NAMBU-LIE GROUPS ENDOURED WITH
MULTIPLICATIVE TENSORS OF TOP ORDER

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ABSTRACT. A multiplicative (Nambu-Poisson) tensor of top order on a Lie group is characterized. As an application, we determine multiplicative structures on 3-dimensional Lie groups.

1. INTRODUCTION

A Nambu-Lie group is defined as a natural generalization of a Poisson Lie group. In fact, if $\eta$ is a multiplicative $k$-vector field on a Lie group $G$ which satisfies fundamental identity, then a pair $(G, \eta)$ is called a Nambu-Lie group. If $k = 2$, then $(G, \eta)$ is especially called a Poisson Lie group [1],[3]. A Nambu-Lie group was studied by J.Grabowski and G.Marmo [2] and I.Vaisman [5]. In [2], they proved that there are no Nambu-Lie structures of order $k \geq 3$ on simple Lie groups. I.Vaisman [5] gave an alternative definition of multiplicativity by defining the $k$-bracket of 1-forms on $G$. In this paper, we characterize the properties of multiplicative Nambu-Poisson tensors of top order (i.e., $n = k$). Note that the word "Nambu-Poisson" is void in this case. As an application of these characterizations, we determine multiplicative (Nambu-Lie) structures defined on 3-dimensional Lie groups.

2. NAMBU-LIE GROUPS

Let $G$ be an $n$-dimensional connected Lie group with the Lie algebra $\mathfrak{g}$. We denote by $\Gamma(\Lambda^k TG)$ the set of $k$-vector fields (or contravariant tensor fields of order $k$) on $G$. Let $\mathcal{F}$ be the set of $C^\infty$-functions on $G$. Each element $\eta$ of $\Gamma(\Lambda^k TG)$ defines a $k$-bracket of functions $f_i \in \mathcal{F}$ as follows.

$$\{f_1, \ldots, f_k\} = \eta(df_1, \ldots, df_k).$$

Since this $k$-bracket satisfies Leibniz rule, we can define a vector field $X_{f_1, \ldots, f_{k-1}}$ by

$$X_{f_1, \ldots, f_{k-1}}(g) = \{f_1, \ldots, f_{k-1}, g\}, \ g \in \mathcal{F}.$$
Definition 2.1. An element $\eta$ of $\Gamma(\Lambda^k TG)$, $k \geq 3$, is called a Nambu-Poisson tensor of order $k$ if $\eta$ satisfies

$$\mathcal{L}_{f_1 \ldots f_{k-1}} \eta = 0$$

for any $f_1, \ldots, f_{k-1} \in \mathcal{F}$.

Note that if $k = n$, every $\eta$ is a Nambu-Poisson tensor [4].

Definition 2.2. An element $\eta$ of $\Gamma(\Lambda^k TG)$ is said to be multiplicative if $\eta$ satisfies

$$\eta_{gh} = L_{g \ast} \eta_h + R_{h \ast} \eta_g$$

for any $g, h \in G$, where $L_g$ and $R_g$ denote, respectively, the left and the right translations. Let $G$ be a Lie group endowed with a multiplicative Nambu-Poisson tensor $\eta$. Then a pair $(G, \eta)$ is called a Nambu-Lie group.

For an element $\Lambda \in \Lambda^k \mathfrak{g}$, we define vector fields $\bar{\Lambda}$ and $\tilde{\Lambda}$ by

$$\bar{\Lambda}_g = L_{g \ast} \Lambda, \quad \tilde{\Lambda}_g = R_{g \ast} \Lambda, \text{ for all } g \in G.$$ 

Then it is clear that $\bar{\Lambda}$ (resp. $\tilde{\Lambda}$) is a left (resp. right) invariant vector field on $G$. Let us recall the following, which was proved by J-H Lu [3].

Proposition 2.1. Let $G$ be a compact (or semisimple) Lie group. Then for every multiplicative $k$-vector field $\eta \in \Gamma(\Lambda^k TG)$, there exists an element $\Lambda \in \Lambda^k \mathfrak{g}$ such that

$$\eta_g = \bar{\Lambda}_g - \tilde{\Lambda}_g$$

for all $g \in G$.

