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On orthocomplemented lattices with Elkan's law

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Abstract

In this short note we answer the problem left open in [3] that

> Are there orthocomplemented lattices different from Boolean algebras verifying (EL) : $(a \cdot b')' = b + a' \cdot b'$ for all a, b ?

That is, we prove that any orthocomplemented lattice satisfying $(a \cdot b')' = b + a' \cdot b'$ is a Boolean algebra. Moreover we show the stronger result that every bounded lattice with the conditions (C1) $1' = 0$ and (EL) is a Boolean algebra.

1 Introduction

It is proved in [3] that each of de Morgan algebras and of orthomodular lattices with Elkan's law (EL) : $(a \cdot b')' = b + a' \cdot b'$ is a Boolean algebra. The equation $(a \cdot b')' = b + a' \cdot b'$ is presented in the theory of fuzzy logic by C.Elkan ([2]). The result means that Boolean algebras are characterized from de Morgan algebras and orthomodular lattices by this equation. As a natural extension of the results, it is left open in [3] that

> Are there orthocomplemented lattices different from Boolean algebras verifying $(a \cdot b')' = b + a' \cdot b'$ for all a, b ?

In other words,

Are orthocomplemented lattices with $(a \cdot b')' = b + a' \cdot b'$ Boolean algebras ?

In this short paper we give an answer to the problem. That is, any oethocomplemented lattice with $(a \cdot b')' = b + a' \cdot b'$ is a Boolean algebra. Moreover we prove the stronger result that any bounded lattice (not necessarily orthocomplemented lattice) satisfying only the conditions

(c1) $1' = 0$

(EL) $(a \cdot b')' = b + a' \cdot b'$

is a Boolean algebra. This is a new charatecrization theorem of Boolean algebras in terms of bounded lattices.

2 Preliminaries

In the following we define pseudo-complemented (de Morgan, orthomodular) lattices to express exactly the open problem to be solved.

Let $\mathcal{L} = (L; \cdot, +, 0, 1)$ be a bounded lattice. A unary operator $'$ satisfying the following condition is called a *pseudo-complement* in [3] : For all $a, b \in L$,

(c0) $0' = 1$

(P1) $a \leq b \implies b' \leq a'$

(P2) $a'' = a$

Usually, the "pseudo-complement" operator "$*$" is not defined by the above but by the following as in [1] : For all $a \in L$, an element a^* is called a pseudo-complement of a if

(pc1) $a \cdot a^* = 0$

(pc2) $a \cdot x = 0 \Longrightarrow x \leq a^*$ for every $x \in L$.

But here we adopted the definition which is used in [3] to avoid confusion.

Let $\mathcal{L} = (L; \cdot, +, ', 0, 1)$ be a bounded lattice with pseudo-complementation in the sense of [3]. That is, $\mathcal{L}$ is a bounded lattice satisfying the conditions (c0), (P1) and (P2). We consider several properties about the unary operator $'$ on L.

(D1-1) $(a \cdot b)' = a' + b'$,

(D1-2) $(a + b)' = a' \cdot b'$ (Duality laws)

(D2-1) $a \cdot (b + c) = a \cdot b + a \cdot c$,

(D2-2) $a + b \cdot c = (a + b) \cdot (a + c)$ (Distributive laws)

(NC) $a \cdot a' = 0$ (Non-contradiction)

(EM) $a + a' = 1$ (Excluded-middle)

(OM) If $a \leq b$ then $b = a + a' \cdot b$ (Orthomodular law)

If a bounded lattice $\mathcal{L}$ with a pseudo-complementation $'$ satisfies (D1) and (D2) ((D1), (NC) and (EM)), then it is called a *de Morgan algebra* (or an *orthomodular lattice*, respectively) ([3]). By **DM** (**OML**, **BA**) we mean the class of all de Morgan algebras (orthomodular lattices, Boolean algebras, respectively). It is proved in [3] that

