On the deformation of A-branes in String theory

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1 A brief sketch

In this paper, we discuss the deformation theory of A-branes in String theory, from the point of view of CR structures and give an outline of our approach. The full paper will appear in another paper. Let $W$ be a Kaehler manifold and let $\omega_W$ be its Kaehler form. Let $M$ be a real hypersurface in $W$. We assume that our $M$ admits an A-brane structure. Namely, there is a real line bundle $L$ on $M$, and a connection $\nabla$ on $L$, satisfying:

1. The curvature of the connection, $F$, is an element of $\Gamma(M, \Lambda^2 \mathcal{F}^*)$, where $\mathcal{F} := T^*M$.
2. $J := \omega^{-1}_W F$ determines a complex structure on $\mathcal{F}$, and $\mathcal{L}$ is a characteristic foliation $\mathcal{L}$, defined by: $p \in M, L_p = \{Y_p, Y_p' \in T_pW, \omega_W(Y_p, Y'_p) = 0, Y'_p \in T_pM\}$.

In this paper, by using the notion of almost CR structures, we reformulate the notion of A-branes. Our $J$ determines an almost CR structure $(M, T'_e)$ on $M$. For this almost CR structure, we prove that $C \otimes \mathcal{L} + T'_e$ is integrable on $M$. And show the deformation complex of A-branes (the Kapustin-Orlov complex)(see (2.7)). This is a natural generalization of the case $M = W$ (Kapustin-Orlov consider the case; A-branes wrap the whole $W$, and obtain the standard $\overline{\partial}$-complex as a deformation complex).

Here we treat A-branes of the type hypersurfaces. Now for a given A-brane, we introduce the notion of family of A-branes, $\{(M, L, \nabla_t)\}_{t \in T}$. In this paper, we introduce the deformation complex of A-branes, and construct the Kodaira-Spencer map for the given family of A-branes. On the parameter space, a complex structure is given. But, we are relying on the Hamilton deformation, so we can't discuss in the complex analytic category (so we have to use that $\{(M, L, \nabla_t)\}_{t \in T}$ depends on $t, \mathcal{C}^\infty$-ly). And because of this fact, we have to discuss in the category, mod $(t^2, t)$.

The author would like to thank Prof.A.Kapustin for allowing me to use the name, the Kapustin-Orlov complex and valuable suggestions during the preparation of this paper (the author learned that Kapustin and his student Yi Li, independently, obtained the integrability of $C \otimes \mathcal{L} + T'_e$).
2 The Kapustin-Orlov complex

In [Kap-Or], Kapustin-Orlov formulate the D-branes of A-type (in their language, A-branes), mathematically. We consider the deformation theory of A-branes in the case real hypersurfaces. For this, we recall the notion of A-branes. Let $W$ be a Kähler manifold. Let $\omega_W$ be its Kähler metric. Let $M$ be a real submanifold of $W$. Then, for this $M$, we have a characteristic foliation $\mathcal{L}$. This is defined by: for $p \in M$,

$$\mathcal{L}_p = \{Y_p, Y_p' \in T_pW, \omega_W(Y_p, Y_p') = 0, Y_p' \in T_pM\}.$$  

By this definition, $\mathcal{L}$ is a subbundle of $TW|_M$ and the rank of $\mathcal{L}$ is $2n - \dim_R M$, because of $\omega_W$ being non-degenerate (here $n$ is the complex dimension of $W$).

**Definition 2.1.** If $p \in M$, $\mathcal{L}_p \subset T_pM$, then $M$ is called coisotropic.

Henceforth we assume that our real submanifold is coisotropic. So, on $M$, we have a quotient bundle

$$\mathcal{F} := \frac{TM}{\mathcal{L}}.$$  

**Definition 2.2.** (A-branes). Let $M$ be a coisotropic submanifold. Then $M$ admits the A-brane if and only if there is a real line bundle $L$ and a connection $\nabla$ of $L$, $(L, \nabla)$ which satisfies

[1] The curvature of the connection, $F$, is an element of $\Gamma(M, \wedge^2 F^*)$,

[2] $J := \omega_W^{-1}F$ determines a "Tao" structure on $M$ (this means that: $J^2 = -1$ and this $J$ is integrable modulo characteristic foliation).

