact (split) type

Then we have the following questions: How and where are the equation (i) and the symmetry of (i) derived from ? What is the essence and the universality of Goursat equations ?

In this note, we look for the answers to these questions by using two dualities in twistor theory : *Lagrange-Grassmann duality* and *Cartan-Legendre duality.*

2. Important viewpoints.

- Reduction to the system of first order ODE.

We have an analogy of Monge geodesics and a Monge flow to geodesics and a geodesic flow.

- Application of twistor theory.

Assuming that the automorphism group is of finite type, in particular A, BD, exceptional types, we can construct the normal Cartan connections and can get lifting theorems and reduction theorems in the twistor theory.

- Another generalization of Monge-Ampère equations of parabolic type.

One generalization of Monge-Ampère equations of parabolic type is decomposable Monge-Ampère systems of Lagrange type ([M-M]). As another generalization, we have Goursat equations.

- Connections with Wolf spaces, Gray spaces.

The twistor diagrams in complex category explained in I.1.2 have relation to Wolf spaces and Gray spaces. A Wolf space X^{4n} is a compact homogeneous quaternion-Kähler manifold with positive curvature. This space X has two kinds of twistor spaces. The one is $M_{\mathbb{C}}^{2n+1}$ called the Salamon twistor space which appears in 3.1. The other is a Gray space $Y_{\mathbb{C}}^{3(n-1)}$ via $N_{\mathbb{C}}^{3n-1}$, which is a compact homogeneous nearly-Kähler manifold with a non-integrable complex structure.

For lack of space, in this note we deal with briefly : constructions of equations in PART I and constructions of solutions in PART II. Details will be appeared elsewhere.

PART I. Constructions of equations

1. Goursat equations.

1.1. Definition.

Let
$$F(x_i, z, p_i, p_{ij}) = 0$$
be a single second order partial differential equation (briefly, a 2nd order PDE). Here x_i are n independent variables of $\mathbb{R}^n$ (or $\mathbb{C}^n$), z a dependent variable of x_i and $p_i = \frac{\partial z}{\partial x_i}, p_{ij} = \frac{\partial^2 z}{\partial x_i \partial x_j}$.

A 2nd order PDE $F(x_i, z, p_i, p_{ij}) = 0$ is called a *Goursat equation* if the followings are satisfied:

(i) the rank of the $n \times n$ matrix $(\frac{\partial F}{\partial p_{ij}})$ is 1, that is, any 2×2 minor is divided by F.

(ii) the Monge characteristic system D (which is defined below) is completely integrable.

If $F = 0$ is of 2 independent variables, the condition (i) means that $F = 0$ is a parabolic type. In (ii), as an example $F = p_{11} = \frac{\partial^2 z}{\partial x_1^2} = 0$, a general solution has a form $z = f(x_2, \cdots, x_n)x_1 + g(x_2, \cdots, x_n)$ and D is spanned by $D = \langle \frac{d}{dx_1}, \frac{\partial}{\partial p_{ij}}{}_{2 \leq i,j \leq n} \rangle$, where $\frac{d}{dx_1} = \frac{\partial}{\partial x_1} + p_1 \frac{\partial}{\partial z} + \sum_{j=1}^n p_{1j} \frac{\partial}{\partial p_j}$.

Let $J^i(\mathbb{R}^n, \mathbb{R})$ be the i-jet space of i-th derivatives of 1-function on $\mathbb{R}^n$. We have the canonical projections:

$$J^2(\mathbb{R}^n, \mathbb{R}) : (x_i, z, p_i, p_{ij}) \longrightarrow J^1(\mathbb{R}^n, \mathbb{R}) : (x_i, z, p_i) \longrightarrow J^0(\mathbb{R}^n, \mathbb{R}) : (x_i, z).$$

The 1-jet space $J^1(\mathbb{R}^n,\mathbb{R})$ has the standard contact structure $\omega = dz - \sum_{i=1}^n p_i dx_i$ and the 2-jet space $J^2(\mathbb{R}^n,\mathbb{R})$ has the standard second contact structure $\omega_0 = dz - \sum_{i=1}^n p_i dx_i, \omega_i = dp_i - \sum_{j=1}^n p_{ij} dx_j$ $(i = 1,\cdots,n)$.

The symbol algebra of $J^2(\mathbb{R}^n,\mathbb{R})$ is

$$\mathfrak{c}^2 = \mathbb{R} \oplus V^* \oplus (V \oplus S^2(V^*)) = \langle \frac{\partial}{\partial z}\rangle \oplus \langle \frac{\partial}{\partial p_i}\rangle \oplus (\langle \frac{d}{dx_i}\rangle \oplus \langle \frac{\partial}{\partial p_{ij}}\rangle),$$

where $\frac{d}{dx_i} = \frac{\partial}{\partial x_i} + p_i \frac{\partial}{\partial z} + \sum_{j=1}^n p_{ij} \frac{\partial}{\partial p_j}$. The dual basis consists of

$$\omega_o \leftrightarrow \frac{\partial}{\partial z},\ \omega_i \leftrightarrow \frac{\partial}{\partial p_i},\ dx_i \leftrightarrow \frac{d}{dx_i},\ dp_{ij} \leftrightarrow \frac{\partial}{\partial p_{ij}}.$$

The distribution $\mathfrak{c}_{-1} = \langle \frac{d}{dx_i}\rangle \oplus \langle \frac{\partial}{\partial p_{ij}}\rangle$ is the annihilation of ω_0, ω_i.

