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EXACT NORMAL FORM FOR (2,5) DISTRIBUTIONS

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1. Introduction

In this work we present a complete solution of the classical problem, going back to E. Cartan, on the classification of generic $(2,5)$-distributions on the level of formal power series.

Namely, we define explicitly a set $\{\mathcal{D}_{C(x)}\}$ of $(2,5)$-distributions parameterized by a function $C(x_1, \ldots, x_5)$ and a set $\mathbf{N}$ of functions of five variables such that the following holds:

a generic $(2,5)$-distribution germ is formally (on the level of formal power series) diffeomorphic to a distribution of the form $\mathcal{D}_{C(x)}, C(x) \in \mathbf{N}$;

two distribution germs $\mathcal{D}_{C(x)}$ and $\mathcal{D}_{\tilde{C}(x)}$ with $C(x)$ $\tilde{C}(x) \in \mathbf{N}$ are formally diffeomorphic if and only if $C(x)$ and $\tilde{C}(x)$ have the same Taylor series.

Our approach is based on the quasi-homogeneous filtration with the natural weights $1,1,2,3,3$. The starting points are as follows:

- all $(2,3,5)$-distributions (i.e. $(2,5)$ distributions with the growth vector $(2,3,5)$) have diffeomorphic quasi-homogeneous 2-jets;
- the classical Cartan invariant is a *complete* invariant in the classification of quasi-homogeneous 3-jets of $(2,3,5)$-distributions.

I started to think about the possibility to obtain an exact normal form after the RIMS Symposium "Developments of Cartan Geometry and Related Mathematical Problems", Kyoto, October 2005. In this Symposium I heard many beautiful talks related to E. Cartan's work [3] written 100 years ago. In particular, the Cartan invariant was explained in [5, 8] by developing the theory [6, 7] of graded nilpotent Lie algebras associated with differential systems and in [9] in terms of the variational-symplectization approach [1, 2]. The coordinate-free constructions of these works allow to obtain many new invariants, see for example [4, 10]. On the other hand, these constructions do not solve, as far as I know, the problem of complete classification of generic $(2,5)$-distributions, i.e. the problem of finding a complete system of independent invariants and realizing them in an exact normal form.

I heard the opinion that constructing an exact normal form without coordinate-free invariants, just using certain methods of step-by-step normalization of formal power series, is a hopeless task. Basing on my own attempts I agree with this opinion if step-by-step means normalizing $(k+1)$-jet with already normalized k-jet and

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the k-jet is understood in usual way – the segment of the Taylor expansion including terms of degree $\leq k$. Under such normalization the variables $x_1, .., x_5$ have the same weights 1. On the other hand it is clear that working with $(2,5)$-distributions the natural weights are $1,1,2,3,3$. I decided to classify $(2,5)$-distributions using the quasi-homogeneous filtration with these weights in the space of vector field germs.

A quasi-homogeneous degree d vector field is a linear combination of quasi-homogeneous monomial vector fields of degree d. The quasi-homogeneous degree of a monomial vector field $x^\alpha \partial/\partial x_i$ is $(\alpha, \lambda) - \lambda_i$, where $\lambda = (\lambda_1, ..., \lambda_5)$ is the tuple of weights. This definition is natural because it is easy to check that the Lie bracket of quasi-homogeneous vector fields of degrees d_1 and d_2 is a quasi-homogeneous vector field of degree $d_1 + d_2$.

With weights $1,1,2,3,3$ the Taylor series of vector fields spanning a $(2,5)$ distribution contains terms of quasi-homogeneous degrees starting with -3. The first, rather simple step is to prove that the quasi-homogeneous (-1)-jets of all $(2,3,5)$-distributions are diffeomorphic. I think that to some extend the quasi-homogeneous (-1)-jet is the nilpotent approximation of a $(2,3,5)$-distribution, but I am not ready to develop this claim in the present work.

The second, also relatively easy step, is to prove that all quasi-homogeneous terms of degree $0,1$ and 2 can be killed by a change of coordinates and change of the frame of vector fields spanning a $(2,3,5)$-distribution. Therefore all $(2,3,5)$ distributions have diffeomorphic quasi-homogeneous 2-jets.

The third step is to realize the classical Cartan invariant as follows: it is the only invariant distinguishing non-diffeomorphic quasi-homogeneous 3-jets. In the original E.Cartan's work [3] this invariant was obtained by rather sophisticated manipulations. Within the quasi-homogeneous approach it appears in a natural way and does not require much calculations. After this step it became clear that an exact normal form can be constructed.

The main final results are formulated in section 2. In this paper we do not present complete proofs (they will be published elsewhere), but in sections 2-9 all results are explained and the outline of the proofs is given.

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2. Main results

We work with germs at $0 \in \mathbb{R}^5$ of C^∞ distributions, but our normal forms hold in the formal category, i.e. on the level of formal power series. The Borel theorem on the realization of any formal power series as the Taylor series of a C^∞ function allows to define the formal equivalence as follows: two germs D and $\tilde{D}$ of 2-distributions are formally equivalent if there exist a couple of vector field germs (V_1, V_2) spanning D, a couple of vector field germs $(\tilde{V}_1, \tilde{V}_2)$ spanning $\tilde{D}$, and a local

C^∞ diffeomorphism $\Phi : (\mathbb{R}^5, 0) \to (\mathbb{R}^5, 0)$ such that the vector fields $\Phi_* V_i$ and $\tilde{V}_i$ have the same Taylor series, $i = 1, 2$. Of course the definition remains the same if one of the frames, either (V_1, V_2) or $(\tilde{V}_1, \tilde{V}_2)$, is fixed.

In subsection 2.1 we present a formal normal form serving for all (2, 3, 5) distributions. In subsection 2.2 we explain, in terms of this normal form, the classical Cartan's tensor and Cartan's invariant. In subsection 2.3 we present a formal normal form for (2, 3, 5) distributions with a non-degenerate Cartan tensor. This normal form is almost exact: it is exact up to a certain discrete group of linear transformations. A completely exact normal form is presented in subsection 2.4.