Now for the submanifold $M$, a CR structure $(M, 0T'')$ is introduced by:

$$0T'' = C \otimes TM \cap T''W|_M,$$

where $C \otimes TM$ means the complexified tangent bundle of $M$. Let $D = \{Y : Y \in TM, Y = X + \overline{X}, X \in 0T''\}$. Then, naturally,

$$D \cong \mathcal{F}.$$  

By this identification, $J$ is defined on $D$, satisfying : $J^2 = -1$. Hence $J$ determines an almost CR structure on $M$. We study this structure. $J$ is defined on $D$. We extend this $J$ on $C \otimes D$, naturally. Set

$$T'_j = \{X : X \in C \otimes D, JX = \sqrt{-1}X\},$$

$$T''_j = \{X' : X' \in C \otimes D, JX' = -\sqrt{-1}X'\}.$$  

Then, as mentioned in [Kap-Or], we have

**Proposition 2.1.**

$$C \otimes D = T'_j + T''_j, T'_j \cap T''_j = 0,$$

$$[\Gamma(M, T'_j), \Gamma(M, T'_j)] \subset \Gamma(M, T'_j) \mod \mathcal{L}.$$  

**Proof.** (0.1) is obvious. We see (0.2). By the definition, $dF = 0$, $d\omega_W = 0$, and

$$\omega_W(X, JX') = F(X, X'), X, X' \in C \otimes D.$$
With these, we compute: for $X_1, X_2 \in \Gamma(M, T'_J)$, $X \in \Gamma(M, C \otimes TM)$,
\begin{align*}
dF(X_1, X_2, X) &= 0, \\
d\omega_{W}(X_1, X_2, X) &= 0. \\
\end{align*}
(2.3) (2.4)

We compute (0.3). Then,
\begin{align*}
X_1 F(X_2, X) - X_2 F(X_1, X) + XF(X_1, X_2) \\
-F([X_1, X_2], X) + F([X_1, X], X_2) - F([X_2, X], X_1) &= 0.
\end{align*}

We rewrite this by using: $\omega_{W}(X, JX') = F(X, X'), X, X' \in C \otimes D$.
\begin{align*}
X_1 \omega_{W}(JX_2, X) - X_2 \omega_{W}(JX_1, X) + X \omega_{W}(JX_1, X_2) \\
-\omega_{W}(J[X_1, X_2], X) + \omega_{W}([X_1, X], JX_2) - \omega_{W}([X_2, X], JX_1) &= 0.
\end{align*}

By $JX_i = \sqrt{-1}X_i, i = 1, 2$, this becomes
\begin{align*}
X_1 \omega_{W}(\sqrt{-1}X_2, X) - X_2 \omega_{W}(\sqrt{-1}X_1, X) + X \omega_{W}(\sqrt{-1}X_1, X_2) \\
-\omega_{W}(J[X_1, X_2], X) + \omega_{W}([X_1, X], \sqrt{-1}X_2) - \omega_{W}([X_2, X], \sqrt{-1}X_1) &= 0.
\end{align*}

While, by (0.4),
\begin{align*}
X_1 \omega_{W}(X_2, X) - X_2 \omega_{W}(X_1, X) + X \omega_{W}(X_1, X_2) \\
-\omega_{W}([X_1, X_2], X) + \omega_{W}([X_1, X], X_2) - \omega_{W}([X_2, X], X_1) &= 0.
\end{align*}

Hence, we have
\begin{align*}
\omega_{W}(J[X_1, X_2], X) &= \omega_{W}(\sqrt{-1}[X_1, X_2], X) \text{ for any } X \in C \otimes D.
\end{align*}

This means that: $[X_1, X_2] \in T'_J$ modulo $\mathcal{L}$. 

