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On the deformation of A-branes in String theory

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1 A brief sketch

In this paper, we discuss the deformation theory of A-branes in String theory, from the point of view of CR structures and give an outline of our approach. The full paper will appear in another paper. Let W be a Kaehler manifold and let ω_W be its Kaehler form. Let M be a real hypersurface in W. We assume that our M admits an A-brane structure. Namely, there is a real line bundle L on M, and a connection ∇ on L, satisfying;

[1] The curvature of the connection, F , is an element of $\Gamma(M, \wedge^2\mathcal{F}^*)$,

[2] $J := \omega_W^{-1}F$ determines a complex structure on $\mathcal{F}$), where $\mathcal{F} := \frac{TM}{\mathcal{L}}$, and $\mathcal{L}$ is a characteristic foliation $\mathcal{L}$, defined by: for $p \in M$, $\mathcal{L}_p = \{Y_p, Y_p \in T_pW, \omega_W(Y_p, Y_p') = 0,\ Y_p' \in T_pM\}$.

In this paper, by using the notion of almost CR structures, we reformulate the notion of A-branes. Our J determines an almost CR structure (M, T_J'') on M. For this almost CR structure, we prove that $C \otimes \mathcal{L} + T_J''$ is integrable on M. And show the deformation complex of A-branes (the Kapustin-Orlov complex)(see (2.7)). This is a natural generalization of the case $M = W$(Kapustin-Orlov consider the case; A-branes wrap the whole W, and obtain the standard $\overline{\partial}$-complex as a deformation complex).

Here we treat A-branes of the type hypersurfaces. Now for a given A-brane, we introduce the notion of family of A-branes, $\{(M, L, \nabla_t)\}_{t\in T}$. In this paper, we introduce the deformation complex of A-branes, and construct the Kodaira-Spencer map for the given family of A-branes. On the parameter space, a complex structure is given. But, we are relying on the Hamilton deformation, so we can't discuss in the complex analytic category(so we have to use that $\{(M, L, \nabla_t)\}_{t\in T}$ depends on t,C^∞-ly). And because of this fact, we have to discuss in the category, mod $(t^2, \bar{t})$.

The author would like to thank Prof.A.Kapustin for allowing me to use the name, the Kapustin-Orlov complex and valuable suggestions during the preparation of this paper(the author learned that Kapustin and his student Yi Li, independently, obtained the integrability of $C \otimes \mathcal{L} + T_J''$).

2 The Kapustin-Orlov complex

In [Kap-Or], Kapustin-Orlov formulate the D-branes of A-type(in their language, A-branes), mathematically. We consider the deformation theory of A-branes in the case real hypersurfaces. For this, we recall the notion of A-branes. Let W be a Kaehler manifold. Let ω_W be its Kaehler metric. Let M be a real submanifold of W. Then, for this M, we have a characteristic foliation $\mathcal{L}$. This is defined by: for $p \in M$,

$$\mathcal{L}_p = \{Y_p, Y_p \in T_pW, \omega_W(Y_p, Y_p') = 0,\ Y_p' \in T_pM\}.$$

By this definition, $\mathcal{L}$ is a subbundle of $TW\mid_M$ and the rank of $\mathcal{L}$ is $2n - dim_R M$, because of ω_W being non-degenerate(here n is the complex dimension of W).

Definition 2.1. *If for $p \in M$, $\mathcal{L}_p \subset T_pM$, then M is called coisotropic.*

Henceforth we assume that our real submanifold is coisotropic. So, on M, we have a quotient bundle

$$\mathcal{F} := \frac{TM}{\mathcal{L}}.$$

Definition 2.2. *(A-branes). Let M be a coisotropic submanifold. Then M admits the A-brane if and only if there is a real line bundle L and a connection ∇ of L, (L, ∇) which satisfies*

[1] The curvature of the connection, F , is an element of $\Gamma(M, \wedge^2\mathcal{F}^*)$,

[2] $J := \omega_W^{-1}F$ determines a "Tac" structure on M(this means that:$J^2 = -1$ and this J is integrable modulo characteristic foliation).