A second order PDE $F = 0$ defines a submanifold $L = F^{-1}(0) = \{F = 0\} \subset J^2$ of codimension 1. We assume that the projection $L \longrightarrow J^1$ is an onto-mapping. Because of a single equation, a subspace $\mathfrak{f} \subset S^2(V^*)$ of codimension 1 is defined in the tangent space at each point of $L \subset J^2$. Dualizing, we have the 1-dimensional subspace $\mathfrak{f}^\perp \subset S^2(V)$. Because of rank $(\frac{\partial f}{\partial p_{ij}}) = 1$ for a Goursat equation, there exists a vector $e \in V$ such that $\mathfrak{f}^\perp = \langle e^2\rangle$. Then we define a *Monge characteristic syatem* D, which is a distribution on L, by

$$D = E \oplus S^2(E^\perp)\ (\subset \mathfrak{c}_{-1}).$$

Here $E = \langle e\rangle$ and $\dim S^2(E^\perp) = \frac{n(n-1)}{2}$.

Because of complete integrability of the Monge characteristic syatem D for a Goursat equation, the leaf space $R = \{F = 0\}/\mathcal{D}$ is locally a manifold. The symbol algebra of $R = \{F = 0\}/\mathcal{D}$ is

$$\mathfrak{m} = W \oplus U \oplus W \otimes U^* \cong J^1(n-1, 2),$$

where $\dim W = 2$, $\dim U = n-1$, $\dim W \otimes U^* = 2(n-1)$. The distribution $U \oplus W \otimes U^*$ is the annihilation of some two 1-forms ϖ_0 (induced by ω_0), ϖ_1.

1.2. Twistor diagram.

Since $D' = S^2(E^\perp) \cong D/E$ is completely integrable, the leaf space $\tilde{R} = \{F = 0\}/\mathcal{D}'$ is locally a manifold. In consideration of the symbol algebra, we have the projections:

$$\{F=0\} \longrightarrow \tilde{R} = \{F=0\}/\mathcal{D}',\quad \tilde{R} \longrightarrow R = \{F=0\}/\mathcal{D},$$
$$\{F=0\} \longrightarrow \tilde{R} = \{F=0\}/\mathcal{D}',\quad \tilde{R} \longrightarrow J^1.$$

Therefore we have the double fibering called the twistor diagram of a Goursat equation:

$$\begin{array}{ccc} & \tilde{R}^{3n} = \{F=0\}/\mathcal{D}' & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ (J^1)^{2n+1} & & R^{3n-1} = \{F=0\}/\mathcal{D}. \end{array}$$

We have a flow called a *Monge flow* on $\tilde{R}$ induced by $E = \langle e \rangle$ on $L = \{F = 0\}$. The space $\tilde{R}$ equipped with the Monge flow is called a *Monge structure.* The space R is the orbit space of the Monge flow on $\tilde{R}$. The total space $\tilde{R}$ is also regarded as a fiber bundle called a *Monge direction bundle* over J^1 with an $(n-1)$-dimensional fiber consisting of direction fields of the Monge flow.

Projecting the Monge flow on $\tilde{R}$ to J^1, we have a unique curve called a *Monge geodesic* such that, for any Monge direction at each point of J^1, a curve passes through the direction at the point. Projecting the fiber $(\subset \tilde{R})$ over J^1 to R, we have an $(n-1)$-dimensional manifold called a *Goursat surface* with 1-dimensional parameter at each point of R.

1.3. Solutions.

A solution surface $S \subset L = \{F = 0\}$ is an n-dimensional integral surface of $\omega_o = \omega_i = 0$. For a Goursat equation, a solution surface has $\dim(T_pS \cap D_p) = 1$ $(p \in S)$.

A solution surface $\tilde{S} \subset \tilde{R}$ is an n-dimensional integral surface of $\varpi_o = \varpi_1 = 0$. It is generated by the 1-dimensional Monge flow. This property characterizes a solution surface of a Goursat equation. It follows that $\pi_1(\tilde{S}) \subset J^1$ is a Legendre subvariety.

1.4. Isomorphism and automorphism.

For the isomorphisms of Goursat equations (cf.[T],[Y]), we have

$$\begin{aligned} &L = \{F = 0\} \sim L' = \{F' = 0\} : \text{contact equivalence} \\ \Longleftrightarrow &\tilde{R}^{3n} \sim \tilde{R'}^{3n} : \text{distribution equivalence} \\ \Longleftrightarrow &R^{3n-1} \sim R'^{3n-1} : \text{distribution equivalence.} \end{aligned}$$

In general, the automorphism of a Goursat equation is of infinite type. We restrict to the situations where the symmetry groups are simple Lie groups of A, BD, exceptional types (no C type). Then the automorphisms are of finite type. We can construct normal Cartan connections and invariants with respect to curvatures.

For the symbol algebra $\mathfrak{m} = W \oplus U \oplus W \otimes U^*$ of $R^{3n-1} = \{F = 0\}/\mathcal{D}$, the automorphism preserving the distribution $U \oplus W \otimes U^*$ of corank 2 is of infinite type and the automorphism preserving the distribution $W \otimes U^*$ is of finite type. As an example, in a 5-dimensional manifold, the automorphism of type $(3, 5)$ distribution is of infinite type and the automorphism of type $(2, 3, 5)$ distribution (called Cartan distribution) is of finite type (G_2' type). Cf. [M].