The following proposition is also mentioned in [Kap-Or].

**Proposition 2.2.**

$\omega_{W}(X_1, X_2) = 0$ for $X_1 \in T'_J, X_2 \in T''_J$.

So, $J$-structure is different from the CR structure, naturally, induced from $W$. Here for the convenience, we give a proof.

**Proof.** We use $\omega_{W}(X, JY) = F(X, Y)$, for any $X, Y \in C \otimes TM$. For $X_1 \in T'_J, X_2 \in T''_J$,
\begin{align*}
\omega_{W}(X_1, JX_2) &= F(X_1, X_2).
\end{align*}

By $JX_2 = -\sqrt{-1}X_2$,
\begin{align*}
\omega_{W}(X_1, -\sqrt{-1}X_2) &= F(X_1, X_2), \\
\text{so,} \\
\omega_{W}(X_1, X_2) &= \sqrt{-1}F(X_1, X_2).
\end{align*}

On the other hand,
\begin{align*}
\omega_{W}(X_2, JX_1) &= F(X_2, X_1).
\end{align*}
So, by $JX_1 = \sqrt{-1}X_1$,
\[ \omega(X_2, X_1) = -\sqrt{-1}F(X_2, X_1). \]
Hence
\[ \omega(X_1, X_2) = -\sqrt{-1}F(X_1, X_2). \]
This means that $\omega_W(X_1, X_2) = 0$. \hfill $\square$

As is mentioned in [Kap-Or], the following corollary follows from this proposition.

**Corollary 2.3.**
\[ \dim_C T'_J = \text{even}. \]

Now we set a $C^\infty$ vector bundle decomposition
\[ C \otimes TM = C \otimes \mathcal{L} + T''_J + T'_J. \]
Here $C \otimes \mathcal{L}$ means the complexified $\mathcal{L}$. While in our case, $(M, T''_J)$ may not be a CR structure (only integrable modulo $\mathcal{L}$). But,

**Proposition 2.4.** $\mathcal{L}$ preserves $J$, namely,
\[ [\Gamma(M, T'_J), \mathcal{L}] \subset \Gamma(M, T'_J) \text{ modulo } \mathcal{L}. \]

**Proof.** By the same ways as in Proposition 2, we see this proposition.
For $X \in T'_J, Y \in T''_J, \zeta \in \mathcal{L}$, as $F, \omega_W$ are closed,
\[
\begin{align*}
&dF(X, Y, \zeta) = 0, \\
&d\omega_W(X, Y, \zeta) = 0.
\end{align*}
\]

By the first equation,
\[
\begin{align*}
XF(Y, \zeta) - YF(X, \zeta) + \zeta F(X, Y) \\
- F([X, Y], \zeta) + F([X, \zeta], Y) - F([Y, \zeta], X) = 0.
\end{align*}
\]

As $\mathcal{L}$ is a characteristic foliation, this becomes
\[ \zeta F(X, Y) + F([X, \zeta], Y) - F([Y, \zeta], JX) = 0. \]

With $\omega_W(X', JY') = F(X', Y')$ for $X', Y' \in C \otimes D$,
\[ \zeta \omega_W(JX, Y) + \omega_W([X, \zeta], JY) - \omega_W([Y, \zeta], JY) = 0. \]

While, by Proposition 2,
\[ \omega_W(JX, Y) = \omega_W(\sqrt{-1}X, Y) = 0. \]
Hence
\[ \omega_W([X, \zeta], -\sqrt{-1}Y) - \omega([Y, \zeta], \sqrt{-1}X) = 0. \quad (2.5) \]
While by the second equation,

\[
X\omega_W(Y, \zeta) - Y\omega_W(X, \zeta) + \zeta\omega_W(X, Y) - \omega_W([X, Y], \zeta) + \omega_W([X, \zeta], Y) - \omega_W([Y, \zeta], X) = 0.
\]

So, by the same way, this becomes

\[
\omega_W([X, \zeta], Y) - \omega([Y, \zeta], X) = 0.
\] (2.6)

With (0.5), (0.6), we have

\[
\omega_W([X, \zeta], Y) = 0, \text{ for } X \in T', Y \in T'.
\]

This means that: the $T'$ part of $[X, \zeta]$ vanishes because of $\omega_W$ being nondegenerate with Proposition 2.2. Hence

\[
[X, \zeta] \in \Gamma(M, T') \text{ modulo } \mathcal{L}.
\]

Now we can state our theorem.