2.1. **Normal form for all** $(2, 3, 5)$ **distributions.** Fix the following couples of vector fields

$$(2.1)\quad \mathcal{A} = \begin{pmatrix} \mathcal{A}_1 \\ \mathcal{A}_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_3} \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \end{pmatrix} = \begin{pmatrix} x_2 \cdot (x_1 \frac{\partial}{\partial x_4} + x_2 \frac{\partial}{\partial x_5}) \\ -x_1 \cdot (x_1 \frac{\partial}{\partial x_4} + x_2 \frac{\partial}{\partial x_5}) \end{pmatrix}$$

$$(2.2)\qquad \mathcal{N} = \mathcal{A} + \mathcal{B}$$

Fix also the following ideal in the ring of function germs.

Notation. By **I** we denote the ideal in the ring of function germs at $0 \in \mathbb{R}^5$ generated by the following monomials of degree 4:

$$(2.3)\qquad x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}, \quad \text{where } \alpha_1 + \alpha_2 \geq 2, \quad \alpha_1 + \alpha_2 + \alpha_3 = 4;$$

$$(2.4)\qquad x_1^2 x_3 x_4, \quad x_2^2 x_3 x_5, \quad x_1 x_2 x_4 x_5.$$

Theorem A. *Any germ at* $0 \in \mathbb{R}^5$ *of a* $(2, 3, 5)$ *distribution is formally equivalent to a distribution spanned by a couple of vector fields of the form*

$$(2.5)\qquad \mathcal{N} + C(x) \cdot \mathcal{B}, \quad C(x) \in \mathbf{I}$$

2.2. **Cartan invariant.** About 100 years ago in the work [3] E. Cartan associated to a $(2, 3, 5)$-distribution a certain homogeneous degree 4 polynomial of two variables x_1, x_2, obtained by rather involved calculations realizing so-called Cartan method, and proved that the equivalence class of this polynomial with respect to the group of linear transformations of the plane $\mathbb{R}^2(x_1, x_2)$ is an invariant of the distribution – it is the same for all diffeomorphic distributions. In terms of normal form (2.5) the Cartan invariant can be defined as follows. Note that for any function $C(x)$ belonging to the ideal **I** one has

$$(2.6)\qquad C(x_1, x_2, 0, 0, 0) = P^{(4)}(x_1, x_2) + o(||(x_1, x_2)||^4),$$

where $P^{(4)}(x_1, x_2)$ is a homogeneous degree 4 polynomial.

Definition 2.1. Let D be the germ at $0 \in \mathbb{R}^5$ of a $(2, 3, 5)$-distribution. Let $C(x)$ be the functional parameter in the normal form (2.5). The homogeneous degree 4 polynomial $P^{(4)}(x_1, x_2)$ defined by (2.6) will be called the Cartan tensor of D.

The following proposition is "isomorphic" to E.Cartan's result in [3].

Proposition 2.2. *If two germs at* 0 *of* $(2, 3, 5)$*-distributions are diffeomorphic then the corresponding Cartan tensors (according to Definition 2.1) are linearly equivalent, i.e. can be brought one to the other by a non-degenerate linear transformation of the plane* $\mathbb{R}^2(x_1, x_2)$.

Proposition 2.2 defines a modulus (Cartan invariant) in the classification of $(2,3,5)$-distributions - the factor-class of the Cartan tensor with respect to the linear equivalence. The explanation of this invariant is given in section 8: it is a complete invariant in the classification of quasi-homogeneous 3-jets of $(2,3,5)$-distributions with respect to the natural weights $1,1,2,3,3$.

Definition 2.3. A homogeneous polynomial $P^{(4)}(x_1,x_2)$ is called non-degenerate if $P^{(4)}(x_1,x_2) \neq A^2(x_1,x_2)B(x_1,x_2)$ for any polynomials $A(x_1,x_2), B(x_1,x_2)$.

It is easy to prove that any non-degenerate homogeneous degree 4 polynomial of two variables can be reduced by a linear change of coordinates to one *and only one* of the polynomials of the set (normal form) $\mathbf{P}^{(4),+} \cup \mathbf{P}^{(4),-}$, where

$\mathbf{P}^{(4),+}$ is the set of all polynomials of the form $x_1^4 + \alpha x_1^2x_2^2 + x_2^4$, $|\alpha| > 2$ and of the form $-x_1^4 - \alpha x_1^2x_2^2 - x_2^4$, $\alpha > 2$;

$\mathbf{P}^{(4),-}$ is the set of all polynomials of the form $x_1^4 + \alpha x_1^2x_2^2 - x_2^4$, $\alpha \in \mathbb{R}$.

Therefore in the non-degenerate case the Cartan invariants are represented by the polynomials of the set $\mathbf{P}^{(4),+} \cup \mathbf{P}^{(4),-}$ and no two different polynomials of this set represent the same Cartan invariant.

Definition 2.4. We will say that a non-degenerate polynomial $P^{(4)}(x_1,x_2)$ has a positive (respectively negative) type if it is linearly equivalent to a polynomial of the set $\mathbf{P}^{(4),+}$ (respectively $\mathbf{P}^{(4),-}$).

2.3. **Exact normal form up to discrete group of involutions.** In this subsection we give a normal form for $(2,3,5)$ distributions with non-degenerate Cartan tensor. This normal form is exact up to a certain discrete group of linear transformations.

Introduce the following subsets $\mathbf{I}_0^+, \mathbf{I}_0^-$ of the ideal $\mathbf{I}$.

The set $\mathbf{I}_0^{\pm}$ of function germs. We will denote by $\mathbf{I}_0^+$ (respectively $\mathbf{I}_0^-$) the subset of the ideal $\mathbf{I}$ consisting of function germs $C(x) \in \mathbf{I}$ satisfying the following conditions:

1. $C(x_1,x_2,0,0,0) = P^{(4)}(x_1,x_2) + o(||(x_1,x_2)||^4)$, where $P^{(4)}(x_1,x_2) \in \mathbf{P}^{(4),+}$ (respectively $P^{(4)}(x_1,x_2) \in \mathbf{P}^{(4),-}$);

2. The Taylor expansion of $C(x)$ does not contain monomials $x_1^3x_3, x_2^3x_3, x_1^4x_4, x_2^4x_5$;

3. The sum (respectively the difference) of the coefficients at the monomials $x_1^4x_3$ and $x_2^4x_3$ in the Taylor expansion of $C(x)$ is equal to 0.