2. Constructions of Goursat equations.

2.1. Legendre cone fields in contact structures, cf. cone fields in CHSS (compact Hermitian symmetric spaces).

Let (M, D) be a contact manifold M of dimension $2n+1$ with a contact structure D. Take a contact form θ such that $\theta = \mathrm{Ker}D$. Let $K(\subset D)$ be an n-dimensional Legendre cone field. For a point $m \in M$, D_m is a symplectic vector space with a symplectic form $d\theta$. Accordingly we can consider a symplectic vector space $(V \cong \mathbb{R}^{2n}(\text{or } \mathbb{C}^{2n}), \Omega)$, which we also regard as a manifold, and consider an n-dimensional Lagrange cone K, that is, an $\mathbb{R}^*$-(or $\mathbb{C}^*$-)invariant Lagrange submanifold.

2.2. Lagrange-Grassmann duality, cf. projective duality and Grassmann duality.

The 2-jet space $J^2(\mathbb{R}^n, \mathbb{R})$ is a fiber bundle over the 1-jet space $J^1(\mathbb{R}^n, \mathbb{R})$. As a generalization, we have a Lagrange Grassmann bundle $L(M)$ with fiber the Lagrange Grassmann manifold of each contact distribution over a contact manifold M.

Let V_1 be a 1-dimensional (isotropic) subspace and V_n a (n-dimensional) Lagrange subspace of V. We have the following twistor diagram called Lagrange-Grassmann duality:

$$\begin{array}{ccc} & I_L = \{V_1 \subset V_n\} & \\ L^{\frac{n(n-1)}{2}} \swarrow & & \searrow P^{n-1} \\ P^{2n-1} = \{V_1\} & & LG^{\frac{n(n+1)}{2}} = \{V_n\} \end{array}$$

The projective space P^{2n-1} has the standard contact structure. The Lagrange Grassmann manifold LG has the standard symmetric matrix coordinates (a_{ij}), $a_{ij} = a_{ji}$.

Fix V_n. Then the set of all V_1 included in V_n is a Legendre plane P^{n-1} of P^{2n-1}. The space I_L is a fiber bundle with fiber P^{n-1} over LG. In other words, LG is regarded as the moduli space of all Legendre planes $\{P^{n-1}\}$. Next, fix V_1. Then the set of all V_n which includes V_1 is an $\frac{n(n-1)}{2}$-dimensional "hyper"plane L. The set of all Legendre planes P^{n-1} through a point in P^{2n-1} is interpreted as an L in LG dually. The set of all L through a point in LG is interpreted as a Legendre plane P^{n-1} in P^{2n-1} dually.

2.3. Intersection variety, cf. tangent variety.

Let $X = P(K)$ be an $(n-1)$-dimensional projectified Lagrange cone in P^{2n-1}. We consider all Legendre planes P^{n-1} across X transversally. Considering dually, we have the two manifolds (varieties).

The one is the $(n-1)$-dimensional dual space $\tilde{X}$ of X obtained by taking the tangent space, which is a Legendre plane, at each point in X.

The other is the space $Z(X)$, called a intersection variety, consisting of all Legendre planes P^{n-1} across X transversally. We see that the space $Z(X)$ is a (codim 1) hypersurface by counting dimensions $n - 1 + \frac{n(n-1)}{2} = \frac{n(n+1)}{2} - 1$ and it is a ruling space of L. Rf. [G-K-Z].

2.4. Resultant, cf. discriminant.

As the intersection variety $Z(X)$ is a hypersurface in LG, $Z(X)$ may be represented by a single definig equation $f(a_{ij}) = 0$. By definition, $f(a_{ij}) = 0$ is nothing but a resultant between X and P^{n-1}.

2.5. Goursat equation, cf. parabolic (degenerate) Monge-Ampère equation.

We will show that the resultant $f(a_{ij}) = 0$ has rank $(\frac{\partial f}{\partial a_{ij}}) = 1$.

For $p \in X = P(K)$, take a Legendre plane $\mathcal{L} = P^{n-1}$ such that

$$T_p X \cap \mathcal{L} = \{p\}.$$

By the projectification of V to $P(V) = P^{2n-1}$, we consider $K, \bar{\mathcal{L}}, \bar{p} \subset V$ such that $K \longrightarrow X, \bar{\mathcal{L}} \longrightarrow \mathcal{L}, \bar{p} \longrightarrow p$. Then we have

$$T_{\bar{p}} K \cap \bar{\mathcal{L}} = \langle \dot{\bar{p}} \rangle.$$

In consideration of

$$T_{\bar{\mathcal{L}}} Z(X) \subset T_{\bar{\mathcal{L}}} \mathrm{Gr}(n, 2n) \cong \mathrm{Hom}(\bar{\mathcal{L}}, V/\bar{\mathcal{L}}),$$

$T_{\bar{\mathcal{L}}} Z(X)$ is regarded as the set of symmetric $n \times n$ matrices with rank $n-1$.

From linear algebra, if A is an $n \times n$ matrix such that rank $n-1$, then the cofactor matrix $\tilde{A}$ of A has rank 1.

Therefore it follows that

$$\mathrm{rank}(\frac{\partial f}{\partial a_{ij}}) = 1.$$

Next expand the above argument from one point $m \in M$ to the whole M. Then from the resultant $f(a_{ij}) = 0$ with $\mathrm{rank}(\frac{\partial f}{\partial a_{ij}}) = 1$, we have a Goursat equation $f(p_{ij}) = 0$.