**Theorem 2.5.** We set $T'' := C \otimes \mathcal{L} + T''$. Then,

\[
[\Gamma(M, T''), \Gamma(M, T'')] \subset \Gamma(M, T'').
\]

By this theorem, we have the deformation complex of A-branes (Kapustin-Orlov complex). Namely, for $u \in \Gamma(M, C)$, we set $\overline{\partial}u$ of $\Gamma(M, (T'')^*)$ by;

\[
\overline{\partial}u(X) = Xu, \text{ for } X \in T''.
\]

By the same way as for ordinary differential forms, we can introduce $\mathcal{F}$ from

\[
\Gamma(M, \wedge^p(T'')^*) \text{ to } \Gamma(M, \wedge^{p+1}(T'')^*).
\]

\[
\mathcal{F} : \Gamma(M, \wedge^p(T'')^*) \to \Gamma(M, \wedge^{p+1}(T'')^*).
\]

Then, by the integrability theorem(Theorem 2.5),

\[
\mathcal{F}^{p+1}\mathcal{F} = 0.
\]

So, we have a deformation complex of A-branes(Kapustin-Orlov complex).

\[
0 \to \Gamma(M, C) \xrightarrow{\overline{\partial}} \Gamma(M, (T'')^*) \xrightarrow{\overline{\partial}} \Gamma(M, \wedge^2(T'')^*) \to \cdots
\] (2.7)

Furthermore, by this theorem, we can introduce a sheaf, $\mathcal{O}_{T''}$, composed of $\overline{\partial}$-closed elements, which are holomorphic in the direction $T''$, and constant in the direction $\mathcal{L}$.  

3 A family of deformations of A-branes

We introduce the notion of a family of deformations of A-branes,

**Definition 3.1.** A set of A-branes \( \{(M, L, \nabla_t), i_t\}_{t \in T} \), where \( T \) is an analytic space with the origin \( o \), is a family of deformations of A-branes if

1. connections \( \nabla_t \) depends on \( t \), \( C^\infty \)-ly, and \( \nabla_o = \nabla \),
2. embeddings \( i_t \) depends on \( t \), \( C^\infty \)-ly, and \( i_o = i \).

Unlike CR structures, we rely on \( C^\infty \) category. Because, in the case symplectic structures, the Hamiltonian deformations play an essential part. We study a family of deformations of A-branes in the case real hypersurfaces. For the embedding \( i_t \), we have the characteristic vector field, \( \xi_t \). By using this vector field, the condition of \( \{(M, L, \nabla_t), i_t\} \) being the A-brane is rewritten as follows.

[1] Let the curvature of the connection \( \nabla_t \), \( F_t \), an element of \( \Gamma(M, \wedge^2 \mathcal{F}_t^*) \),

[2] Let \( J_t := (i_t^* \omega_W)^{-1} F_t \). Then, \( J_t^2 = -1 \) on \( \mathcal{F}_t \), where

\[ \mathcal{F}_t := \frac{TM}{\mathcal{L}_t}, \]

and \( \mathcal{L}_t \) is generated by \( \xi_t \). While the inclusion map induces a bundle isomorphism map \( \rho_t \) from \( D \) to \( \frac{TM}{\mathcal{L}_t} \), induced by the inclusion map ; \( D \) to \( TM \). The structure defined by \( J_t \) induces an almost CR structure on \( D \) by;

\[ J_t' := \rho_t^{-1} J_t \rho_t. \]

Henceforth, we use the same notation \( J_t \) for \( J_t' \) and we regard \( J_t \) as an almost CR structure on \( D \). Therefore \([1]_t, [2]_t \) are written as

[1] The curvature of the connection \( \nabla_t \), \( F_t \), satisfies \( F_t(\xi_t, Y) = 0 \) for \( Y \in D \),

[2] Let \( J_t := (i_t^* \omega_W)^{-1} F_t \). Then, \( J_t^2 = -1 \) on \( D \).