Using the above proposition, we show the following theorem.

Theorem 2.2. Let $(G, \eta)$ be an $n$-dimensional compact or semisimple Nambu-Lie group, and let $\eta$ of top order. Then $\eta = 0$.

Proof. By Proposition 2.1, there exists an element $\Lambda \in \Lambda^n \mathfrak{g}$ such that $\eta = \bar{\Lambda} - \tilde{\Lambda}$. For all $g, h \in G$,

$$Ad_g \bar{\Lambda}_h = R_{g^{-1} \ast} L_g \bar{\Lambda}_h = R_{g^{-1} \ast} \bar{\Lambda}_{gh}.$$ 

On the other hand, since $G$ is a unimodular Lie group, we have

$$Ad_g \bar{\Lambda}_h = (\det Ad_g) \bar{\Lambda}_{ghg^{-1}} = \bar{\Lambda}_{ghg^{-1}}.$$ 

Hence we obtain that $R_{g^{-1} \ast} \bar{\Lambda}_{gh} = \bar{\Lambda}_{ghg^{-1}}$. This means that a left invariant vector field $\bar{\Lambda}$ is also a right invariant vector field. i.e., $R_{h \ast} \bar{\Lambda}_g = \bar{\Lambda}_{gh}$. This equation induces

$$R_{h \ast} \bar{\Lambda}_g = R_{h \ast} L_g \ast \Lambda = L_{g \ast} R_{h \ast} \Lambda = \bar{\Lambda}_{gh} = L_{g \ast} L_{h \ast} \Lambda.$$ 

Thus we have $R_{h \ast} \Lambda = L_{h \ast} \Lambda$ for all $h \in G$, and this means $\eta = \bar{\Lambda} - \tilde{\Lambda} = 0$. 

\[\square\]
Let $\eta$ be a Nambu-Poisson tensor of order $k$ on $G$. Then $\eta$ defines a bundle mapping
\[ \eta : T^*G \times \cdots \times T^*G \longrightarrow TG \]
given by
\[ <\beta, \eta(\alpha_1, \ldots, \alpha_{k-1})> = \eta(\alpha_1, \ldots, \alpha_{k-1}, \beta), \]
where all the arguments are covectors.

For such a tensor $\eta$, I.Vaisman [5] defined a $k$-bracket of 1-forms by
\[ \{\alpha_1, \ldots, \alpha_k\} = d(\eta(\alpha_1, \ldots, \alpha_k)) + \sum_{j=1}^{k}(-1)^{k+j}i(\eta(\alpha_1, \ldots, \overline{\alpha_j}, \ldots, \alpha_k))d\alpha_j, \]
where $\alpha_j$ ($j=1, \ldots, k$) are 1-forms on $G$.

The following theorem proved by I.Vaisman [5] gives one of the characterizations of Nambu-Lie groups.

**Theorem 2.3.** If $G$ is a connected Lie group endowed with a Nambu-Poisson tensor field $\eta$ which vanishes at the unit $e$ of $G$, then $(G, \eta)$ is a Nambu-Lie group if and only if the $k$-bracket of any $k$ left (right) invariant 1-forms of $G$ is a left (right) invariant 1-form.

Using Theorem 2.3, we characterize a multiplicative tensor $\eta$ of top order. Let $\mathfrak{g}$ be a Lie algebra of $G$ with a basis $X_1, \ldots, X_n$. We also denote the extended left invariant vector fields induced from $X_i$ by the same letter. Since $\eta$ is of top order, $\eta$ has an expression $\eta = f X_1 \wedge \cdots \wedge X_n$ for some $f \in \mathcal{F}$. Let $\omega_i$ ($i=1, \ldots, n$) be left invariant 1-forms dual to $X_i$. Under these notations we prove

**Theorem 2.4.** Let $\eta = f X_1 \wedge \cdots \wedge X_n$, $f \in \mathcal{F}$ be a tensor of top order on $G$. (Recall that such a tensor is always a Nambu-Poisson tensor.) Then $\eta$ is multiplicative if and only if $f(e) = 0$ and
\[ X_i f + \left( \sum_{k=1}^{n} C_{ik}^k \right) f = q_i, \quad i = 1, \ldots, n, \]
where $\{C_{ij}^k\}$ are structure constants of $\mathfrak{g}$ with respect to the basis $X_1, \ldots, X_n$, and $q_i$ ($i=1, \ldots, n$) are some constants.