We also need the involutions

$$i_1 : \; x_1 \to -x_1, \;\; x_2 \to x_2, \; x_3 \to -x_3, \; x_4 \to x_4, \; x_5 \to -x_5$$

$$i_2 : \; x_1 \to x_1, \;\; x_2 \to -x_2, \; x_3 \to -x_3, \; x_4 \to -x_4, \; x_5 \to x_5$$

$$i_3 : \; x_1 \to x_2, \;\; x_2 \to x_1, \; x_3 \to -x_3, \; x_4 \to -x_5, \; x_5 \to -x_4$$

Note that the involutions i_1, i_2 commute and the group generated by these involutions consists of 3 non-identity transformations; the group generated by i_1, i_2, i_3 consists of 7 non-identity transformations.

Theorem B.

(i). The germ at 0 *of any* $(2,3,5)$*-distribution germ with a non-degenerate Cartan tensor of positive (respectively negative) type is formally equivalent to a distribution spanned by a couple of vector fields*

$$\mathcal{N} + C(x)\cdot\mathcal{B}, \quad C(x)\in \mathbf{I}_0^+ \quad (\textit{respectively } C(x)\in \mathbf{I}_0^-). \tag{2.7}$$

(ii). Two distribution germs spanned by vector fields $\mathcal{N}+C(x)\cdot\mathcal{B}$ *and* $\mathcal{N}+\tilde{C}(x)\cdot\mathcal{B}$ *with* $C(x),\tilde{C}(x)\in \mathbf{I}_0^+$ *(respectively* $C(x),\tilde{C}(x)\in\mathbf{I}_0^-$*) are formally equivalent if and only if the Taylor series of* $\tilde{C}(x)$ *can be brought to the Taylor series of* $C(x)$ *by a linear change of coordinates of the group generated by the involutions* i_1,i_2,i_3 *(respectively the group generated by the involutions* i_1,i_2*).*

2.4. **Exact normal form.** The action of the discrete group of transformations in Theorem **B** can be easily "killed" by replacing the set $\mathbf{I}_0^\pm$ by a subset $(\mathbf{I}_0^\pm)^\# \subset \mathbf{I}_0^\pm$ consisting of function-germs $C(x)\in\mathbf{I}_0^\pm$ whose Taylor series satisfy certain conditions of the form of inequalities. To present these condition one has to chose an open subset O^+ (respectively O^-) of the set of $(2,3,5)$ distributions with a non-degenerate Cartan tensor of positive (respectively negative) type. Consider, for example, the following open sets.

The open sets $O^\pm$ **in the space of** $(2,5)$ **distribution germs.** We denote by O^+ (respectively O^-) the open set consisting of $(2,3,5)$ distribution germs at the origin satisfying the following conditions:

1. The Cartan tensor is non-degenerate and has positive (respectively negative) type;

2. Let b_1,b_2 be the coefficients at the monomials $x_1^2x_2x_3$ and $x_1x_2^2x_3$ in the Taylor expansion of $C(x)$ in the normal form (2.7). Then $b_1\neq 0, b_2\neq 0, b_1\neq b_2$ (respectively $b_1\neq 0, b_2\neq 0$).

For this choice of $O^\pm$ the set $(\mathbf{I}_0^\pm)^\#\subset\mathbf{I}_0^\pm$ is as follows.

The set $(\mathbf{I}_0^\pm)^\#$ **of function germs.** By $(\mathbf{I}_0^-)^\#$ we denote the subset of the set $\mathbf{I}_0^-$ consisting of function germs $C(x)\in\mathbf{I}_0^-$ such that the coefficients b_1,b_2 at $x_1^2x_2x_3$ and $x_1x_2^2x_3$ in the Taylor expansion of $C(x)$ are positive. By $(\mathbf{I}_0^+)^\#$ we denote the subset of the set $\mathbf{I}_0^+$ consisting of function germs $C(x)\in\mathbf{I}_0^+$ such that the same coefficients satisfy the condition $b_1>b_2>0$.

Theorem C.

(i). Any $(2,3,5)$ *distribution germ in the open set* O^+ *(respectively* O^-*) is formally equivalent to a distribution spanned by a couple of vector fields of the form*

$$\mathcal{N}+C(x)\cdot\mathcal{B}, \quad C(x)\in(\mathbf{I}_0^+)^\# \quad (\textit{respectively } C(x)\in(\mathbf{I}_0^-)^\#). \tag{2.8}$$

(ii). This normal form is exact: two distributions spanned by vector fields of the form $\mathcal{N}+C(x)\cdot\mathcal{B}$ *and* $\mathcal{N}+\tilde{C}(x)\cdot\mathcal{B}$ *with* $C(x),\tilde{C}(x)\in(\mathbf{I}_0^+)^\#$ *(respectively* $C(x),\tilde{C}(x)\in(\mathbf{I}_0^-)^\#$*) are formally equivalent if and only if the function germs* $C(x)$ *and* $\tilde{C}(x)$ *have the same Taylor series.*

3. Quasi-homogeneity

The explanation of results of section 2 requires the quasi-homogeneous filtration in the space of functions and vector fields with the natural for (2, 3, 5)-distributions weights 1, 1, 2, 3, 3.

Fix a coordinate system $x_1, ..., x_n$ and positive numbers $\lambda_1, ..., \lambda_n$ which will be called the weights of $x_1, .., x_n$. In the definitions below by quasi-homogeneity we mean quasi-homogeneity with respect to these weights.

• The quasi-homogeneous degree of a monomial x^α is the number $(\alpha, \lambda) = \alpha_1\lambda_1 + \cdots + \alpha_n\lambda_n$;

• The quasi-homogeneous degree of a monomial vector field $x^\alpha \frac{\partial}{\partial x_j}$ is the number $(\alpha, \lambda) - \lambda_j$.

• A function (vector field) is called quasi-homogeneous of degree d if it is a linear combination with numerical coefficients of monomials (monomial vector fields) of quasi-homogeneous degree d. By definition the zero function (vector field) is quasi-homogeneous of any degree.

• The quasi-homogeneous r-jet of a function (a vector field) is its r-equivalence class, where the r-equivalence is as follows: two functions (vector fields) are r-equivalent if they have the same segment of Taylor expansions containing monomials (monomial vector fields) of degree $\leq r$. Usually the r-jet will be identified with this segment of Taylor expansion. The quasi-homogeneous r-jet will be denoted j^r_{qh}.

One can easily check the following properties.

