数理解析研究所講究録
1503 巻 2006 年 129-131

Solid angles of high dimensional spheres

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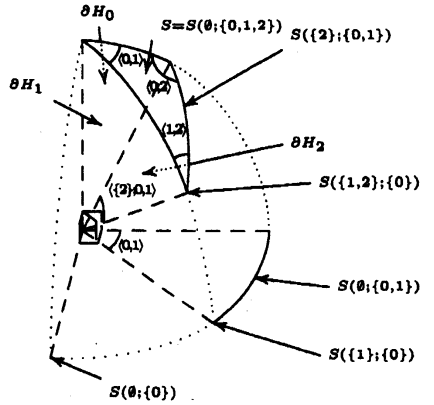
Spherical simplices. A *spherical simplex* S is an intersection of n pieces of half spaces $H_k = \{\vec{x} \in \mathbb{R}^n : \vec{p}_k \cdot \vec{x} \geqq 0\}$ and the unit sphere $\mathbf{S}^{n-1}$ in the n-dimensional Euclidean space $\mathbb{R}^n$, i.e., $S = \{\vec{x} \in \mathbf{S}^{n-1} : \vec{p}_k \cdot \vec{x} \geqq 0 \ ({}^\forall k = 0, \ldots, n-1)\}$. See the figure for $n = 3$:



where

$$S(I;J) = \{\vec{x} \in (\sum_{k \in I \cup J} \mathbb{R} \cdot \vec{p}_k) \cap \mathbf{S}^{n-1} : \vec{p}_k \cdot \vec{x} = 0 \ ({}^\forall k \in I),\ \vec{p}_k \cdot \vec{x} \geqq 0 \ ({}^\forall k \in J)\},$$

for disjoint subsets I and non-empty J of $\{0, \ldots, n-1\}$ and $\langle I; i, j \rangle$ is the dihedral angle between H_i and H_j subtended by $S(I;J)$.

Volumes of spherical simplices for odd n. The function f_m is the $(m-1)$-dimensional *Schläfli's normalized volume function*, i.e.,

$$f_m = \frac{\mathrm{Vol}_{m-1}(\cdot)}{\mathrm{Vol}_{m-1}(\mathbf{S}^{m-1})/2^m} = \frac{\mathrm{Vol}_{m-1}(\cdot)}{\pi^{m/2}/(2^{m-1}\Gamma(m/2))}$$

with $f_0(\cdot) = 1$. For $n = 3$, it is well-known that

$$\mathrm{Vol}_2(S) = \sum_{\substack{I \subseteqq \{0,1,2\} \\ \sharp I = 2}} \mathrm{Vol}_1(S(\emptyset; I)) - \pi, \quad \text{which is equivalent to} \quad f_3(S) = \sum_{\substack{I \subseteqq \{0,1,2\} \\ \sharp I = 2}} f_2(S(\emptyset; I)) - 2.$$

Schläfli generalized it to *Schläfli's reduction formula* for odd n:

$$f_n(S) = \sum_{k=0}^{\frac{n-1}{2}} (-1)^k a_k \sum_{\substack{I \subseteqq \{0,\ldots,n-1\} \\ \sharp I = n-1-2k}} f_{n-1-2k}(S(\emptyset; I)),$$

This is an abstract and the details will be published elsewhere. The title in Japanese is "高次元球面の立体角について".

where a_k is such that $\tan x = \sum_{k=0}^{\infty} a_k \frac{x^{2k+1}}{(2k+1)!}$. It is due to *Schläfli's differential formula:*

$$df_n(S) = \sum_{I \subseteqq \{0,\dots,n-1\}}^{\sharp I = 2} f_{n-2}(S(I;I^c))\, df_2(S(\emptyset;I)),$$

which is also valid for even n, where $I^c = \{0,\dots,n-1\} \setminus I$. However, it is difficult to represent $f_n(S)$ for even n.

Volumes of orthogonal spherical simplices for $n = 4$. A spherical simplex S is called *orthogonal* if $\langle i,j \rangle = \frac{\pi}{2}$ for $|j-i| \geqq 2$. Coxeter gave the formula representing the volume of orthogonal S for $n = 4$:

$$\frac{\pi^2}{2} f_4(S) = \sum_{k=1}^{\infty} \frac{X^k}{k^2}(\cos 2k\alpha - \cos 2k\beta + \cos 2k\gamma - 1) - \alpha^2 + \beta^2 - \gamma^2,$$

where $\alpha = \frac{\pi}{2} - \langle 0,1 \rangle$, $\beta = \langle 1,2 \rangle$, $\gamma = \frac{\pi}{2} - \langle 2,3 \rangle$, and

$$X = \frac{\sqrt{\cos^2\alpha \cos^2\gamma - \cos^2\beta} - \sin\alpha \sin\gamma}{\sqrt{\cos^2\alpha \cos^2\gamma - \cos^2\beta} + \sin\alpha \sin\gamma}.$$

Volumes of spherical simplices for all n. Aomoto gave the formula representing $f_n(S)$:

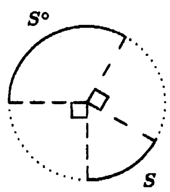
$$2^n \cdot \frac{1}{\sqrt{\det B}} \frac{\pi^{n/2} f_n(S)}{2^{n-1}\Gamma(\frac{n}{2})} / n = \sum_{(\sigma_{i,j})_{0 \leqq i \lneqq j \lneqq n}} \frac{\prod_{i \lneqq j} (-2b_{i,j})^{\sigma_{i,j}}}{\prod_{i \lneqq j} \sigma_{i,j}!} \cdot \frac{\prod_{k=0}^{n-1} \Gamma(\frac{\sigma_{0,k} + \cdots + \sigma_{k-1,k} + \sigma_{k,k+1} + \cdots + \sigma_{k,n-1} + 1}{2})}{\Gamma(\frac{n}{2}+1)},$$

where each entry of $(\sigma_{i,j})_{0 \leqq i \lneqq j \lneqq n}$ runs over all non-negative integers, $B = (b_{i,j})_{j=0,\dots,n-1}^{i=0,\dots,n-1} = K^{-1} \cdot ((-\cos\langle i,j \rangle)_{j=0,\dots,n-1}^{i=0,\dots,n-1})^{-1} \cdot K^{-1}$, $K = (\delta_j^i \cdot \Delta(0,\dots,\widehat{i}\dots,n-1)/\Delta(0,\dots,n-1))_{j=0,\dots,n-1}^{i=0,\dots,n-1}$, $\Delta(i_0,\dots,i_{r-1}) = \det\left((-\cos\langle k,l \rangle)_{l=i_0,\dots,i_{r-1}}^{k=i_0,\dots,i_{r-1}}\right)$, δ_j^i is Kronecker's delta, and the circumflex indicates that the term below it has been omitted.

Polars of spherical simplices. For a spherical simplex S, the polar S° is

$$S^\circ = \{\vec{y} \in \mathbb{S}^{n-1} : \vec{x} \cdot \vec{y} \leqq 0,\ {}^\forall \vec{x} \in S\}.$$

See the figure for $n = 2$.



Main result. For all n, we have simple formulae:

$$\sum_{I \subseteqq \{0,\dots,n-1\}}^{\sharp I:\ \text{even}} f_{n-\sharp I}(S(I;I^c))\, f_{\sharp I}((S(\emptyset;I))^\circ) = 2^{n-1},$$

$$\sum_{I \subseteqq \{0,\dots,n-1\}}^{\sharp I:\ \text{odd}} f_{n-\sharp I}(S(I;I^c))\, f_{\sharp I}((S(\emptyset;I))^\circ) = 2^{n-1}.$$

Application. From the former, we can get a formula which represents $f_n(S)+f_n(S^\circ)$ by the volumes of lower dimensional spherical simplices for even n, e.g.,

$$f_2(S)+f_2(S^\circ)=2, \quad \text{which is trivial for } n=2,$$

$$f_4(S)+f_4(S^\circ)=8-\sum_{I\subsetneqq\{0,1,2,3\}}^{\sharp I=2} f_2(S(I;I^c))f_2((S(\emptyset;I))^\circ),$$

which is already known by Schläfli for $n=4$,

$$f_6(S)+f_6(S^\circ)=32-\sum_{I\subsetneqq\{0,\dots,5\}}^{\sharp I=2} f_4(S(I;I^c))f_2((S(\emptyset;I))^\circ)-\sum_{I\subsetneqq\{0,\dots,5\}}^{\sharp I=4} f_2(S(I;I^c))f_4((S(\emptyset;I))^\circ),$$

which seems to be new result for $n=6$, and so on. For odd n, from the former again, we can also represent $f_n(S)$ distinct from Schläfli's reduction formula, e.g.,

$$f_3(S)=4-\sum_{I\subsetneqq\{0,1,2\}}^{\sharp I=2} 1\cdot f_2((S(\emptyset;I))^\circ), \quad \text{which is trivial for } n=3,$$

$$f_5(S)=16-\sum_{I\subsetneqq\{0,\dots,4\}}^{\sharp I=2} f_3(S(I;I^c))f_2((S(\emptyset;I))^\circ)-\sum_{I\subsetneqq\{0,\dots,4\}}^{\sharp I=4} 1\cdot f_4((S(\emptyset;I))^\circ),$$

which seems to be new result for $n=5$, and so on.

Sketch of proof. We have Schläfli's differential formulae for lower dimensional spherical simplices and polars:

$$df_{n-\sharp I}(S(I;I^c))=\sum_{K\subsetneqq I^c}^{\sharp K=2} f_{n-\sharp I-2}(S(I\cup K;(I\cup K)^c))df_2(S(I;K)),$$
$$df_{\sharp J}((S(\emptyset;J))^\circ)=-\sum_{K\subsetneqq J}^{\sharp K=2} f_{\sharp J-2}((S(\emptyset;J\setminus K))^\circ)df_2(S(J\setminus K;K)).$$

From differential formulae above, we can calculate the derivations of the left-hand sides of the main results, which are equal to zero. So the left-hand sides are constant, which are determined by the values of the spherical simplex whose all half-spaces are pairwise perpendicular. For such a spherical simplex, all terms of the left-hand side are equal to 1, so, the constant values are 2^{n-1}. □

References

[A] K. Aomoto, *Analytic structure of Schläfli function*, Nagoya Math. J. **68** (1977), 1–16.

[C] H. S. M. Coxeter, *The functions of Schläfli and Lobatschefsky*, Quart. J. of Math. **6** (1935), 13– 29, Reprinted in: H. S. M. Coxeter, *The beauty of geometry: Twelve essays*, Dover, New York, 1968.

[S] L. Schläfli, *On the multiple integral* $\int^n dxdy\dots dz$, *whose limits are* $p_1=a_1x+b_1y+\dots+h_1z>0$, $p_2>0,\dots,p_n>0$, *and* $x^2+y^2+\dots+z^2<1$, Quart. J. of Math. **2** (1858), 269–301, **3** (1860), 54– 68, 97–108.

[一] 一松信, 高次元の正多面体, 日本評論社, 数セミ・ブックス 7, 1983.

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