Now for the submanifold M, a CR structure $(M, {}^0T'')$ is introduced by:

$${}^0T'' = C \otimes TM \cap T''W \mid M,$$

where $C \otimes TM$ means the complexfied tangent bundle of M. Let $D = \{Y : Y \in TM, Y = X + \overline{X},\ X \in {}^0T''\}$. Then, naturally,

$$D \cong \mathcal{F}.$$

By this identification, J is defined on D, satisfying : $J^2 = -1$. Hence J determines an almost CR structure on M. We study this structure. J is defined on D. We extend this J on $C \otimes D$, naturally. Set

$$T_J' = \{X : X \in C \otimes D,\ JX = \sqrt{-1}X\},$$
$$T_J'' = \{X' : X' \in C \otimes D,\ JX' = -\sqrt{-1}X'\}.$$

Then, as mentioned in [Kap-Or], we have

Proposition 2.1.

$$C \otimes D = T_J' + T_J'',\ T_J' \cap T_J'' = 0, \tag{2.1}$$
$$[\Gamma(M, T_J'), \Gamma(M, T_J')] \subset \Gamma(M, T_J')\ mod\ \mathcal{L}. \tag{2.2}$$

Proof. (0.1) is obvious. We see (0.2). By the definition, $dF = 0$, $d\omega_W = 0$, and

$$\omega_W(X, JX') = F(X, X'),\ X, X' \in C \otimes D.$$

With these, we compute : for $X_1, X_2 \in \Gamma(M, T'_J)$, $X \in \Gamma(M, C \otimes TM)$,

$$dF(X_1, X_2, X) = 0, \tag{2.3}$$
$$d\omega_W(X_1, X_2, X) = 0. \tag{2.4}$$

We compute (0.3). Then,

$$X_1 F(X_2, X) - X_2 F(X_1, X) + X F(X_1, X_2)$$
$$-F([X_1, X_2], X) + F([X_1, X], X_2) - F([X_2, X], X_1) = 0.$$

We rewrite this by using : $\omega_W(X, JX') = F(X, X'), X, X' \in C \otimes D$.

$$X_1 \omega_W(JX_2, X) - X_2 \omega_W(JX_1, X) + X\omega_W(JX_1, X_2)$$
$$-\omega_W(J[X_1, X_2], X) + \omega_W([X_1, X], JX_2) - \omega_W([X_2, X], JX_1) = 0.$$

By $JX_i = \sqrt{-1}X_i, i = 1, 2$, this becomes

$$X_1 \omega_W(\sqrt{-1}X_2, X) - X_2 \omega_W(\sqrt{-1}X_1, X) + X\omega_W(\sqrt{-1}X_1, X_2)$$
$$-\omega_W(J[X_1, X_2], X) + \omega_W([X_1, X], \sqrt{-1}X_2) - \omega_W([X_2, X], \sqrt{-1}X_1) = 0.$$

While, by (0.4),

$$X_1 \omega_W(X_2, X) - X_2 \omega_W(X_1, X) + X\omega_W(X_1, X_2)$$
$$-\omega_W([X_1, X_2], X) + \omega_W([X_1, X], X_2) - \omega_W([X_2, X], X_1) = 0.$$

Hence, we have

$$\omega_W(J[X_1, X_2], X) = \omega_W(\sqrt{-1}[X_1, X_2], X) \text{ for any } X \in C \otimes D.$$

This means that: $[X_1, X_2] \in T'_J$ modulo $\mathcal{L}$. □

The following proposition is also mentioned in [Kap-Or].

Proposition 2.2.

$$\omega_W(X_1, X_2) = 0 \textit{ for } X_1 \in T'_J, X_2 \in T''_J.$$

So, J-structure is different from the CR structure, naturally, induced from W. Here for the convenience, we give a proof.

Proof. We use $\omega_W(X, JY) = F(X, Y)$, for any $X, Y \in C \otimes TM$. For $X_1 \in T'_J, X_2 \in T''_J$,

$$\omega_W(X_1, JX_2) = F(X_1, X_2).$$

By $JX_2 = -\sqrt{-1}X_2$,

$$\omega_W(X_1, -\sqrt{-1}X_2) = F(X_1, X_2),$$

so,

$$\omega_W(X_1, X_2) = \sqrt{-1}F(X_1, X_2).$$

On the other hand,

$$\omega_W(X_2, JX_1) = F(X_2, X_1).$$

So, by $JX_1 = \sqrt{-1}X_1$,

$$\omega(X_2, X_1) = -\sqrt{-1}F(X_2, X_1).$$

Hence

$$\omega(X_1, X_2) = -\sqrt{-1}F(X_1, X_2).$$

This means that $\omega_W(X_1, X_2) = 0$. □

As is mentioned in [Kap-Or], the following corollary follows from this proposition.