Proposition 3.1. *Let f_1, f_2 be quasi-homogeneous functions of degrees $d(f_1), d(f_2)$ and let V_1, V_2 be quasi-homogeneous vector fields of degrees $d(V_1), d(V_2)$.*

(i). $f_1 f_2$ is a quasi-homogeneous function of degree $d(f_1) + d(f_2)$;

(ii). $f_1 V_1$ is a quasi-homogeneous vector field of degree $d(f_1) + d(V_1)$;

(iii). $[V_1, V_2]$ is a quasi-homogeneous vector field of degree $d(V_1) + d(V_2)$.

4. Convention and notations. The group G

If D is a $(2, 3, 5)$ distribution then in suitable coordinates

$$D(0) = span(\partial/\partial x_1, \partial/\partial x_2),\ D^2(0) = D(0) + span(\partial/\partial x_3),$$
$$D^3(0) = D^2(0) + span(\partial/\partial x_4, \partial/\partial x_5).$$

Therefore the natural weights in the study of (2, 3, 5)-distributions are 1, 1, 2, 3, 3.

Convention. In what follows we work in a fixed local coordinate system $x_1, ..., x_5$ and fix the weights

$$\text{weight } (x_1) = \text{ weight } (x_2) = 1,\ \text{weight } (x_3) = 2,\ \text{weight } (x_4) = \text{ weight } (x_5) = 3.$$

The quasi-homogeneity means the quasi-homogeneity with respect to these weights.

Notation. By $\mathbf{V}$ we denote the space of germs at 0 of vector field on $\mathbb{R}^5$; by $\mathbf{V}^{(i)}$ we denote the subspace of $\mathbf{V}$ consisting of quasi-homogeneous degree i vector fields.

Example. Throughout the paper we will use the couples of vector fields $\mathcal{A}, \mathcal{B}, \mathcal{N}$ in (2.1), (2.2). One has

$$\mathcal{A}, \mathcal{B}, \mathcal{N} \in (\mathbf{V}^{(-1)})^2.$$

We need few more notations:

F: the space of all function germs at 0; $\mathbf{F}^{(i)}$ - the subspace of **F** consisting of quasi-homogeneous degree i functions;

$\mathbf{M}_{2,2}$: the space of all 2×2 matrices whose entries are function germs, $\mathbf{M}_{2,2}^{(i)}$ - the subspace of $\mathbf{M}_{2,2}$ consisting of matrices whose entries belong to $\mathbf{F}^{(i)}$.

G: the group consisting of pairs $((H), \Phi)$, where $(H) \in \mathbf{M}_{2,2}$ is a matrix-function such that $det(H)(0) \neq 0$ and $\Phi : (\mathbb{R}^5, 0) \to (\mathbb{R}^5, 0)$ is a local diffeomorphism.

The group **G** acts in the space $\mathbf{V}^2$: $((H), \Phi).\xi = (H) \cdot \Phi_* \xi, \quad \xi \in \mathbf{V}^2$. The problem of classification of $(2,5)$-distributions coincides with the problem of the classification of the space $\mathbf{V}^2$ with respect to this action. Two couples $\xi, \tilde{\xi} \in \mathbf{V}^2$ will be called **G**-equivalent if they belong to one orbit of the action of **G**.

5. The quasi-homogeneous (-1)-jet.

Proposition 5.1. *Any couple $\xi \in \mathbf{V}^2$ spanning a $(2,3,5)$ distribution is $\mathbf{G}$-equivalent to a couple whose Taylor series has the form*

$$\mathcal{N} + \xi^{(0)} + \xi^{(1)} + \xi^{(2)} + \cdots, \quad \xi^{(i)} \in (\mathbf{V}^{(i)})^2. \tag{5.1}$$

The couple $\mathcal{N} \in (\mathbf{V}^{(-1)})^2$ is the quasi-homogeneous degree (-1) part in (5.1). It satisfies the following conditions:

1. The couple of vector fields $\mathcal{N}$ spans a $(2,3,5)$-distribution;
2. If $V_1, ..., V_4$ is one of the vector fields of the couple N then $[V_1, [V_2, [V_3, V_4]]] = 0$.

It is worth to note that the proof of these properties does not require calculations, they are direct corollaries of Proposition 3.1. I believe that the couple $\mathcal{N}$ is the nilpotent approximation of a $(2,3,5)$-distribution, i.e. *the nilpotent approximation of a $(2,3,5)$-distribution is its quasi-homogeneous (-1)-jet.* Nevertheless this claim, if it is correct, requires additional explanation.

6. The stationary group of the (-1)-jet. Subgroups $\mathbf{G}^+$ and $Symm^0(\mathcal{N})$

Now we will determine the group of transformations preserving the normal form (5.1). The group **G** can be decomposed onto three parts – the "negative", the "zero", and the "positive" parts.

Definition 6.1. Let $((H), \Phi) \in \mathbf{G}$. Write the matrix (H) and the diffeomorphism Φ in the form

$$(H) = I + (h), \quad \Phi : x_i \to x_i + \phi_i(x).$$

The subgroups $\mathbf{G}^-, \mathbf{G}^0$ and $\mathbf{G}^+$ (the negative, zero, and the positive parts of **G**) are distinguished by the following conditions, where w_i is the weight of x_i ($w_1 = w_2 = 1, w_3 = 2, w_4 = w_4 = 3$) and $i = 1, ..., 5$:

$\mathbf{G}^-$: (h) is the zero matrix and $\phi_i(x)$ is a polynomial containing no terms of quasi-homogeneous degree $\geq w_i$;

$\mathbf{G}^0$: (h) is a constant matrix and $\phi_i(x)$ is a quasi-homogeneous polynomial of degree w_i;

$\mathbf{G}^+$: $h(0) = 0$ and $\phi_i(x)$ is a function with zero quasi-homogeneous w_i-jet.