We also have the following twistor diagram:

$$\begin{array}{ccc} & Q^{3n} & \\ P(K) \swarrow & & \searrow P^1 \\ M^{2n+1} & & N^{3n-1} \end{array}$$

Summarizing the above arguments, we have the following.

Theorem 1. *Let (M, D) be a contact manifold M of dimension $2n+1$ with a contact structure D and $K (\subset D)$ a Legendre cone field. Then, by the above construction via*

$$\text{(i) } K \text{ (cone)}, \text{(ii) } X = P(K) \text{ (projective cone)}, \text{(iii) } \tilde{X} \text{ (dual space)},$$
$$\text{(iv) } Z(X) \text{ (intersection variety)}, \text{(v) } f(a_{ij}) = 0 \text{ (resultant)},$$

we have a Goursat equation

$$f(p_{ij}) = 0.$$

The dual space $\tilde{X}$ may have relation to an involutive system.

3. Automorphisms of finite type.

3.1. Homogeneous Ansatz.

In a twistor diagram

$$\begin{array}{ccc} & Q^{3n} & \\ P(K) \swarrow & & \searrow P^1 \\ M^{2n+1} & & N^{3n-1} \end{array}$$

we assume that a group G_0 acts on $X = P(K)$ transitively and G, which is the prolongation of G_0, also acts on M, Q, N transitively. We assume that G is a simple Lie group of A, BD, or exceptional type.

3.2. Contact structures of finite type.

We study contact structures of finite type for three equivalent manners:
(i) infinitesimal automorphisms $\mathfrak{g}_0, \mathfrak{g}$,
(ii) ($2n$-dimensional) contact distribution D,
(iii) (($n-1$)-dimensional) projectified n-dimensional cubic cone $P(K)$
Remark that C type contact structure, which is called the projective contact structure, has no cubic cone structure. Compare to various dimensional cone strucutres in CHSS which is the first kind flag manifolds (e.g., quadratic cone structures in conformal structures).

∘ A type called the Lagrange contact structure

(i) $\mathfrak{g}_0 = \mathfrak{gl}(n, \mathbb{C}) \oplus \mathbb{C}$
$\mathfrak{g} = \mathfrak{sl}(n+2, \mathbb{C}) = \mathfrak{g}_{-2}$ (dim $= 1$) $\oplus\, \mathfrak{g}_{-1}$ (dim $= 2n$) $\oplus\, \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$
(ii) $D = D_1 \oplus D_2$ (D_i: Lagrangian)
(iii) $P^{n-1}(D_1) \cup P^{n-1}(D_2)$ (D_i: degenerate cones)

∘ BD type called the Lie contact structure

(i) $\mathfrak{g}_0 = \mathfrak{o}(n, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathbb{C}$
$\mathfrak{g} = \mathfrak{o}(n+4, \mathbb{C}) = \mathfrak{g}_{-2}$ (dim $= 1$) $\oplus\, \mathfrak{g}_{-1}$ (dim $= 2n$) $\oplus\, \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$
(ii) $D = W \otimes V$ (rank $W = 2$, rank $V = n$ for (V, g))
(iii) $P^1(W) \times Q^{n-2}(V)$ (null) $\subset P^1(W) \times P^{n-1}(V) \hookrightarrow P^{2n-1}(W \otimes V)$
It is called a Segre cone, which is reducible (linear + quadratic).

∘ exceptional type — cubic cones are irreducible.

· G_2 type

(i) $\mathfrak{g}_0 = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathbb{C}$, $\mathfrak{g} = \mathfrak{g}_2^{\mathbb{C}}$, $\dim(\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}) = 5 = 1 + 4$
(ii) $D = S^3(V)$ (rank $V = 2$)
(iii) $P^1(V)$ called twisted cubic curve, Veronese curve.

We omit the F_4, E_6, E_7, E_8 types.

4. Explicit examples of Goursat equations.

In the Lagrange-Grassmann duality

$$\begin{array}{ccc} & I_L = \{V_1 \subset V_n\} & \\ \swarrow & & \searrow \\ P^{2n-1} = \{V_1\} & & LG^{\frac{n(n+1)}{2}} = \{V_n\}, \end{array}$$

we take inhomogeneous coordinates $(x_i, y_i) = (x_1, \cdots, x_{n-1}, x_n = 1, y_1, \cdots, y_{n-1}, y_n)$ in P^{2n-1} and (a_{ij}), $a_{ij} = a_{ji}$, in $LG^{\frac{n(n+1)}{2}}$.

We have the incidence relations (twistor equations):

$$\begin{cases} y_i = \sum_{j=1}^{n-1} a_{ij}x_j + b_i, \quad (a_{ij} = a_{ji}) \quad 1 \le i \le n-1, \\ y_n = z = \sum_{i=1}^{n-1} b_i x_i + c. \end{cases}$$

In P^{2n-1}, the incidence relations represent a Legendre plane with respect to the contact form $\omega = dz + \sum_{i=1}^{n}(x_i dy_i - y_i dx_i)$.

○ A type

The group $GL(n,\mathbb{C})(\subset Sp(n,\mathbb{C}))$ acts on $V = \mathbb{C}^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n \cong D_m$ $(m \in M)$.