1. The only de Morgan algebras in which the law $(a \cdot b')' = b + a' \cdot b'$ holds are those that are Boolean algebras. (Theorem 1)
2. If in an orthomodular lattice L the law $(a \cdot b')' = b + a' \cdot b'$ holds for all $a, b \in L$, then L is a Boolean algebra. (Theorem 3)

For the sake of simplicity, we call an equation

$$(a \cdot b')' = b + a' \cdot b'$$

the Elkan's law (EL). We simply denote these statements by informal expression :

$$\mathbf{DM} + (\mathrm{EL}) = \mathbf{BA} \quad \text{and} \quad \mathbf{OML} + (\mathrm{EL}) = \mathbf{BA}$$

3 Orthocomplemented Lattices

Let $(L, \cdot, +, 0, 1)$ be a bounded lattice. A unary operator $' : L \to L$ is called an *orthocomplementation* in [3] when it satisfies, for all $a, b \in L$,

(c0) $0' = 1$

(c1) $1' = 0$

(P1) $a \leq b \Longrightarrow b' \leq a'$

(P2) $a'' = a$.

We also note that usually an operator $*$ is called *orthocomplentation* as in [1] if it satisfies

(OC1) $a \cdot a^* = 0$

(OC2) $a + a^* = 1$

besides those of the above. But, we here do not assume the extra axioms (OC1) and (OC2) in the definition of orthocomplementation. That is, by an orthocomplementation operator we mean the operator $'$ satisfying (c0), (c1), (P1) and (P2) according to the original paper on which we based.

A bounded lattice with an orthocomplementation (satisfying only (c0), (c1), (P1) and (P2)) is called an *orthocomplemented* lattice and we denote the class of orthocomplemented lattices by **OCL**. Then, the problem raised in [3] is represented by

$$\mathbf{OCL} + (\mathrm{EL}) = \mathbf{BA}\ ?$$

In this note we show that every orthocomplemented lattice with Elkan's law is a Boolean algebra. That is, the class **OCL**+ (EL) of all orthocomplemented lattices with Elkan's law coincides with the class **BA** of all Boolean algebras. Moreover we show that any bounded lattice with (c1) and (EL) is a Boolean algebra.

In the following, let L be an orthocomplemented lattice with Elkan's law (EL), which is called simply orthocomplemented lattice with EL.

Proposition 1. *For each $a \in L$, we have*

(EM) $a + a' = 1$

(NC) $a \cdot a' = 0$

Proof. Let L be an orthocomplemented lattice with EL. If we take $a = 0$ and $b = a$ in the equation (EL), then we have

$$(0 \cdot a')' = a + 0' \cdot a'.$$

Since $0 \cdot a' = 0$ and $0' \cdot a = 1 \cdot a = a$, it follows that

$$\begin{aligned} 1 = 0' = (0 \cdot a')' &= a + 0' \cdot a' \\ &= a + 1 \cdot a' \\ &= a + a' \end{aligned}$$

Concerning to the another equation (NC), since

$$a \cdot b' = (a \cdot b')'' = (b + a' \cdot b')',$$

if we take $b = a$ then we get

$$a \cdot a' = (a + a' \cdot a')' = (a + a')' = 1' = 0.$$

□

Proposition 2. *For all $a, b \in L$,*

(D1-1) $(a + b)' = a' \cdot b'$

(D1-2) $(a \cdot b)' = a' + b'$.

Proof. Case (D1-1) : $(a + b)' = a' \cdot b'$.