It is easy to show that $\mathbf{G}^-, \mathbf{G}^0$ and $\mathbf{G}^+$ are subgroups of $\mathbf{G}$ and for any $g \in G$ one has unique decomposition

$$g = g^+ \cdot g^0 \cdot g^-, \quad g^- \in \mathbf{G}^-, \; g^0 \in \mathbf{G}^0, \; g^+ \in \mathbf{G}^+. \tag{6.1}$$

Proposition 6.2. *Let ξ be any couple of vector fields of form (5.1) and let $g \in \mathbf{G}$. If $j_{qh}^{-1}(g.\xi) = j_{qh}^{-1}\xi = \mathcal{N}$ then the decomposition (6.1) contains no negative part: $g^- = id$. If $g \in \mathbf{G}^+$ then $j_{qh}^{-1}(g.\xi) = j_{qh}^{-1}\xi = \mathcal{N}$.*

It is not hard to prove that the group $\mathbf{G}^0$ acts in the space $(\mathbf{V}^{(i)})^2$, for any i:

$$g \in \mathbf{G}^0, \;\; \xi \in (\mathbf{V}^{(i)})^2 \;\; \Longrightarrow \;\; g.\xi \in (\mathbf{V}^{(i)})^2.$$

Notation. By $Symm^0(\mathcal{N})$ we denote the subgroup of $\mathbf{G}^0$ preserving $\mathcal{N}$.

Proposition 6.2 implies the following statement.

Proposition 6.3. *Two couples of form (5.1) are* $\mathbf{G}$*-equivalent if and only if they are equivalent with respect to the action of the group $\mathbf{G}^+ \cdot Symm^0(\mathcal{N})$.*

The group $Symm^0(\mathcal{N})$ can be calculated. Note that the group $\mathbf{G}^0$ contains non-linear transformations. It is not hard to prove that the group $Symm^0(\mathcal{N})$ contains linear transformations only and they have the following form.

Proposition 6.4. *The group $Symm^0(\mathcal{N})$ is isomorphic to the group of constant non-singular 2×2 matrices. The isomorphism is as follows: $Q \to g^0_Q = ((Q^t), L_Q) \in Symm^0(N)$, where Q is an arbitrary constant non-singular 2×2 matrix and L_Q is the following linear transformation of $\mathbb{R}^5$:*

$$\begin{aligned} L_Q: \;\; (x_1, x_2)^t &\to Q \cdot (x_1, x_2)^t, \quad x_3 \to (detQ) \cdot x_3, \\ (x_4, x_5)^t &\to detQ \cdot Q \cdot (x_4, x_5)^t. \end{aligned} \tag{6.2}$$

Notations. Given a constant non-singular 2×2 matrix Q we will use the notation L_Q for the linear transformation (6.2) of $\mathbb{R}^5$ and the notation g^0_Q for the transformation $((Q^t), L_Q) \in Symm^0(\mathcal{N})$.

Proposition 6.4 states that any transformation $g^0 \in Symm^0(\mathcal{N})$ is defined by a constant non-singular 2×2 matrix Q: $g^0 = g^0_Q$. A priori the transformations of the group $Symm^0(\mathcal{N})$ preserve $\mathcal{N} = \mathcal{A} + \mathcal{B}$. By Proposition 6.4 any transformation of the group $Symm^0(\mathcal{N})$ is linear, therefore we have the following corollary.

Corollary 6.5. *Any transformation of the group $Symm^0(\mathcal{N})$ preserves each of the couples $\mathcal{A}, \mathcal{B} \in (\mathbf{V}^{(-2)})^2$.*

This corollary implies the following statement which will be used in the sequel (it is the key point for the normalization with respect to the group $Symm^0(\mathcal{N}) \cdot \mathbf{G}^+$ after normalization with respect to the group $\mathbf{G}^+$).

Proposition 6.6. *The transformation $g^0_Q \in Symm^0(\mathcal{N})$ brings a couple of vector fields of the form $\mathcal{N} + C(x) \cdot \mathcal{B}$ to the couple $\mathcal{N} + C(L_Q(x)) \cdot \mathcal{B}$.*

7. The infinitesimal operators $\mathbf{L}^{(i)}_{\mathcal{N}}$.

In this subsection we "forget" about the group $Symm^0(\mathcal{N})$ (till the next subsection) and present a normal form to which any couple of vector fields with the quasi-homogeneous (-1)-jet $\mathcal{N}$ can be reduced by a transformation of the the group

$\mathbf{G}^+$. We present this normal form in terms of infinitesimal operators defined by $\mathcal{N}$. We need the Lie algebra of the group $\mathbf{G}^+$ and the exponential map.

Notation. By $\mathbf{M}_{2,2}^+$ we denote the subspace of $\mathbf{M}_{2,2}$ consisting of 2×2 matrix functions (h) such that $h(0) = 0$. By $\mathbf{V}^+$ we denote the subspace of $\mathbf{V}$ consisting of vector fields with zero quasi-homogeneous 0-jet.

The space $\mathbf{M}_{2,2}^+ \times \mathbf{V}^+$ can be treated as the Lie algebra of the group $\mathbf{G}^+$. Given

$$\lambda = ((h), Z) \in \mathbf{M}_{2,2}^+ \times \mathbf{V}^+$$

define the 1-parameter family $((H_t), \Phi_t) \in \mathbf{G}$ by the system of ODEs

$$\frac{d\Phi_t}{dt} = Z(\Phi_t), \quad \frac{dH_t}{dt} = H_t \cdot h(\Phi_t)$$

and the initial conditions $\Phi_0 = id, H_0 = I$. One can prove that $((H_t), \Phi_t) \in \mathbf{G}^+$. The family $((H_t), \Phi_t)$ will be called the flow of λ. The map sending λ to $((H_1), \Phi_1)$ will be called the exponential map:

$$exp : \mathbf{M}_{2,2}^+ \times \mathbf{V}^+ \to \mathbf{G}^+, \quad \exp(\lambda) = ((H_1), \Phi_1)).$$

Given a couple $\xi \in \mathbf{V}^2$ define the infinitesimal linear operator

$$\mathbf{L}_\xi : \mathbf{M}_{2,2}^+ \times \mathbf{V}^+ \to \mathbf{V}^2, \quad \mathbf{L}_\xi(\lambda) = (d/dt)|_{t=0}(exp(t\lambda)).\xi.$$

In other words $\mathbf{L}_\xi$ is the differential of the map $\mathbf{G} \ni g \to g.\xi$ at the point $id \in G$. It is easy to calculate

$$\mathbf{L}_\xi((h), Z) = (h) \cdot \xi + [\xi, Z].$$

The normalization of couples of vector fields of form (5.1) is tied with the linear operator $\mathbf{L}_\mathcal{N}$.