Corollary 2.3.

$$dim_C T'_J = even.$$

Now we set a C^∞ vector bundle decomposition

$$C \otimes TM = C \otimes \mathcal{L} + T''_J + T'_J.$$

Here $C \otimes \mathcal{L}$ means the complexfied $\mathcal{L}$. While in our case, (M, T''_J) may not be a CR structure(only integrable modulo $\mathcal{L}$). But,

Proposition 2.4. *$\mathcal{L}$ preserves J, namely,*

$$[\Gamma(M, T'_J), \mathcal{L}] \subset \Gamma(M, T'_J) \ modulo\ \mathcal{L}.$$

Proof. By the same ways as in Proposition 2, we see this proposition.
For $X \in T'_J, Y \in T''_J, \zeta \in \mathcal{L}$, as F, ω_W are closed,

$$dF(X, Y, \zeta) = 0,$$
$$d\omega_W(X, Y, \zeta) = 0.$$

By the first equation,

$$XF(Y, \zeta) - YF(X, \zeta) + \zeta F(X, Y)$$
$$-F([X, Y], \zeta) + F([X, \zeta], Y) - F([Y, \zeta], X) = 0.$$

As $\mathcal{L}$ is a characteristic foliation, this becomes

$$\zeta F(X, Y) + F([X, \zeta], Y) - F([Y, \zeta], JX) = 0.$$

With $\omega_W(X', JY') = F(X', Y')$ for $X', Y' \in C \otimes D$,

$$\zeta\omega_W(JX, Y) + \omega_W([X, \zeta], JY) - \omega_W([Y, \zeta], JY) = 0.$$

While, by Proposition 2,

$$\begin{aligned} \omega_W(JX, Y) &= \omega_W(\sqrt{-1}X, Y) \\ &= 0. \end{aligned}$$

Hence

$$\omega_W([X, \zeta], -\sqrt{-1}Y) - \omega([Y, \zeta], \sqrt{-1}X) = 0. \tag{2.5}$$

While by the second equation,

$$X\omega_W(Y,\zeta) - Y\omega_W(X,\zeta) + \zeta\omega_W(X,Y)$$
$$-\omega_W([X,Y],\zeta) + \omega_W([X,\zeta],Y) - \omega_W([Y,\zeta],X) = 0.$$

So, by the same way, this becomes

$$\omega_W([X,\zeta],Y) - \omega([Y,\zeta],X) = 0. \tag{2.6}$$

With $(0.5), (0.6)$, we have

$$\omega_W([X,\zeta],Y) = 0, \text{ for } X \in T'_J\ Y \in T''_J$$

This means that: the T''_J part of $[X,\zeta]$ vanishes because of ω_W being nondegenerate with Proposition 2.2. Hence

$$[X,\zeta] \in \Gamma(M,T'_J) \text{ modulo } \mathcal{L}.$$

□

Now we can state our theorem.

Theorem 2.5. *We set* $T'' := C \otimes \mathcal{L} + T''_J$. *Then,*

$$[\Gamma(M,T''),\Gamma(M,T'')] \subset \Gamma(M,T'').$$

By this theorem, we have the deformation complex of A-branes (Kapustin-Orlov complex). Namely, for $u \in \Gamma(M,C)$, we set $\overline{\partial}u$ of $\Gamma(M,(T'')^*)$ by;

$$\overline{\partial}u(X) = Xu, \text{ for } X \in T''.$$

By the same way as for ordinary differntial forms, we can introduce $\overline{\partial}^p$ from $\Gamma(M,\wedge^p(T'')^*)$ to $\Gamma(M,\wedge^{p+1}(T'')^*)$.

$$\overline{\partial}^p : \Gamma(M,\wedge^p(T'')^*) \to \Gamma(M,\wedge^{p+1}(T'')^*).$$

Then, by the integrability theorem(Theorem 2.5),

$$\overline{\partial}^{p+1}\overline{\partial}^p = 0.$$

So, we have a deformation complex of A-branes(Kapustin-Orlov complex).

$$0 \to \Gamma(M,C) \xrightarrow{\overline{\partial}} \Gamma(M,(T'')^*) \xrightarrow{\overline{\partial}^1} \Gamma(M,\wedge^2(T'')^*) \to \cdots \tag{2.7}$$

Furthermore, by this theorem, we can introduce a sheaf, $\mathcal{O}_{T''}$, composed of $\overline{\partial}$-closed elements, which are holomorphic in the direction T''_J, and constant in the direction $\mathcal{L}$.

3 A family of deformations of A-branes

We introduce the notion of a family of deformations of A-branes,

Definition 3.1. *A set of A-branes* $\{(M, L, \nabla_t), i_t\}_{t \in T}$, *where* T *is an analytic space with the origin* o, *is a family of deformations of A-branes if*

(1) *connections* ∇_t *depends on* t, $\mathbf{C}^\infty$*-ly, and* $\nabla_o = \nabla$,

(2) *embeddings* i_t *depends on* t, $\mathbf{C}^\infty$*-ly, and* $i_o = i$.