We see why we call this complex a deformation complex of A-branes.

**Definition 3.2.** The quartets of A-branes, \( \{(M, L), \xi \}, \{(M, L', \nabla'), i'\} \) are equivalent if there is a gauge transform of the line bundle \( L \) (we write this bundle map by \( q \)), and there is a Hamiltonian diffeomorphism map of \( W \), defined by a \( C^\infty \) function \( g \) (we write it by \( V_g \)), satisfying;

1. the composition of maps \( V_g \) and \( i \), \( V_g i = i' \),
2. \( V_g^* q^* \nabla = \nabla' \)

Next we introduce an equivalence relation for a family of deformations of A-branes.

**Definition 3.3.** The family of deformations of A-branes, \( \{(M, L, \nabla_t), i_t\}_{t \in T} \), \( \{(M', L', \nabla'_t), i'_t\}_{t \in S} \) are equivalent if there is a local biholomorphic map \( h \) from \( T \) to \( S \) satisfying : \( h(o) = o \), there is a gauge transform of the line bundle \( L \) (we write this bundle map by \( q \)), and there is a Hamiltonian diffeomorphism map of \( W \), defined by a \( C^\infty \) function \( g_t \) (we write it by \( V_{g_t} \)), satisfying;

1. the composition of maps \( V_{g_t} \) and \( i_t \), \( V_{g_t} i_t = i'_t(h(t)) \),
2. \( V_{g_t}^* q_t^* \nabla_t = \nabla'_{h(t)} \)
4 The infinitesimal case

For a family of deformations of A-branes, \( \{(M, L, \nabla_t), i_t\}_{t \in T} \), we can introduce the Kodaira-Spencer map, like the case the deformation theory of complex structures.

**Theorem 4.1.**

\[ \frac{\partial}{\partial t} \{(M, L, \nabla_t), i_t\}_{t \in T} \big|_{t=0} \]

determines an element of \( \text{Ker} \overline{\partial}^{(1)}/\text{Im} \overline{\partial} \) (the first cohomology of the differential complex (2.7)).

**Definition 4.1.** Let \( W \) be a Kaehler manifold and \( \{(M, L, \nabla)\} \) be an A-brane in \( W \). Let \( \{\nabla_t\}_{t \in T} \) be a family of connections of \( L \), satisfying \( L_0 = L \). Let \( \xi_t \) of a section of \( \Gamma(M, TW | M) \), satisfying that \( \xi_0 = 0 \) and \( \xi_t \) can be extended to a neighborhood of \( M \), and let \( i_t \) be the embedding map, induced by \( \xi_t \). If the following holds, then \( \{(M, L_t, \nabla_t)\}_{t \in T} \) is called an infinitesimal deformation of A-branes.

1. The curvature of the connection \( \nabla_t \), \( F_t \), satisfies \( F_t(\xi_t, Y) \equiv 0 \) for \( Y \in D \), mod \( (t^2, \overline{t}) \).

2. Let \( J_t := (i_t^* \omega_W)^{-1} F_t \). Then, \( J_t^2 \equiv -1 \) mod \( (t^2, \overline{t}) \) on \( D \).

With this correspondence, we have

**Theorem 4.2.** For \( \phi \in \Gamma(M, (C \otimes \mathcal{L} + T_J'')^*) \), satisfying \( \overline{\partial}^{(1)} \phi = 0 \), on \( M \), we can set a family of deformations of A-branes, infinitesimally.

In a forthcoming paper, the proof is given.

**References**