Notation. The restriction of the operator $\mathbf{L}_\mathcal{N}$ to the space $\mathbf{M}_{2,2}^{(i)} \times \mathbf{V}^{(i)}, i \geq 1$, will be denoted $\mathbf{L}_\mathcal{N}^{(i)}$.

Since $\mathcal{N} \in (\mathbf{V}^{(-1)})^2$ then by Proposition 3.1 the image of the operator $\mathbf{L}_\mathcal{N}^{(i)}$ belongs to the space $(\mathbf{V}^{(i-1)})^2$. We obtain a one-index family of linear operators

$$\mathbf{L}_\mathcal{N}^{(i)} : \mathbf{M}_{2,2}^{(i)} \times \mathbf{V}^{(i)} \to (\mathbf{V}^{(i-1)})^2, \quad i \geq 1,$$

$$\mathbf{L}_\mathcal{N}^{(i)}((h), Z)) = (h) \cdot \mathcal{N} + [\mathcal{N}, Z].$$

Proposition 7.1. *Let $i \geq 1$ and let $\lambda = ((h), Z) \in \mathbf{M}_{2,2}^{(i)} \times \mathbf{V}^{(i)}$. Let ξ be any couple of vector fields with the quasi-homogeneous (-1)-jet $\mathcal{N}$. Then*

$$j_{qh}^{i-1}(exp\lambda).\xi = j_{qh}^{i-1}\xi + \mathbf{L}_\mathcal{N}^{(i)}(\lambda).$$

This proposition easily implies the following corollary.

Proposition 7.2. *Fix any complementary subspaces $\mathbf{W}^{(i-1)}$ for the image of the operator $\mathbf{L}_\mathcal{N}^{(i)}$ in $(\mathbf{V}^{(i-1)})^2$:*

$$(\mathbf{V}^{(i-1)})^2 = Image(\mathbf{L}_\mathcal{N}^{(i)}) \oplus \mathbf{W}^{(i-1)}, \quad i \geq 1.$$

Any couple of form (5.1) is formally $\mathbf{G}^+$-equivalent to a couple of the form

$$\mathcal{N} + \tilde{\xi}^{(0)} + \tilde{\xi}^{(1)} + \tilde{\xi}^{(2)} + \cdots, \quad \tilde{\xi}^{(i)} \in \mathbf{W}^{(i)}. \tag{7.1}$$

If $\xi^{(0)}, \ldots, \xi^{(s)} \in W^{(i)}$ then the couple (5.1) is formally $\mathbf{G}^+$-equivalent to (7.1) with $\tilde{\xi}^{(0)} = \xi^{(0)}, \ldots, \tilde{\xi}^{(s)} = \xi^{(s)}$.

In view of Propositions 7.1 and 7.2 it is worth to answer the following questions: for which i the operator $\mathbf{L}_{\mathcal{N}}^{(i)}$ is surjective? injective? To guess the answer let us find the difference

$$\triangle_i = dim(\mathbf{V}^{(i-1)})^2 - dim\mathbf{M}_{2,2}^{(i)} \times \mathbf{V}^{(i)}$$

between the dimensions of the target and the source space of the operator $\mathbf{L}_{\mathcal{N}}^{(i)}$. One can expect that if $\triangle_i < 0$ then the opeartor $\mathbf{L}_{\mathcal{N}}^{(i)}$ is surjective and its kernel has dimension $|\triangle_i|$, and if $\triangle_i > 0$ then the operator $\mathbf{L}_{\mathcal{N}}^{(i)}$ is injective. Theorem 7.3 below confirms that this is true. Note that

$$dim(\mathbf{V}^{(i)}) = 2dim\mathbf{F}^{i+1} + dim\mathbf{F}^{(i+2)} + 2dim\mathbf{F}^{(i+3)}, \quad dim\mathbf{M}_{2,2}^{(i)} = 4dim\mathbf{F}^{(i)}$$

(recall that $\mathbf{F}^{(i)}$ denotes the space of quasi-homogeneous degree i functions). It follows

$$\triangle_i = 3dim\mathbf{F}^{(i+2)} - 2dim\mathbf{F}^{(i+3)}.$$

Calculating $dim\mathbf{F}^{(i)}$ we obtain the following table

i	$dim\mathbf{F}^{(i+2)}$	$dim\mathbf{F}^{(i+3)}$	$\triangle_i$
1	8	13	-2
2	13	20	-1
3	20	31	-2
4	31	44	5
5	44	64	4
≥ 6	...	...	≥ 24

Table 1. The difference $\triangle_i$ between the dimensions of the target and the source space of the operator $\mathbf{L}_{\mathcal{N}}^{(i)}$

This table explains (though of course does not prove) the following

Proposition 7.3.
(i) The operator $\mathbf{L}_{\mathcal{N}}^{(i)}$ is surjective if and only if $i \leq 3$.

(ii) The operator $\mathbf{L}_{\mathcal{N}}^{(i)}$ is injective if and only if $i \geq 4$. One has

$$dimKer\mathbf{L}_{\mathcal{N}}^{(1)} = 2, \; dimKer\mathbf{L}_{\mathcal{N}}^{(2)} = 1, \; dimKer\mathbf{L}_{\mathcal{N}}^{(3)} = 2.$$

8. Quasi-homogeneous 3-jet. Cartan invariant.

The complementary subspace for the image of the operator $\mathbf{L}_{\mathcal{N}}^{(4)}$ in $(\mathbf{V}^{(3)})^2$ can be easily calculated. Table 1 suggest that it is 5-dimensional.

Lemma 8.1. $(\mathbf{V}^{(3)})^2 = Image(\mathbf{L}_{\mathcal{N}}^{(4)}) \oplus \{P^{(4)}(x_1, x_2) \cdot \mathcal{B}\}$, *where $P^{(4)}(x_1, x_2)$ is an arbitrary homogeneous degree 4 polynomial of two variables.*

This lemma and Propositions 7.2 and 7.3 imply that the quasi-homogeneous 3-jet of any couple of vector fields with the quasi-homogeneous (-1)-jet $\mathcal{N}$ is $\mathbf{G}^+$-equivalent to a quasi-homogeneous 3-jet of the form $\mathcal{N} + P^{(4)}(x_1, x_2) \cdot \mathcal{B}$, where $P^{(4)}(x_1, x_2)$ is a homogeneous degree 4 polynomial. Using Proposition 7.1 it is not hard to prove a stronger statement.