Unlike CR structures, we rely on $\mathbf{C}^\infty$ category. Because, in the case symplectic structures, the Hamiltonian deformations play an essential part. We study a family of deformations of A-branes in the case real hypersurfaces. For the embedding i_t, we have the characteristic vector field, ξ_t. By using this vector field, the condition of $\{(M, L, \nabla_t), i_t\}$ being the A-brane is rewritten as follows.

$[1]_t$ The curvature of the connection ∇_t, F_t , is an element of $\Gamma(M, \wedge^2 \mathcal{F}_t^*)$,

$[2]_t$ Let $J_t := (i_t^* \omega_W)^{-1} F_t$. Then,$J_t^2 = -1$ on $\mathcal{F}_t$, where

$$\mathcal{F}_t := \frac{TM}{\mathcal{L}_t},$$

and $\mathcal{L}_t$ is generated by ξ_t. While the inclusion map induces a bundle isomorphism map ρ_t; from D to $\frac{TM}{\mathcal{L}_t}$, induced by the inclusion map ; D to TM. The structure defined by J_t induces an almost CR structure on D by;

$$J'_t := \rho_t^{-1} J_t \rho_t.$$

Henceforth, we use the same notation J_t for J'_t and we regard J_t as an almost CR structure on D. Therefore $[1]_t$,$[2]_t$ are written as

$[1]'_t$ The curvature of the connection ∇_t, F_t , satisfies $F_t(\xi_t, Y) = 0$ for $Y \in D$,

$[2]'_t$ Let $J_t := (i_t^* \omega_W)^{-1} F_t$. Then, $J_t^2 = -1$ on D.

We see why we call this complex a deformation complex of A-branes.

Definition 3.2. *The quartets of A-branes,* $\{(M, L, \nabla), i\}$, $\{(M, L', \nabla'), i'\}$ *are equivalent if there is a gauge transform of the line bundle L (we write this bundle map by q), and there is a Hamiltonian diffeomorphism map of W, defined by a* C^∞ *function g (we write it by* V_g*), satisfying;*

(1) *the composition of maps* V_g *and* i, $V_g i = i'$,

(2) $V_g^* q^* \nabla = \nabla'$

Next we introduce an equivalence relation for a family of deformations of A-branes.

Definition 3.3. *The family of deformations of A-branes,* $\{(M, L, \nabla_t), i_t\}_{t \in T}$, $\{(M, L', \nabla'_s), i'_s\}_{s \in S}$ *are equivalent if there is a local biholomorphic map h from T to S satisfying :* $h(o) = o$, *there is a gauge transform of the line bundle L (we write this bundle map by q), and there is a Hamiltonian diffeomorphism map of W, defined by a* C^∞ *function* g_t *(we write it by* V_{g_t}*), satisfying;*

(1) *the composition of maps* V_{g_t} *and* i_t, $V_{g_t} i_t = i'_{h(t)}$,

(2) $V_{g_t}^* q_t^* \nabla_t = \nabla'_{h(t)}$

4 The infinitesimal case

For a family of deformations of A-branes, $\{(M, L, \nabla_t), i_t\}_{t\in T}$, we can introduce the Kodaira-Spencer map, like the case the deformation theory of complex structures.

Theorem 4.1.
$$\frac{\partial}{\partial t}\{(M, L, \nabla_t), i_t\}_{t\in T}\mid_{t=o}$$

determines an element of $Ker\ \overline{\partial}^{(1)}/Im\ \overline{\partial}$ *(the first cohomology of the differential complex* (2.7)*).*

Definition 4.1. *Let W be a Kaehler manifold and $\{(M, L, \nabla)\}$ be an A-brane in W. Let $\{\nabla_t\}_{t\in T}$ be a family of connections of L, satisfying $L_o = L$. Let ξ_t of a section of $\Gamma(M, TW\mid_M)$, satisfying that ;$\xi_o = 0$ and ξ_t can be extended to a neighborhood of M, and let i_t be the embedding map, induced by ξ_t. If the following holds, then $\{(M, L_t, \nabla_t)\}_{t\in T}$ is called an infnitesimal deformation of A-branes.*

$[1]'_t$ *The curvature of the connection ∇_t, F_t , satisfies $F_t(\xi_t, Y) \equiv 0$ for $Y \in D$, mod $(t^2, \bar{t})$*

$[2]'_t$ *Let $J_t := (i_t^*\omega_W)^{-1}F_t$. Then, $J_t^2 \equiv -1$ mod $(t^2, \bar{t})$ on D.*

With this correspondence, we have

Theorem 4.2. *For $\phi \in \Gamma(M, (C \otimes \mathcal{L} + T''_J)^*)$, satisfying ; $\overline{\partial}^{(1)}\phi = 0$, on M, we can set a family of deformations of A-branes, infinitesimally.*

In a forthcomming paper, the proof is given.

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