Spherical simplices. A spherical simplex $S$ is an intersection of $n$ pieces of half spaces $H_k = \{ \vec{x} \in \mathbb{R}^n : \tilde{p}_k \cdot \vec{x} \geq 0 \}$ and the unit sphere $S^{n-1}$ in the $n$-dimensional Euclidean space $\mathbb{R}^n$, i.e., $S = \{ \vec{x} \in S^{n-1} : \tilde{p}_k \cdot \vec{x} \geq 0 \ (\forall k = 0, \ldots, n-1) \}$. See the figure for $n = 3$:

$$S(I; J) = \{ \tilde{x} \in (\sum_{k \in I \cup J} \mathbb{R} \cdot \tilde{p}_k) \cap \mathbb{R}^{n-1} : \tilde{p}_k \cdot \vec{x} = 0 (\forall k \in I), \tilde{p}_k \cdot \vec{x} \geq 0 (\forall k \in J) \}$$

for disjoint subsets $I$ and non-empty $J$ of $\{0, \ldots, n-1\}$ and $\langle I; i, j \rangle$ is the dihedral angle between $H_i$ and $H_j$ subtended by $S(I; J)$.

Volumes of spherical simplices for odd $n$. The function $f_m$ is the $(m-1)$-dimensional Schl"afli's normalized volume function, i.e.,

$$f_m = \frac{\text{Vol}_{m-1}(\cdot)}{\text{Vol}_{m-1}(S^{m-1})/2^m} = \frac{\text{Vol}_{m-1}(\cdot)}{\pi^{m/2}/(2^{m-1}\Gamma(m/2))}$$

with $f_0(\cdot) = 1$. For $n = 3$, it is well-known that

$$\text{Vol}_2(S) = \sum_{I \subseteqq \{0,1,2\}}^{|I|=2} \text{Vol}_1(S(\emptyset; I)) - \pi,$$

which is equivalent to $f_3(S) = \sum_{I \subseteqq \{0,1,2\}}^{|I|=2} f_2(S(\emptyset; I)) - 2$.

Schl"afli generalized it to Schl"afli's reduction formula for odd $n$:

$$f_n(S) = \sum_{k=0}^{n-1} (-1)^k a_k \sum_{I \subseteqq \{0,\ldots,n-1\}}^{|I|=n-1-2k} f_{n-2k}(S(\emptyset; I)),$$

This is an abstract and the details will be published elsewhere. The title in Japanese is “高次元球面の立体角について”.

Solid angles of high dimensional spheres

SATÔ, Kenzi
佐藤 健治
where \( a_k \) is such that \( \tan x = \sum_{k=0}^{\infty} a_k \frac{x^{2k+1}}{(2k+1)!} \). It is due to Schlafli's differential formula:

\[
d f_n(S) = \sum_{I \subseteq \{0, \ldots, n-1\}}^{I \neq \emptyset} f_{n-2}(S(I; I^c)) df_{2}(S(\emptyset; I)),
\]

which is also valid for even \( n \), where \( I^c = \{0, \ldots, n-1\} \setminus I \). However, it is difficult to represent \( f_n(S) \) for even \( n \).

**Volumes of orthogonal spherical simplices for \( n = 4 \).** A spherical simplex \( S \) is called orthogonal if \( \langle i, j \rangle = \frac{\pi}{2} \) for \( |j-i| \geq 2 \). Coxeter gave the formula representing the volume of orthogonal \( S \) for \( n = 4 \):

\[
\frac{\pi^2}{2} f_4(S) = \sum_{k=1}^{\infty} \frac{X^k}{k^2} (\cos 2k\alpha - \cos 2k\beta + \cos 2k\gamma - 1) - \alpha^2 + \beta^2 - \gamma^2,
\]

where \( \alpha = \frac{\pi}{2} - \langle 0, 1 \rangle \), \( \beta = \langle 1, 2 \rangle \), \( \gamma = \frac{\pi}{2} - \langle 2, 3 \rangle \), and

\[
X = \frac{\sqrt{\cos^2 \alpha \cos^2 \gamma - \cos^2 \beta} - \sin \alpha \sin \gamma}{\sqrt{\cos^2 \alpha \cos^2 \gamma - \cos^2 \beta} + \sin \alpha \sin \gamma}.
\]