**Proof.** By Theorem 2.3, we know that $\eta$ is multiplicative if and only if $\eta_e = 0$ and
\[ \{\omega_1, \ldots, \omega_n\} = d(\eta(\omega_1, \ldots, \omega_n)) + \sum_{k=1}^{n}(-1)^{n+k}i(\eta(\omega_1, \ldots, \omega_k, \ldots, \omega_n))d\omega_k \\
= df + f \sum_{k=1}^{n}i(X_k)d\omega_k = df + f \left( \sum_{\alpha,k=1}^{n} C_{\alpha k}^k \omega_\alpha \right) \]
is a left invariant 1-form. Since \(<X_i,\{\omega_1,\ldots,\omega_n\}>\) is constant for any \(X_i\), we have
\[
<X_i,\{\omega_1,\ldots,\omega_n\}> = X_i f + \left(\sum_{k=1}^{n} C_{ik}^{k}\right) f = q_i, \quad i = 1,\ldots,n.
\]

\[\square\]

### 3. Examples

In this section, as an application of Theorem 2.4, we calculate Nambu-Lie group structures (i.e., multiplicative Nambu-Poisson tensors) of order 3 on 3-dimensional simply connected Lie groups. Since such tensors are of top degree, we have only to see whether they are multiplicative or not.

Throughout this section, we denote by \(G\) the simply connected Lie groups corresponding to Lie algebras \(g\). Linearly independent three left invariant vector fields are denoted by \(X, Y, Z\). Then \(\eta \in \Gamma(A^3TG)\) is written as \(\eta = fX \wedge Y \wedge Z, \ f \in C^\infty(G)\). It is well-known that there are 9 types of 3-dimensional Lie algebras. If \(g\) is not a simple Lie algebra, its corresponding simply connected Lie group has global coordinates \(x, y, z\). Hence a function \(f\) can be considered to be defined on \(\mathbb{R}^3(x, y, z)\).

**Type 1.** \([g, g] = 0\). Namely \(g\) is an abelian Lie algebra. The corresponding Lie group \(G\) is given by
\[
G = \left\{ \begin{pmatrix} e^x & 0 & 0 \\ 0 & e^y & 0 \\ 0 & 0 & e^z \end{pmatrix} \right| x, y, z \in \mathbb{R} \right\}.
\]

Using these coordinates \(x, y, z\), left invariant vector fields are written as \(X = \frac{\partial}{\partial x}, \ Y = \frac{\partial}{\partial y}, \ Z = \frac{\partial}{\partial z}\). By Theorem 2.4, a function \(f(x, y, z)\) must satisfy \(f(0, 0, 0) = 0\), and \(\frac{\partial f}{\partial x} = a, \ \frac{\partial f}{\partial y} = b, \ \frac{\partial f}{\partial z} = c\), where \(a, b, c\) are some constants. Hence \(f = ax + by + cz\), and
\[
\eta = (ax + by + cz) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}
\]
gives a Nambu-Lie group structure on \(G\).

By the similar method, we can get the results for other types.

**Type 2.** \(\dim[g, g] = 1\). There are 2 cases as follows.

**Case (1).** \(g\) = Heisenberg Lie algebra. \(g\) is characterized by the condition \([g, g] \subset 1\)-dimensional center. The corresponding Lie group \(G\) is given by
\[
G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right| x, y, z \in \mathbb{R} \right\}.
\]

A Nambu-Lie group structure on \(G\) is given by
\[
\eta = (ax + by) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}.
\]
Case (2). A Lie algebra \( \mathfrak{g} \) endowed with a property \([\mathfrak{g}, \mathfrak{g}] \not\subset \) the center of \( \mathfrak{g} \). The corresponding Lie group \( G \) is given by

\[
G = \left\{ \begin{pmatrix} e^{y+z} & 0 & xe^y \\ 0 & e^y & 0 \\ 0 & 0 & e^y \end{pmatrix} \bigg| x, y, z \in \mathbb{R} \right\}.
\]

\[
\eta = \left\{ ax + c(e^z - 1) \right\} \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}
\]
gives a Nambu-Lie group structure on \( G \).