For

$$X = P^{n-1} \subset P^{2n-1} : (x_1, \cdots, x_{n-1}, 1, 0, \cdots, 0),$$

the dual space $\tilde{X}$ of X is one point $\{a_{ij} = 0\}$ and the intersection variety $Z(X) \subset LG^{\frac{n(n+1)}{2}}$ has the defining equation by the resultant

$$R_X = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & & \\ \vdots & & \ddots & \vdots \\ a_{1n} & & \cdots & a_{nn} \end{vmatrix} = 0 \quad \curvearrowleft GL(n,\mathbb{C}).$$

From the resultant $R_X = 0$, we have a Goursat equation

$$Hess = \begin{vmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{12} & p_{22} & & \\ \vdots & & \ddots & \vdots \\ p_{1n} & & \cdots & p_{nn} \end{vmatrix} = 0 \ \subset J^2 \quad \curvearrowleft SL(n+2,\mathbb{C}).$$

○ BD type

The group $SL(2,\mathbb{C}) \otimes O(n,\mathbb{C})(\subset Sp(n,\mathbb{C}))$ acts on $W \otimes V = \mathbb{C}^2 \otimes \mathbb{C}^n = \mathbb{C}^n \oplus \mathbb{C}^n \cong D_m$ $(m \in M)$.

For

$$P^1 \times Q^{n-2} \longrightarrow X \subset P^{2n-1}$$

$$([u,v],[x_i]), \sum_{i=1}^{n} x_i^2 = 0 \longmapsto \begin{bmatrix} ux_1 & \cdots & ux_{n-1} & ux_n \\ vx_1 & \cdots & vx_{n-1} & vx_n \end{bmatrix} \subset \begin{bmatrix} x_1 & \cdots & x_{n-1} & x_n \\ y_1 & \cdots & y_{n-1} & y_n \end{bmatrix},$$

taking inhomogeneous coordinates $u = 1, x_n = 1$, we ask for the resultant $R_X = 0$ which is the condition for having common solutions of n quadratic equations

$$\begin{cases} (\sum_{k=1}^{n-1} b_k x_k)x_i - (\sum_{j=1}^{n-1} a_{ij}x_j - cx_i) - b_i = 0, \quad 1 \le i \le n-1, \\ \sum_{i=1}^{n-1} x_i^2 = 0. \end{cases}$$

But $R_X = 0$ is very complicated even though in the case $n = 3$ (cf. [K-S-Z]).

∘ G_2 type

The group $SL(2,\mathbb{C})(\subset Sp(2,\mathbb{C}))$ acts on $S^3(V) = S^3(\mathbb{C}^2) = \mathbb{C}^4 \cong D_m$ $(m \in M)$.
For

$$P^1 \supset \mathbb{C} \longrightarrow X \subset P^3, \quad x \longmapsto (x, y = -\frac{1}{2}x^2, z = \frac{1}{6}x^3),$$

which is a Legendre curve with respect to $\omega = dz + xdy - ydx$, the dual curve $\tilde{X}$ of X is

$$a = -x, \quad b = \frac{1}{2}x^2, \quad c = -\frac{1}{3}x^3$$

and the resultant $R_X = 0$, which is the condition for having common solutions of two equations

$$f_1 = -\frac{1}{2}x^2 - ax - b = 0, \quad f_2 = \frac{1}{6}x^3 - bx - c = 0$$

is as follows: using the Sylvester determinant,

$$R_X = R(f_1, f_2) = \begin{vmatrix} -\frac{1}{2} & -a & -b & & \\ & -\frac{1}{2} & -a & -b & \\ & & -\frac{1}{2} & -a & -b \\ \frac{1}{6} & 0 & -b & -c & \\ & \frac{1}{6} & 0 & -b & -c \end{vmatrix}$$
$$= -\frac{1}{8}c^2 - \frac{4}{9}b^3 + \frac{1}{2}abc + \frac{1}{6}a^2(b^2 - ac) = 0.$$

Multiplying it by -72, we have the intersection variety $Z(X)$ with a definig equation

$$9c^2 + 32b^3 - 36abc - 12a^2(b^2 - ac) = 0.$$

Taking $a \to t, b \to s, c \to r$, we have a Goursat equation

$$9r^2 + 32s^3 - 36rst + 12t^2(rt - s^2) = 0.$$

This is nothing but the equation (i) in Introduction.

PART II. Constructions of solutions

1. G_2' twistor diagram

1.1. Imaginary split octonions.

Let V^7 be the imaginary split octonions $\mathrm{Im}\mathbb{O}'$. Let g be the inner product of type $(3,4)$ and ϕ the associative 3-form:

$$\phi(x, y, z) = g(xy, z).$$

Then we have

$$G_2' = \{g \in \mathrm{GL}(V) \mid g^*\phi = \phi\}.$$

Take a basis $\{e_i\}$ and coordinates

$$x = x_1e_1 + x_2e_2 + x_3e_3 + y_1e_5 + y_2e_6 + y_3e_7 + ze_4 = (x_1, x_2, x_3, y_1, y_2, y_3, z)$$

such that g has the matrix representation

$$g = \begin{pmatrix} O & I & 0 \\ I & O & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2dx_1dy_1 + 2dx_2dy_2 + 2dx_3dy_3 + dz^2.$$

Then we have

$$\begin{aligned} \phi &= \omega_{415} + \omega_{426} + \omega_{437} + \omega_{567} - \omega_{123} \\ &= dz \wedge (dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + dx_3 \wedge dy_3) + dy_1 \wedge dy_2 \wedge dy_3 - dx_1 \wedge dx_2 \wedge dx_3, \end{aligned}$$

where $\omega_{ijk} = e_i^* \wedge e_j^* \wedge e_k^*$, $\{e_i^*\}$ is the dual basis of $\{e_i\}$. Rf. [B].

