On orthocomplemented lattices with Elkan’s law

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Abstract
In this short note we answer the problem left open in [3] that
Are there orthocomplemented lattices different from Boolean
algebras verifying (EL) : \((a \cdot b')' = b + a' \cdot b'\) for all \(a, b\)?

That is, we prove that any orthocomplemented lattice satisfying \((a \cdot b')' = b + a' \cdot b'\) is a Boolean algebra. Moreover we show the stronger
result that every bounded lattice with the conditions (C1) \(1' = 0\) and
(EL) is a Boolean algebra.

1 Introduction

It is proved in [3] that each of de Morgan algebras and of orthomodular
lattices with Elkan’s law (EL) : \((a \cdot b')' = b + a' \cdot b'\) is a Boolean algebra.
The equation \((a \cdot b')' = b + a' \cdot b'\) is presented in the theory of fuzzy logic
by C.Elkan ([2]). The result means that Boolean algebras are characterized
from de Morgan algebras and orthomodular lattices by this equation. As a
natural extension of the results, it is left open in [3] that

Are there orthocomplemented lattices different from Boolean al-
gebras verifying \((a \cdot b')' = b + a' \cdot b'\) for all \(a, b\)?

In other words,
Are orthocomplemented lattices with \((a \cdot b)' = b + a' \cdot b'\) Boolean algebras?

In this short paper we give an answer to the problem. That is, any orthocomplemented lattice with \((a \cdot b)' = b + a' \cdot b'\) is a Boolean algebra. Moreover we prove the stronger result that any bounded lattice (not necessarily orthocomplemented lattice) satisfying only the conditions

\begin{itemize}
  \item[(c1)] \(1' = 0\)
  \item[(EL)] \((a \cdot b)' = b + a' \cdot b'\)
\end{itemize}

is a Boolean algebra. This is a new characterization theorem of Boolean algebras in terms of bounded lattices.

2 Preliminaries

In the following we define pseudo-complemented (de Morgan, orthomodular) lattices to express exactly the open problem to be solved.

Let \(\mathcal{L} = (L;\cdot,+,'\cdot,0,1)\) be a bounded lattice. A unary operator \('\) satisfying the following condition is called a pseudo-complement in [3]: For all \(a, b \in L\),

\begin{itemize}
  \item[(c0)] \(0' = 1\)
  \item[(P1)] \(a \leq b \implies b' \leq a'\)
  \item[(P2)] \(a'' = a\)
\end{itemize}

Usually, the "pseudo-complement" operator "\(\ast\)" is not defined by the above but by the following as in [1]: For all \(a \in L\), an element \(a^{\ast}\) is called a pseudo-complement of \(a\) if

\begin{itemize}
  \item[(pc1)] \(a \cdot a^{\ast} = 0\)
  \item[(pc2)] \(a \cdot x = 0 \implies x \leq a^{\ast}\) for every \(x \in L\).
\end{itemize}

But here we adopted the definition which is used in [3] to avoid confusion.

Let \(\mathcal{L} = (L;\cdot,+,'\cdot,0,1)\) be a bounded lattice with pseudo-complementation in the sense of [3]. That is, \(\mathcal{L}\) is a bounded lattice satisfying the conditions (c0), (P1) and (P2). We consider several properties about the unary operator \('\) on \(L\).
(D1-1) \((a \cdot b)' = a' + b'\),
(D1-2) \((a + b)' = a' \cdot b'\) (Duality laws)
(D2-1) \(a \cdot (b + c) = a \cdot b + a \cdot c\),
(D2-2) \(a + b \cdot c = (a + b) \cdot (a + c)\) (Distributive laws)
(NC) \(a \cdot a' = 0\) (Non-contradiction)
(EM) \(a + a' = 1\) (Excluded-middle)
(OM) If \(a \leq b\) then \(b = a + a' \cdot b\) (Orthomodular law)

If a bounded lattice \(\mathcal{L}\) with a pseudo-complementation \('\) satisfies (D1) and
(D2) ((D1), (NC) and (EM)), then it is called a de Morgan algebra (or an
orthomodular lattice, respectively) ([3]). By DM (OML, BA) we mean the
class of all de Morgan algebras (orthomodular lattices, Boolean algebras,
respectively). It is proved in [3] that