Since $a \cdot b' \leq a$, we have

$$a' \leq (a \cdot b')' = b + a' \cdot b'$$

and hence

$$a' + b \leq b + a' \cdot b' \leq b + a'.$$

This implies

$$a' + b = b + a' \cdot b' = (a \cdot b')'.$$

In this formula, if we replace b by b' then we have

$$a' + b' = (a \cdot b'')' = (a \cdot b)'.$$

It follows from this formula that

$$(a + b)' = (a'' + b'')' = (a' \cdot b')'' = a' \cdot b'.$$

The other case can be proved similarly. □

Proposition 3. *Every orthocomplemented lattice with EL satisfies the orthomodular law : $a \leq b$ implies $b = a + a' \cdot b$.*

Proof. In the formula (EL) : $(a \cdot b')' = b + a' \cdot b'$, if we replace a by b' and b by a simultaneously, then we have $(b' \cdot a')' = a + (b')' \cdot a'$, that is,

$$a + b = (a' \cdot b')' = a + a' \cdot b.$$

If $a \leq b$, since $a + b = b$, then we conclude $b = a + a' \cdot b$. Thus the orthomodular law holds in each orthocomplemented lattice with EL. □

In the theory of orthomodular lattices, the binary relation (called *commutativity*) C defined by

$$xCy \iff x = x \cdot y + x \cdot y'$$

plays an important role ([1]). From the definition we have

(a) If $a \leq b$, then aCb.

(b) If aCb, then aCb'.

It is well known that if $\{x, y, z\}$ is a distributive triple, that is, one of the elements of the set is commutative with the other two elements, then all possible forms of the two distributive laws with x, y and z hold. Thus, for example, if we have xCy and xCz then we can conclude that

$$\begin{aligned} x \cdot (y+z) &= x \cdot y + x \cdot z, \\ y \cdot (x+z) &= y \cdot x + y \cdot z, \\ z \cdot (x+y) &= z \cdot x + z \cdot y, \\ x + (y \cdot z) &= (x+y) \cdot (x+z), \\ &\vdots \end{aligned}$$

It is easy to prove that

Proposition 4. *For all $a, b \in L$, we have aCb, that is, $a = a \cdot b + a \cdot b'$. Thus, every element $a \in L$ is commutative with all elements in L, so L is a distributive lattice.*

From the above propositions, we can conclude that

Theorem 1. *Any orthocomplemented lattice with EL is a Boolean algebra.*

4 Stronger result

By careful investigation of the proof in the previouis section, we can prove the stronger result about the open problem.

Theorem 2. *Let $\mathcal{L} = (L; \cdot, +, 0, 1)$ be a bounded lattice. If a unary operator $'$ satisfies only the following conditions, then L is a Boolean algebra :*

(c1) $1' = 0$

(EL) $(a \cdot b')' = b + a' \cdot b'$

Proof. In order to prove our statement, it suffices to verify that (P1),(P2),(NC) and (EM) hold in any bounded lattice with (c1) and (EL).

At first we show (P2). In the equation (EL), if we put $a = 1$ and $b = a$ simultaneously, then we have

$$(1 \cdot a')' = a + 1' \cdot a'.$$

Since $1 \cdot a' = a'$ and $0 \cdot a' = 0$, it follows from (c1) that

$$a'' = a + 0 = a.$$

Concerning (EM) : $a + a' = 1$, if we take $a = 0$ and $b = a$ in (EL), then we have $(0 \cdot a')' = a + 0' \cdot a'$. Since $0' = 1$ as proved in the above, we have

$$1 = 0' = (0 \cdot a')' = a + 0' \cdot a' = a + a'.$$

For (NC) : $a \cdot a' = 0$, if we take $b = a$ in (EL), then $(a \cdot a')' = a + a' \cdot a' = a + a' = 1$. Thus we get

$$a \cdot a' = (a \cdot a')'' = 1' = 0.$$

At last, we prove that (P1) : $a \leq b \implies b' \leq a'$. Suppose that $a \leq b$. Since $a \cdot b' \leq b \cdot b' = 0$, we have $a \cdot b' = 0$. If we replace a by a' in (EL), then

$$\begin{aligned}(a' \cdot b')' &= b + a'' \cdot b' \\ &= b + a \cdot b' \\ &= b + 0 = b.\end{aligned}$$

This yields $a' \cdot b' = (a' \cdot b')'' = b'$ and hence $b' \leq a'$. □

References

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