Type 3. \( \dim[\mathfrak{g}, \mathfrak{g}] = 2. \) \( \mathfrak{g}^{(2)} = 0 \). There are 4 cases as follows.

Case (1). Left invariant vector fields \( X, Y, Z \) satisfy \([X, Y] = 0, \) \([X, Z] = -X, \) \([Y, Z] = -X - Y \). The corresponding Lie group \( G \) is given by

\[
G = \left\{ \begin{pmatrix} e^{-z} & xe^{-2z} \\ 0 & ye^{-2z} \\ 0 & e^{-2z} \end{pmatrix} \bigg| x, y, z \in \mathbb{R} \right\}.
\]

We know that

\[
\eta = c(e^{2x} - 1) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}
\]
gives a Nambu-Lie group structure on \( G \).

Case (2). Left invariant vector fields \( X, Y, Z \) satisfy \([X, Y] = 0, \) \([X, Z] = -X, \) \([Y, Z] = -Y \). The corresponding Lie group \( G \) is given by

\[
G = \left\{ \begin{pmatrix} e^{-z} & xe^{-2z} \\ 0 & ye^{-2z} \\ 0 & e^{-2z} \end{pmatrix} \bigg| x, y, z \in \mathbb{R} \right\}.
\]

We have

\[
\eta = c(e^{2x} - 1) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}.
\]

Case (3). Let \( \mathfrak{g} \) be a Lie algebra endowed with the following bracket relations. \([X, Y] = 0, \) \([X, Z] = -X, \) \([Y, Z] = -qY, \) \((q \neq 0, 1) \). The corresponding Lie group \( G \) is given by

\[
G = \left\{ \begin{pmatrix} e^{-qz} & xe^{-(q+1)z} \\ 0 & ye^{-(q+1)z} \\ 0 & e^{-(q+1)z} \end{pmatrix} \bigg| x, y, z \in \mathbb{R} \right\}.
\]

We have

\[
\eta = c(e^{(q+1)x} - 1) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}.
\]

Case (4). Let \( \mathfrak{g} \) be a Lie algebra endowed with the following bracket relations. \([X, Y] = 0, \) \([X, Z] = -Y, \) \([Y, Z] = X - qY, \) \((q^2 < 4) \). The
corresponding Lie group $G$ has rather complicated expression. Put $k = q/2, \ p = \sqrt{1 - k^2} = \sqrt{4 - q^2}/2$. Then $G$ is given by

$$G = \left\{ \begin{pmatrix} \frac{1}{p} e^{-kz} (-k \sin(pz) + p \cos(pz)) & -\frac{1}{p} e^{-kz} \sin(pz) & x e^{-2kz} \\ \frac{1}{p} e^{-kz} \sin(pz) & \frac{1}{p} e^{-kz} (p \cos(pz) + k \sin(pz)) & y e^{-2kz} \\ 0 & 0 & e^{-2kz} \end{pmatrix} \right\} \quad x, y, z \in \mathbb{R}.$$ 

Then

$$\eta = \begin{cases} \frac{c}{q} (e^{qx} - 1) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}, & q \neq 0 \\ c x \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}, & q = 0 \end{cases}$$

gives a Nambu-Lie group structure on $G$.

Type 4. $\dim [\mathfrak{g}, \mathfrak{g}] = 3$. It is well-known that such Lie algebras are simple, and there are 2 cases. The corresponding simply connected Lie groups are $G_1 = SU(2)$ and $G_2 = SL(2, \mathbb{R})$, where $SL(2, \mathbb{R})/\mathbb{Z} \cong SL(2, \mathbb{R})$. Since $G_1$ is compact, and $G_2$ is semisimple, we have $\eta = 0$ by Theorem 2.2.

REFERENCES