1. The only de Morgan algebras in which the law \((a \cdot b')' = b + a' \cdot b'\)
holds are those that are Boolean algebras. (Theorem 1)
2. If in an orthomodular lattice \(L\) the law \((a \cdot b')' = b + a' \cdot b'\)
holds for all \(a, b \in L\), then \(L\) is a Boolean algebra. (Theorem 3)

For the sake of simplicity, we call an equation

\[(a \cdot b')' = b + a' \cdot b'\]

the Elkan’s law (EL). We simply denote these statements by informal ex-
pression:

\[
\text{DM} + (\text{EL}) = \text{BA} \quad \text{and} \quad \text{OML} + (\text{EL}) = \text{BA}
\]

3 Orthocomplemented Lattices

Let \((L, ', +, 0, 1)\) be a bounded lattice. A unary operator \(' : L \to L\) is called
an orthocomplementation in [3] when it satisfies, for all \(a, b \in L\),

1. \(0' = 1\)
2. \(1' = 0\)
3. \(a \leq b \implies b' \leq a'\)
4. \(a'' = a\).
We also note that usually an operator $\ast$ is called orthocomplementation as in [1] if it satisfies

\begin{align*}
(OC1) \quad a \cdot a^* &= 0 \\
(OC2) \quad a + a^* &= 1
\end{align*}

besides those of the above. But, we here do not assume the extra axioms (OC1) and (OC2) in the definition of orthocomplementation. That is, by an orthocomplementation operator we mean the operator $\ast$ satisfying (c0), (c1), (P1) and (P2) according to the original paper on which we based.

A bounded lattice with an orthocomplementation (satisfying only (c0), (c1), (P1) and (P2)) is called an orthocomplemented lattice and we denote the class of orthocomplemented lattices by OCL. Then, the problem raised in [3] is represented by

$$\text{OCL } + \text{(EL) } = \text{BA} ?$$

In this note we show that every orthocomplemented lattice with Elkan's law is a Boolean algebra. That is, the class OCL + (EL) of all orthocomplemented lattices with Elkan's law coincides with the class BA of all Boolean algebras. Moreover we show that any bounded lattice with (c1) and (EL) is a Boolean algebra.

In the following, let $L$ be an orthocomplemented lattice with Elkan's law (EL), which is called simply orthocomplemented lattice with EL.

**Proposition 1.** For each $a \in L$, we have

\begin{align*}
(EM) \quad a + a' &= 1 \\
(NC) \quad a \cdot a' &= 0
\end{align*}

*Proof.* Let $L$ be an orthocomplemented lattice with EL. If we take $a = 0$ and $b = a$ in the equation (EL), then we have

$$(0 \cdot a')' = a + 0' \cdot a'. $$

Since $0 \cdot a' = 0$ and $0' \cdot a = 1 \cdot a = a$, it follows that

$$1 = 0' = (0 \cdot a')' = a + 0' \cdot a'$$

$$= a + 1 \cdot a'$$

$$= a + a'$$
Concerning to the another equation (NC), since
\[ a \cdot b' = (a \cdot b)' = (b + a' \cdot b'), \]
if we take \( b = a \) then we get
\[ a \cdot a' = (a + a' \cdot a')' = (a + a')' = 1' = 0. \]

**Proposition 2.** For all \( a, b \in L \),

(D1.1) \( (a + b)' = a' \cdot b' \)

(D1.2) \( (a \cdot b)' = a' + b' \).

*Proof.* Case (D1-1) : \( (a + b)' = a' \cdot b' \).

Since \( a \cdot b' \leq a \), we have
\[ a' \leq (a \cdot b')' = b + a' \cdot b' \]
and hence
\[ a' + b \leq b + a' \cdot b' \leq b + a'. \]

This implies
\[ a' + b = b + a' \cdot b' = (a \cdot b)'). \]

In this formula, if we replace \( b \) by \( b' \) then we have
\[ a' + b' = (a \cdot b'')' = (a \cdot b'). \]

It follows from this formula that
\[ (a + b)' = (a'' + b'')' = (a' \cdot b'')'' = a' \cdot b'. \]

The other case can be proved similarly.