**Volumes of spherical simplices for all \( n \).** Aomoto gave the formula representing \( f_n(S) \):

\[
2^n \cdot \frac{1}{\sqrt{\det B}} \frac{\pi^{n/2} f_n(S)}{2^{n-1} \Gamma\left(\frac{n}{2}\right)} = \sum_{(\sigma, j) \subseteq \{0, \ldots, n-1\}}^{(\sigma, j) \neq \emptyset} \prod_{i \leq j} (-2b_{i,j})^{\sigma_i,j} \prod_{k=0}^{n-1} \Gamma\left(\frac{\sigma_0i_0 + \cdots + \sigma_{n-1}i_{n-1} - 1}{2}\right) \prod_{i \neq j} \sigma_i,j!, \Gamma\left(\frac{n}{2} + 1\right),
\]

where each entry of \( (\sigma, j) \subseteq \{0, \ldots, n-1\} \) runs over all non-negative integers, \( B = (b_{i,j})_{i=0}^{n-1} \) = \( K^{-1} \cdot ((-\cos(i,j))_{i=0}^{n-1} - 1) \cdot K^{-1}, \) \( K = (\delta_j^i \cdot \Delta(i_0, \ldots, i_{r-1}) / \Delta(i_0, \ldots, n-1))_{i=0}^{n-1}, \ Delta(i_0, \ldots, n-1) = det\left((-\cos(k, l))_{i=0}^{n-1} \right), \) \( \delta_j^i \) is Kronecker's delta, and the circumflex indicates that the term below it has been omitted.

**Polars of spherical simplices.** For a spherical simplex \( S \), the polar \( S^o \) is

\[
S^o = \{ \tilde{y} \in S^{n-1} : \tilde{x} \cdot \tilde{y} \leq 0, \forall \tilde{x} \in S \}.
\]

See the figure for \( n = 2 \).

![Diagram](https://example.com/diagram)

**Main result.** For all \( n \), we have simple formulae:

\[
\begin{align*}
\text{if: even} & \sum_{I \subseteq \{0, \ldots, n-1\}}^{I \neq \emptyset} f_{n-I}(S(I; I^c)) df_{I}(S(\emptyset; I)) = 2^{n-1}, \\
\text{if: odd} & \sum_{I \subseteq \{0, \ldots, n-1\}}^{I \neq \emptyset} f_{n-I}(S(I; I^c)) df_{I}(S(\emptyset; I)) = 2^{n-1}.
\end{align*}
\]
Application. From the former, we can get a formula which represents $f_n(S) + f_n(S^o)$ by the volumes of lower dimensional spherical simplices for even $n$, e.g.,

$$f_2(S) + f_2(S^o) = 2, \quad \text{which is trivial for } n = 2,$$

$$f_4(S) + f_4(S^o) = 8 - \sum_{I \subseteq \{0,1,2,3\}} f_2(S(I;I^c))f_2((S(\emptyset;I))^o),$$

which is already known by Schläfli for $n = 4$,

$$f_6(S) + f_6(S^o) = 32 - \sum_{I \subseteq \{0,\ldots,5\}} f_2(S(I;I^c))f_2((S(\emptyset;I))^o) - \sum_{I \subseteq \{0,\ldots,5\}} f_2(S(I;I^c))f_4((S(\emptyset;I))^o),$$

which seems to be new result for $n = 6$, and so on. For odd $n$, from the former again, we can also represent $f_n(S)$ distinct from Schläfli's reduction formula, e.g.,

$$f_3(S) = 4 - \sum_{I \subseteq \{0,1,2\}} 1 \cdot f_2((S(\emptyset;I))^o), \quad \text{which is trivial for } n = 3,$$

$$f_5(S) = 16 - \sum_{I \subseteq \{0,\ldots,4\}} f_3(S(I;I^c))f_2((S(\emptyset;I))^o) - \sum_{I \subseteq \{0,\ldots,4\}} 1 \cdot f_4((S(\emptyset;I))^o),$$

which seems to be new result for $n = 5$, and so on.

Sketch of proof. We have Schläfli's differential formulae for lower dimensional spherical simplices and polars:

$$df_{n-1}(S(I; I^c)) = \sum_{K \subseteq I^c} f_{n-2}(S(I \cup K; (I \cup K)^c)) df_2(S(I; K)),$$

$$df_1((S(\emptyset; J))^o) = - \sum_{K \subseteq J} f_{1-2}(S(\emptyset; J \backslash K))^o df_2(S(J \backslash K; K)).$$

From differential formulae above, we can calculate the derivations of the left-hand sides of the main results, which are equal to zero. So the left-hand sides are constant, which are determined by the values of the spherical simplex whose all half-spaces are pairwise perpendicular. For such a spherical simplex, all terms of the left-hand side are equal to 1, so, the constant values are $2^n - 1$. □

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