Proposition 8.2. *The quasi-homogeneous 3-jet of a couple of vector fields with the quasi-homogeneous (-1)-jet $\mathcal{N}$ is $\mathbf{G}^+$-equivalent to one* **and only one** *of the quasi-homogeneous 3-jets of the form*

$$\mathcal{N} + P^{(4)}(x_1, x_2) \cdot \mathcal{B}, \tag{8.1}$$

where $P^{(4)}(x_1, x_2)$ is a homogeneous degree 4 polynomial.

According to our definition in section 2 the homogeneous degree 4 polynomial $P^{(4)}(x_1, x_2)$ in this normal form is called the Cartan tensor. In order to explain this definition we have to analyze the action in the space of quasi-homogeneous 3-jets of the whole group $\mathbf{G}$.

Recall that by Proposition 6.3 two couples of vector fields with the quasi-homogeneous (-1)-jet $\mathcal{N}$ are $\mathbf{G}$-equivalent if and only if they are equivalent with respect to the action of the group $Symm^0(\mathcal{N}) \cdot \mathbf{G}^+$. Let $g^0 \in Symm^0(\mathcal{N})$. By Proposition 6.4 $g^0 = g^0_Q$ for some non-singular constant 2×2 matrix Q. By Proposition 6.6 the transformation g^0_Q brings a couple of the form $\mathcal{N} + C(x) \cdot \mathcal{B}$ to the couple $\mathcal{N} + C(L_Q(x)) \cdot \mathcal{B}$. If $C(x)$ depends on x_1, x_2 only then $C(L_Q(x))$ is the function obtained from $C(x)$ by the linear transformation of $\mathbb{R}^2$ with the matrix Q. We obtain the following statement.

Proposition 8.3. *Let Q be a non-singular constant 2×2. The transformation $g^0_Q \in Symm^0(\mathcal{N})$ brings a couple of vector fields of form (8.1) to the couple of the form $\mathcal{N} + \tilde{P}^{(4)}(x_1, x_2) \cdot \mathcal{B}$, where $\tilde{P}^{(4)}(x_1, x_2)$ is the polynomial obtained from $P^{(4)}(x_1, x_2)$ by a linear transformation with the matrix Q.*

This means that the group $Symm^0(\mathcal{N})$ preserves the normal form (8.1). Propositions 5.1, 8.2 and 8.3 imply the following statement on the classification of quasi-homogeneous 3-jets with respect to the whole group $\mathbf{G}$.

Proposition 8.4. *The quasi-homogeneous 3-jet of a couple of vector fields spanning a $(2, 3, 5)$ distribution is $\mathbf{G}$-equivalent to a quasi-homogeneous 3-jet of form (8.1). Two quasi-homogeneous 3-jets of this form, with Cartan tensors $P^{(4)}(x_1, x_2)$ and $\tilde{P}^{(4)}(x_1, x_2)$, are $\mathbf{G}$-equivalent if and only if the Cartan tensors can be brought one to the other by a linear non-degenerate transformation of the plane $\mathbb{R}^2(x_1, x_2)$.*

Proposition 2.2 is a part of Proposition 8.4. Proposition 8.4 implies that the Cartan invariant (see section 2.2) is *a complete* invariant in the classification of quasi-homogeneous 3-jets of $(2, 3, 5)$ distributions.

If the Cartan tensor $P^{(4)}(x_1, x_2)$ is non-degenerate and has positive (respectively negative) type then it is linearly equivalent to one and only one of the polynomials of the set $\mathbf{P}^{(4),+}$ (respectively $\mathbf{P}^{(4),-}$), see section 2.2. Therefore the exact normal form for the quasi-homogeneous 3-jets is as follows.

Proposition 8.5. *The quasi-homogeneous 3-jet of a couple of vector fields spanning a $(2, 3, 5)$-distribution with a non-degenerate Cartan tensor of positive (respectively negative) type is $\mathbf{G}$-equivalent to one and only one of the quasi-homogeneous 3-jets of the form $\mathcal{N} + P^{(4)}(x_1, x_2) \cdot \mathcal{B}$, where $P^{(4)}(x_1, x_2) \in \mathbf{P}^{(4),+}$ (respectively $P^{(4)}(x_1, x_2) \in \mathbf{P}^{(4),-}$).*

9. Proof of Theorems A,B,C (outline)

To prove Theorem **A** we calculate the complementary subspace for the images of the operator $\mathbf{L}_{\mathcal{N}}^{(i)}$ in $(\mathbf{V}^{(i-1)})^2$ for *any* $i \geq 1$. Let

$$\mathbf{W}^{(i-1)} = \left\{C(x)\cdot \mathcal{B},\ \ C(x) \in \mathbf{F}^{(i)} \cap \mathbf{I}\right\}, \quad i \geq 1, \tag{9.1}$$

where $\mathbf{I}$ is the ideal defined in section 2.

Proposition 9.1. $(\mathbf{V}^{(i-1)})^2 = Image(\mathbf{L}_{\mathcal{N}}^{(i)}) \oplus \mathbf{W}^{(i-1)}, \quad i \geq 1.$

Remarks. Lemma 8.1 is a particular case of Proposition 9.1 – the space $\mathbf{F}^{(4)} \cap \mathbf{I}$ is exactly the space of homogeneous degree 4 polynomials $P^{(4)}(x_1, x_2)$. Note that $\mathbf{W}^{(0)} = \mathbf{W}^{(1)} = \mathbf{W}^{(2)} = \{0\}$ because any function in the ideal $\mathbf{I}$ has zero quasi-homogeneous 3-jet. Therefore Proposition 7.3, (i) is a corollary of Proposition 9.1.

Theorem **A** is a direct corollary of Propositions 5.1, 7.2, and 9.1. Theorem **C** is a simple corollary of Theorem **B**. In what follows we give the outline of the proof of Theorem **B**. At first we reduce Theorem **B** to the following statement.

Proposition 9.2.