For the interior product of ϕ, we have

$$\begin{aligned} i_{e_i}\phi &= -dz \wedge dy_i - dx_j \wedge dx_k \quad (e_i,\ i = 1, 2, 3) \\ i_{e_i}\phi &= dz \wedge dx_i + dy_j \wedge dy_k \quad (e_i,\ i = 5, 6, 7) \\ i_{e_4}\phi &= dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + dx_3 \wedge dy_3, \end{aligned}$$

where (i, j, k) is the even permutation of $(1, 2, 3)$.

1.2. Twistor diagram.

In V^7, let V_1 be a 1-dimensional null subspace, V_2 a 2-dimensional null subspace such that $i_{e_i}\phi$ $(1 \le i \le 7)$ vanish, V_3 a 3-dimensional null subspace such that ϕ vanishes. Putting $L^6 = \{(V_1, V_2) \mid V_1 \subset V_2\}$, $M^5 = \{\text{all } V_1\}$, $N^5 = \{\text{all } V_2\}$, we have the G_2' twistor diagram:

$$\begin{tikzcd} & L^6 = \{V_1 \subset V_2\} \arrow[dl, "P^1"', "\pi_1"] \arrow[dr, "\pi_2", "P^1"'] & \\ M^5 = \{V_1\} & & N^5 = \{V_2\}. \end{tikzcd}$$

Fix V_2. Then the set of all V_1 included in V_2 is a line P^1 in M^5 called a *Goursat line* P^1_G. The space L^6 is a fiber bundle with fiber P^1 over N^5. In other words, N^5 is regarded as the moduli space of all Goursat lines $\{P^1_G\}$. Next, fix V_1. Then the set of all V_2 which includes V_1 is a line P^1 in N^5 called a *Monge line* P^1_M. The space L^6 is a fiber bundle with fiber P^1 over M^5. In other words, M^5 is regarded as the moduli space of all Monge lines $\{P^1_M\}$. The bundle L^6 is called the *Goursat direction bundle* over M^5, and also is called the *Monge direction bundle* over N^5.

1.3. Two geometric structures.

In the Grassmann manifold $G_{2,7}$ which consists of all 2-dimensional subspaces in V, we take inhomogeneous coordinates

$$\left\{\begin{pmatrix} 1 & b_1 & a_2 & a_3 & b_2 & 0 & e \\ 0 & d_1 & c_2 & c_3 & d_2 & 1 & f \end{pmatrix}\right\}.$$

Here we exchange coordinates in V for $(x_1, y_1, x_2, x_3, y_2, y_3, z)$.

Take inhomogeneous coordinates $(x_1 = 1, y_1, x_2, x_3, y_2, y_3, z)$ in the projective space $P^6(V)$. From $y_1 + x_2y_2 + x_3y_3 + z^2 = 0$, the space $M^5(\subset P^6)$ is represented by the graph of

$$y_1 = -x_2y_2 - x_3y_3 - z^2.$$

Namely we have local coordinates (x_2, x_3, y_2, y_3, z) in M.

From $i_{e_i}\phi$, we consider

$$\begin{cases} \theta_1 = dz - y_2dy_3 + y_3dy_2 \\ \theta_2 = dx_2 + zdy_3 - y_3dz \\ \theta_3 = dx_3 - zdy_2 + y_2dz. \end{cases}$$

Putting

$$\begin{cases} Y_2 = \dfrac{\partial}{\partial y_2} - y_3\dfrac{\partial}{\partial z} - y_3^2\dfrac{\partial}{\partial x_2} + (z + y_2y_3)\dfrac{\partial}{\partial x_3} \\ Y_3 = \dfrac{\partial}{\partial y_3} + y_2\dfrac{\partial}{\partial z} - y_2^2\dfrac{\partial}{\partial x_3} - (z - y_2y_3)\dfrac{\partial}{\partial x_2}, \end{cases}$$

we can canonically consider a distribution

$$D_M = \mathrm{Ann}\{\theta_1, \theta_2, \theta_3\} = \langle Y_2, Y_3\rangle$$

on M. From

$$[Y_2, Y_3] = 2\frac{\partial}{\partial z} + 4y_3\frac{\partial}{\partial x_2} - 4y_2\frac{\partial}{\partial x_3} =: Z,$$
$$[Y_2, Z] = -4\frac{\partial}{\partial x_3} =: X_3, \quad [Y_3, Z] = 4\frac{\partial}{\partial x_2} =: X_2,$$

D_M is of type $(2,3,5)$ distribution which is called the Cartan distribution.

We can canonically consider a conformal structure g of type $(2,3)$ on M defined by

$$g = \theta_2dy_2 + \theta_3dy_3 + \theta_1^2.$$

It follows that $\langle Y_2, X_2\rangle = 1, \langle Y_3, X_3\rangle = 1, \langle Z, Z\rangle = 1$, otherwise 0, and D_M is a null plane.

From nullity and $i_{e_i}\phi$, the space $N^5(\subset G_{2,7})$ is represented by the graph of

$$b_1 = -e^2 - a_2f,\ b_2 = f,\ c_3 = d_2e - f^2,\ c_2 = -e,\ d_1 = -a_3 - a_2d_2 - \frac{1}{2}ef.$$

Namely we have local coordinates (a_2, a_3, d_2, e, f) in N.

