Circular Codes and Petri Nets

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Abstract

The purpose of this paper is to investigate the relationship between limited codes and Petri nets. For a given Petri net with an initial marking \( \mu \), we can naturally define an automaton \( A \) which has the initial marking \( \mu \) as an initial state, the reachability set \( \text{Re}(\mu) \) as a set of states, and the set of transitions as a set of inputs. We can define prefix codes by considering the set of firing sequences which arrive from the positive initial marking of a Petri net to a certain subset of the reachability set [10,12]. The set \( M \) of all positive firing sequences which start from the positive initial marking \( \mu \) of a Petri net and reach \( \mu \) itself forms a pure monoid. Our main interest is in the base \( D \) of \( M \). The family of pure monoids contains the family of very pure monoids, and the base of a very pure monoid is a circular code. Therefore, we can expect that \( D \) may be a circular code. Here, for "small" Petri nets, we discuss under what conditions \( D \) is circular.

Key words: Petri net, Code, Prefix code, Circular code, Limited code.

1. Introduction

Let \( A \) be an alphabet, \( A^* \) the free monoid over \( A \), and \( 1 \) the empty word. A word \( v \in A^* \) is a left factor of a word \( u \in A^* \) if there is a word \( w \in A^* \) such that \( u = vw \). The left factor \( v \) of \( u \) is called proper if \( v \neq u \). A right factor and a proper right factor of a word are defined in a symmetric manner.

For a word \( w \in A^* \) and a letter \( x \in A \) we let \( |w|_x \) denote the number of \( x \) in \( w \). The length of \( w \) is the number of letters in \( w \). A non-empty subset \( C \) of \( A^+ \) is said to be a code if for \( x_1, \ldots, x_p, y_1, \ldots, y_q \in C, p, q \geq 1, \)

\[
x_1 \cdots x_p = y_1 \cdots y_q \text{ implies } p=q \text{ and } x_1 = y_1, \ldots, x_p = y_p.
\]

A subset \( M \) of \( A^* \) is a submonoid of \( A^* \) if \( M^2 \subseteq M \) and \( 1 \in M \). Every submonoid \( M \) of a free monoid has a unique minimal set of generators

\[
C = (M - \{1\}) - (M - \{1\})^2.
\]

\( C \) is called the base of \( M \).

This is the abstract and the details will be published elsewhere.
A submonoid $M$ is right unitary in $A^*$ if for all $u, v \in A^*$, 
\[ u, uv \in M \implies v \in M. \]

$M$ is called left unitary in $A^*$ if it satisfies the dual condition. A submonoid $M$ is biunitary if it is both left and right unitary.

**Definition 1.1.** Let $M$ be a submonoid of a free monoid $A^*$, and $C$ its base. If $CA^* \cap C = \emptyset$ (resp. $A^*C \cap C = \emptyset$), then $C$ is called a prefix (resp. suffix) code over $A$. $C$ is called a bifix code if it is a prefix and suffix code.

A submonoid $M$ of $A^*$ is right unitary (resp. biunitary) if and only if its minimal set of generator is a prefix code (bifix code) ([1, p.46]).

**Definition 1.2.** A Petri net is a 5-tuple, $PN = (P, A, F, W, \mu_0)$ where:
- $P = \{p_1, p_2, \ldots, p_m\}$ is a finite set of places,
- $A = \{t_1, t_2, \ldots, t_n\}$ is a finite set of transitions,
- $F \subseteq (P \times A) \cup (A \times P)$ is a set of arcs,
- $W : F \rightarrow \{1, 2, \ldots\}$ is a weight function,
- $\mu_0 : P \rightarrow \{0, 1, 2, \ldots\}$ is the initial marking,
- $P \cap A = \emptyset$ and $P \cup A \neq \emptyset$.