**Proposition 3.** Every orthocomplemented lattice with EL satisfies the orthomodular law : \( a \leq b \) implies \( b = a + a' \cdot b \).

*Proof.* In the formula (EL) : \( (a \cdot b')' = b + a' \cdot b' \), if we replace \( a \) by \( b' \) and \( b \) by \( a \) simultaneously, then we have \( (b' \cdot a')' = a + (b')' \cdot a' \), that is,
\[ a + b = (a' \cdot b')' = a + a' \cdot b. \]

If \( a \leq b \), since \( a+b = b \), then we conclude \( b = a+a' \cdot b \). Thus the orthomodular law holds in each orthocomplemented lattice with EL.
In the theory of orthomodular lattices, the binary relation (called \textit{commutativity}) $C$ defined by
\[xCy \iff x = x \cdot y + x \cdot y'\]
plays an important role ([1]). From the definition we have

(a) If $a \leq b$, then $aCb$.  
(b) If $aCb$, then $aCb'$.  

It is well known that if \{x, y, z\} is a distributive triple, that is, one of the elements of the set is commutative with the other two elements, then all possible forms of the two distributive laws with $x, y$ and $z$ hold. Thus, for example, if we have $xCy$ and $xCz$ then we can conclude that
\[
x \cdot (y + z) = x \cdot y + x \cdot z, \\
y \cdot (x + z) = y \cdot x + y \cdot z, \\
z \cdot (x + y) = z \cdot x + z \cdot y, \\
x + (y \cdot z) = (x + y) \cdot (x + z),
\]
\[\vdots\]

It is easy to prove that

**Proposition 4.** For all $a, b \in L$, we have $aCb$, that is, $a = a \cdot b + a \cdot b'$. Thus, every element $a \in L$ is commutative with all elements in $L$, so $L$ is a distributive lattice.

From the above propositions, we can conclude that

**Theorem 1.** Any orthocomplemented lattice with $EL$ is a Boolean algebra.

## 4 Stronger result

By careful investigation of the proof in the previous section, we can prove the stronger result about the open problem.

**Theorem 2.** Let $\mathcal{L} = (L; \cdot, +, 0, 1)$ be a bounded lattice. If a unary operator $'$ satisfies only the following conditions, then $L$ is a Boolean algebra:
Proof. In order to prove our statement, it suffices to verify that (P1), (P2), (NC) and (EM) hold in any bounded lattice with (c1) and (EL).

At first we show (P2). In the equation (EL), if we put $a = 1$ and $b = a$ simultaneously, then we have
\[(1 \cdot a')' = a + 1' \cdot a'.\]
Since $1 \cdot a' = a'$ and $0 \cdot a' = 0$, it follows from (c1) that
\[a'' = a + 0 = a.\]
Concerning (EM) : $a + a' = 1$, if we take $a = 0$ and $b = a$ in (EL), then we have $(0 \cdot a')' = a + 0' \cdot a'$. Since $0' = 1$ as proved in the above, we have
\[1 = 0' = (0 \cdot a')' = a + 0' \cdot a' = a + a'.\]
For (NC) : $a \cdot a' = 0$, if we take $b = a$ in (EL), then $(a \cdot a')' = a + a' \cdot a' = a + a' = 1$. Thus we get
\[a \cdot a' = (a \cdot a')'' = 1' = 0.\]
At last, we prove that (P1) : $a \leq b \rightarrow b' \leq a'$. Suppose that $a \leq b$. Since $a \cdot b' \leq b \cdot b' = 0$, we have $a \cdot b' = 0$. If we replace $a$ by $a'$ in (EL), then
\[(a' \cdot b')' = b + a'' \cdot b'\]
\[= b + a \cdot b'\]
\[= b + 0 = b.\]
This yields $a' \cdot b' = (a' \cdot b')'' = b'$ and hence $b' \leq a'$.

References