We can canonically consider a contact structure on N defined by

$$\omega = da_3 + d_2 da_2 - edf + 2fde.$$

We will show in the next section that the contact distribution D_N defined by ω is equipped with a 2-dimensional cone field K of degree 3. The contact distribution D_N is spanned by

$$D_N = \mathrm{Ker}\omega = \langle \frac{\partial}{\partial a_2} - d_2\frac{\partial}{\partial a_3}, \frac{\partial}{\partial d_2}, \frac{\partial}{\partial e} - 2f\frac{\partial}{\partial a_3}, \frac{\partial}{\partial f} + e\frac{\partial}{\partial a_3}\rangle.$$

2. Cartan-Legendre duality

2.1. Goursat lines.

A Goursat line P_G^1 in M^5 is represented by

$$\begin{aligned} &y_3 = t, \quad y_2 = d_2 t + f, \quad z = ft + e, \\ &x_3 = (d_2 e - f^2)t + a_3, \quad x_2 = -et + a_2, \quad (\lambda = d_2) \end{aligned}$$

with respect to a parameter t and constants d_2, f, e, a_3, a_2. It follows that P_G^1 is tangent to the Cartan distribution D_M. It is a rigid singular curve on D_M (cf.[M]).

2.2. Monge lines.

A Monge line P_M^1 in N^5 is represented by

$$\begin{aligned} &d_2 = \lambda, \quad f = y_2 - y_3\lambda, \quad e = z - y_2 y_3 + y_3^2\lambda, \\ &a_3 = x_3 + y_2^2 y_3 - (z + y_2 y_3)y_3\lambda, \quad a_2 = x_2 + (z - y_2 y_3)y_3 + y_3^3\lambda, \quad (t = y_3) \end{aligned}$$

with respect to a parameter λ and constants y_3, y_2, z, x_3, x_2. Taking derivatives with respect to λ, we have

$$(\dot{d}_2, \dot{f}, \dot{e}, \dot{a}_3, \dot{a}_2) = (1, -y_3, y_3^2, -(z + y_2 y_3)y_3, y_3^3).$$

If we consider coordinates with respect to the basis $\{\frac{\partial}{\partial a_2} - d_2\frac{\partial}{\partial a_3}, \frac{\partial}{\partial d_2}, \frac{\partial}{\partial e} - 2f\frac{\partial}{\partial a_3}, \frac{\partial}{\partial f} + e\frac{\partial}{\partial a_3}\}$ in D_N, then we have

$$(y_3^3, 1, y_3^2, -y_3).$$

It is a Veronese (a twisted cubic) curve with respect to a parameter $y_3 = t$ in the projectified $P(V_N)$ at each point of N. We also have a 2-dimensional cone field K of degree 3 on N given by

$$(y_3^3 s, s, y_3^2 s, -y_3 s).$$

2.3. Cartan-Legendre duality.

We have the G_2' twistor diagram with coordinates:

$$\begin{array}{ccc} & L^6 & \\ P^1 : \lambda \swarrow \pi_1 & & \pi_2 \searrow P^1 : t \\ M^5 : (x_2, x_3, y_2, y_3 = t, z) & & N^5 : (a_2, a_3, d_2 = \lambda, e, f) \end{array}$$

We call the correspondence

$$(x_2, x_3, y_2, y_3, z, \lambda) \leftrightsquigarrow (a_2, a_3, d_2, e, f, t)$$

by the above-mentioned each six relations on L^6 the *Cartan-Legendre duality (transformation)* (or briefly, *C-L duality*).

3. Constructions of solutions

3.1. Construction by C-L duality.

We construct solutions of the Goursat equation (i) in Introduction of G_2' type by the Cartan-Legendre duality.

A D_M-curve is a curve on M whose tangent vectors are tangent to the Cartan distribution D_M. It is called a *Goursat curve.*

Take a Goursat curve l in M which is not a Goursat line. We transform l into N via the Cartan-Legendre duality in two ways.

One way is to first consider a surface $S = \pi_1^{-1}(l)$ on L.

Taking the C-L duality, from transforming the fiber direction λ to $d_2 = \lambda$ direction, we have S generated by the Monge flow with a parameter λ.

Projecting it by π_2 onto N, we have $S' = \pi_2(S)$ which is ruled by Monge lines.

The other way is to construct the dual curve h in N which is a Monge curve. A *Monge curve* is a curve whose tangent vectors are tangent to the cone field K $(\subset D_N)$.

From l, we consider the tangent vector $(m, l'_{|m})$ at m in l. This is regarded as an element of L which is the Goursat direction bundle (called a Goursat lift, cf. a Legendre lift).

Projecting it onto N by π_2, we have a Monge curve $h = \pi_2(l')$ called the *dual curve* of a Goursat curve l.

As another interpretation, by the G_2' twistor diagram, a point m in M corresponds to a Monge line P_N^1 in N, and a Goursat line $P_M^1(m) \sim l'_{|m}$ in M to a point n in N. Here the tangent vector $l'_{|m}$ is identified with an embedded tangent line, i.e., a Goursat line $P_M^1(m)$.

Given a Goursat curve l in M, then we have a Monge curve h in N which is the dual curve of l by way of $(m, P_M^1(m))$ $(m \in l)$ and $(P_N^1(n), n)$ $(n \leftrightsquigarrow P_M^1(m))$.