(i) Any couple of vector fields whose quasi-homogeneous 3*-jet has the form* $\mathcal{N} + P^{(4)}(x_1, x_2)\cdot\mathcal{B}$ *with* $P^{(4)}(x_1, x_2) \in \mathbf{P}^{(4),\pm}$ *is* $\mathbf{G}^+$*-equivalent to a couple of the form* $\mathcal{N} + C(x)\cdot\mathcal{B}$*, where* $C(x) \in \mathbf{I}_0^{\pm}$ *(either all signs are + or all signs are −).*

(ii) This normal form is exact with respect to the group $\mathbf{G}^+$*: two couples of the form* $\mathcal{N} + C(x)\cdot\mathcal{B}$ *and* $\mathcal{N} + \tilde{C}(x)\cdot\mathcal{B}$ *with* $C(x), \tilde{C}(x) \in \mathbf{I}_0^+$ *or* $C(x), \tilde{C}(x) \in \mathbf{I}_0^-$ *are formally* $\mathbf{G}^+$*-equivalent if and only if the Taylor series of* $C(x)$ *and* $\tilde{C}(x)$ *coincide.*

Here $\mathbf{I}_0^{\pm}$ is the set of function germs defined in section 2.3. Theorem **B**, (i) is a direct corollary of Propositions 8.5 and 9.2, (i). Theorem **B**, (ii) is almost logical corollary of Proposition 9.2, (ii), the results of sections 6 and 8, and the following observations:

1. It is easy to show that the linear transformation of $\mathbb{R}^2$ with a matrix Q preserves a polynomial of the set $\mathbf{P}^{(4),+}$ (respectively $\mathbf{P}^{(4),-}$) if and only if Q belongs to the group generated by the matrices

$$Q_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},\ Q_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ Q_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(respectively to the group generated by the matrices Q_1, Q_2).

2. It is clear that $L_{Q_1} = i_1,\ L_{Q_2} = i_2,\ L_{Q_3} = i_3$, where i_1, i_2, i_3 are the involutions defined in section 2.3.

3. The involutions i_1, i_2 preserve $\mathbf{I}_0^+$ and $\mathbf{I}_0^-$. The involution i_3 preserves $\mathbf{I}_0^+$.

The proof of Proposition 9.2 is based on the infinitesimal-quasi-homogeneous techniques for the normalization of a couple of vector fields with a fixed quasi-homogeneous 3-jet $\mathcal{N} + \xi^{(3)}$. Fix, as in section 7, a complementary subspace $\mathbf{W}^{(i)}$ for the image of the operator $\mathbf{L}_{\mathcal{N}}^{(i+1)}$ in $(\mathbf{V}^{(i)})^2$: $(\mathbf{V}^{(i)})^2 = Image(\mathbf{L}_{\mathcal{N}}^{(i+1)}) \oplus \mathbf{W}^{(i)}$. Let $\pi_i : (\mathbf{V}^{(i)})^2 \to \mathbf{W}^{(i)}$ be the projection according to this direct sum. Recall that by Proposition 7.3, (ii) the operators $\mathbf{L}_{\mathcal{N}}^{(1)}, \mathbf{L}_{\mathcal{N}}^{(2)}, \mathbf{L}_{\mathcal{N}}^{(3)}$ have kernels of dimension 2, 1, 2 and the kernels of the operators $\mathbf{L}_{\mathcal{N}}^{(i)}$, $i \geq 4$ are trivial. The latter is the main point in the proof of Theorem 9.2.

Given $\xi^{(3)} \in (\mathbf{V}^{(3)})^2$ consider the following three linear operators:

$$(9.2)\qquad \mathbf{T}^{(i)}_{\mathcal{N},\xi^{(3)}} : Ker\mathbf{L}^{(i)}_{\mathcal{N}} \to \mathbf{W}^{(i+3)}, \quad \mathbf{T}^{(i)}_{\mathcal{N},\xi^{(3)}}(\lambda) = \pi_{i+3}\mathbf{L}_{\xi^{(3)}}(\lambda), \quad i = 1, 2, 3.$$

Here $\mathbf{L}_\xi$ is the infinitesimal operator defined by $\xi \in (\mathbf{V})^2$, see section 7. Fix complementary subspaces $\mathbf{U}^{(i)}$ for the images of these operators in $\mathbf{W}^{(i+3)}$:

$$\mathbf{W}^{(i+3)} = \left(Image\ \mathbf{T}^{(i)}_{\mathcal{N},\xi^{(3)}}\right) \oplus \mathbf{U}^{(i+3)}, \quad i = 1, 2, 3.$$

Proposition 9.3. *Fix* $\xi^{(3)} \in (V^{(3)})^2$.

(i) Any couple of vector fields with the quasi-homogeneous 3-jet $\mathcal{N} + \xi^{(3)}$ *is formally* $\mathbf{G}^+$*-equivalent to a couple of the form*

$$(9.3)\quad \mathcal{N} + \xi^{(3)} + \xi^{(4)} + \cdots, \quad \xi^{(4)} \in \mathbf{U}^{(4)}, \xi^{(5)} \in \mathbf{U}^{(5)}, \xi^{(6)} \in \mathbf{U}^{(6)}, \ \xi^{(i\geq 7)} \in \mathbf{W}^{(i)}.$$

(ii) If the operators (9.2) are injective then this normal form is exact with respect to the group $\mathbf{G}^+$*: two couples of form (9.3) are formally* $\mathbf{G}^+$*-equivalent only if they have the same Taylor series.*

Now to prove Proposition 9.2 one has to analyze operators (9.2) with $\xi^{(3)} = P^{(4)}(x_1, x_2) \cdot \mathcal{B}$, where $P^{(4)}(x_1, x_2) \in \mathbf{P}^{(4),\pm}$.

Proposition 9.4. *Let* $\xi^{(3)} = P^{(4)}(x_1, x_2) \cdot \mathcal{B}$*, where* $P^{(4)}(x_1, x_2) \in \mathbf{P}^{(4),\pm}$*. Let* $\mathbf{W}^{(i)}$ *be the subspaces (9.1) (see Proposition 9.1).*

1. The operators (9.2) are injective.

2. $\mathbf{W}^{(i+3)} = Image\left(\mathbf{T}^{(i)}_{\mathcal{N},\xi^{(3)}}\right) \oplus \{C(x) \cdot \mathcal{B},\ C(x) \in \mathbf{F}^{(i+4)} \cap \mathbf{I}_0^{\pm}\}$ *(the sign in* $I_0^\pm$ *is the same as the sign in the assumption of the proposition).*

Theorem 9.2 is a direct corollary of Propositions 9.3 and 9.4.

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