Summing up two ways, for a given Goursat curve l in M, we have a tangent developable surface S' in N ruled by Monge lines along the dual curve h :

$$\begin{aligned} &l \longrightarrow S = \pi_1^{-1}(l) \longrightarrow S' = \pi_2(S), \\ &l \longrightarrow l' \longrightarrow h = \pi_2(l'). \end{aligned}$$

Theorem 2. *The tangent developable surface S' in N constructed above is a solution surface of the Goursat equation of G_2' type.*

3.2. Explicit representaions.

Let $l = l(s)$ be a Goursat curve in M such that it is represented by, for a smooth function $f(s)$,

$$y_3 = s,\ y_2 = f''(s),\ z = 2f' - sf'',\ x_3 = 2f'f'' - 3\int (f'')^2 ds,\ x_2 = -6f + 4sf' - s^2 f''.$$

Then the surface S' in N constructed above is a (s, λ)-surface represented by

$$\begin{aligned} &d_2 = \lambda,\ f = f'' - s\lambda,\ e = 2f' - 2sf'' + s^2\lambda, \\ &a_3 = 2f'f'' + s(f'')^2 - 3\int (f'')^2 ds - 2sf'\lambda,\ a_2 = -6f + 6sf' - 3s^2 f'' + s^3\lambda. \end{aligned}$$

It follows that the locus of singular points to the π_2-projection is $\lambda = f'''(s)$. It is nothing but the dual curve of l in N.

Proposition 1. *The tangent developable surface S' in N constructed above has, as a generic singularity, a cuspidal edge along the dual curve l with type $(1, 2, 3, 4, 5)$, i.e., each ordinary point. Moreover, for the front mapping $(d_2, f, e, a_3, a_2) \longmapsto (f, a_2, a_3)$ (or (e, a_2, a_3)), we have a tangent developable surface along a space curve with type $(2, 4, 5)$ (or $(3, 4, 5)$) at some point.*

3.3. Solutions not obtained by Goursat curves.

For the Cartan distribution $D_1 = D_M$ on M, let us consider the derived system of D_1:

$$D_2 = \partial D_1 = D_1 + [D_1, D_1] \subset TM.$$

It is a type $(3, 5)$ distribution.

Take a D_2-curve l in M. Lifting it as $S = \pi_1^{-1}(l)$ to L, transforming S by the C-L duality, and projecting S onto N by π_2, we have $S' = \pi_2(S)$ which is ruled by Monge lines, as well as a Goursat curve that is a D_1-curve.

Theorem 3. *The ruled surface S' in N constructed above is a solution surface of the Goursat equation of G_2' type.*

Proposition 2. *A generic ruled surface S' is a smooth surface without the dual curve of l.*

4. Generalization.

From the G_2' twistor diagram, we can extend M^5 with the G_2' Goursat structure to M^{3n-1} with a distribution D_M of rank $2(n-1)$ called a *Goursat structure*, and N^5 with the G_2' contact structure to N^{2n+1} with a contact distribution D_N and a cubic Legendre cone field K. We have the following twistor diagram:

$$\begin{array}{ccc} & L^{3n} & \\ P^1 \swarrow & & \searrow F^{n-1} \\ M^{3n-1} & & N^{2n+1}. \end{array}$$

Remark that the left-sided and right-sided base spaces are reversed to those in I.3.1, and the notations of M and N are also reversed.

We assume that a Lie group G acts on M, L, N transitively. If we take a simple Lie group G of A, BD, or exceptional type, then we have the twistor diagram for each type. But there is not a twistor diagram for C type.

Lifting an $(n-1)$-dimensional Goursat surface (see I.1.2.) which is a D_M-surface in M, taking the generalized C-L duality, and projecting onto N, we have an n-dimensional ruled surface in N which is ruled by Monge lines. It is a solution surface of the Goursat equation of each A, BD, exceptional type.

References

[B] R. Bryant, *Élie Cartan and geometric duality*, Lecture at Institute Élie Cartan, 1998.

[C] E. Cartan, *Les systèmes de Pfaff à cinq variables et les équations aux dérivées partielles du second ordre*, Ann. Ec. Normale **27** (1910), pp.109–192.

[G-K-Z] I.M. Gelfand, M.M. Kapranov and A.V. Zelevinsky, *Discriminants, resultants and multidimensional determinants*, Birkhäuser, Boston, 1994.

[G] E. Goursat, *Leçon sur l'intégration des équations aux dérivées partielles du second ordre à deux variables indépendantes* I,II, Hermann, Paris, 1898.

[I-L] T.A. Ivey and J.M. Landsberg, *Cartan for beginners: differential geometry via moving frames and exterior differential systems*, American Mathematical Society, 2003.

[K-S-Z] M.M. Kapranov, B. Sturmfels and A.V. Zelevinsky, *Chow polytopes and general resultants*, Duke Math. J. **67** (1992), pp.189–218.

[M-M] Y. Machida and T. Morimoto, *On decomposable Monge-Ampère equations*, Lobachevskii J. Math. **3** (1999), pp.185–196.

[M] R. Montgomery, *A tour of subriemannian geometries, their geodesics and applications*, Mathematical Surveys and Monographs 91, American Mathematical Society, 2002.

[T] A. Tsuchiya, *Geometric theory of partial differential equations of second order*, Lecture Note at Nagoya University (in Japanese), 1981.

[Y] K. Yamaguchi, *G_2-geometry of overdetermined systems of second order*, in Analysis and geometry in several complex variables (Katata,1997), Trends Math., Birkhäuser, Boston (1999), pp.289–314.