We use the following notations for a pre-set and a post-set: 
\[ t = \{p | (p, t) \in F\}, \quad t^{-} = \{p | (t, p) \in F\}, \]

In this paper we shall assume that a Petri net has no isolated transitions, i.e., no $t$ such that $t \cup t^{-} = \emptyset$. A marking $\mu_0$ can be represented by a vector:
\[ \mu_0 = (\mu_0(p_1), \mu_0(p_2), \ldots, \mu_0(p_n)), \quad p_i \in P, \quad n = |P|. \]

For every $t \in A$ the vector $\Delta t$ is defined by
\[ \Delta t = (\Delta t(p_1), \Delta t(p_2), \ldots, \Delta t(p_n)), \quad n = |P|, \]

where
\[ \Delta t(p) = \begin{cases} 
-W(p, t) + W(t, p) & \text{if } p \in t \cap t^{-}, \\
-W(p, t) & \text{if } p \in t - t^{-}, \\
W(t, p) & \text{if } p \in t - t^{-}, \\
0 & \text{if } p \notin t \cup t^{-}. 
\end{cases} \]

A transition $t \in A$ is said to be enabled in $\mu_0$, if $W(p, t) \leq \mu_0(p)$ for all $p \in t$. A firing of an enabled transition $t$ removes $W(p_1, t)$ tokens from each input place $p_1 \in t$, and adds $W(t, p_2)$ tokens to each output place $p_2 \in t$. Firing of an enabled transition $t$ at $\mu_0$ produces a new
marking $\mu_1$ such that

$$
\mu_1(p) = \begin{cases} 
\mu_0(p) - W(p, t) & \text{if } p \in t - t, \\
\mu_0(p) + W(t, p) & \text{if } p \in t \cdot t, \\
\mu_0(p) - W(p, t) + W(t, p) & \text{if } p \in t \cdot t, \\
\mu_0(p) & \text{otherwise.}
\end{cases}
$$

If we obtain the marking $\mu'$ that results from a firing of $t$ at $\mu$, we write $\delta(\mu, t) = \mu'$. A word $w = t_1t_2\ldots t_r, (t_i \in A)$, of transitions is said to be a (firing) sequence from $\mu_0$ if there exist markings $\mu_i, 1 \leq i \leq r$, such that $\delta(\mu_{i-1}, t_i) = \mu_i$ for all $i, (1 \leq i \leq r)$. In this case, $\mu_r$ is reachable from $\mu_0$ by $w$ and we write $\delta(\mu_0, w) = \mu_r$. The set of all possible markings reachable from $\mu_0$ is denoted by $Re(\mu_0)$, and the set of all possible sequences from $\mu_0$ is denoted by $Seq(\mu_0)$.

The function $\delta : Re(\mu_0) \times T \rightarrow Re(\mu_0)$ is called a next-state function of a Petri net $PN$. We note that the above condition for $r = 0$ is understood to be $\mu_0 \in Re(\mu_0)$.

A marking $\mu$ is said to be positive if $\mu(p) > 0$ for all $p \in P$. A sequence $t_1t_2\ldots t_n \in Seq(\mu_0)$, $(t_i \in T)$, is called a positive sequence from $\mu_0$ if $\delta(\mu_0, t_1t_2\ldots t_i)$ is positive for all $i, (1 \leq i \leq n)$. The set of all positive sequences from $\mu_0$ is denoted by $PSeq(\mu_0)$.

By $PRe(\mu_0)$ we denote the set of all positive markings reachable from $\mu_0$; $PRe(\mu_0) = \{\delta(\mu_0, w) | w \in PSeq(\mu_0)\}$.

2. Some codes related to Petri nets

For a Petri net $PN = (P, T, F, W, \mu_0)$ and a subset $X \subseteq Re(\mu_0)$ we can define a deterministic automaton $A(PN)$ as follows: $Re(\mu_0), T, \delta : Re(\mu_0) \times T \rightarrow Re(\mu_0), \mu_0$, and $X$, are regarded as a state set, an input set, a next-state function, an initial state, and a final set of $A(PN)$, respectively. By using such automata, in [10,12] we defined four kinds of prefix codes and examined fundamental properties of these codes.

Let $PN = (P, A, F, W, \mu)$ be a Petri net. The set

$$
Stab(PN) = \{w | w \in Seq(\mu) \text{ and } \delta(\mu, w) = \mu\}
$$

forms a submonoid of $A^*$. If $Stab(PN) \neq \{1\}$, then we denote the base of $Stab(PN)$ by $S(PN)$. Since $S(PN)A^+ \cap S(PN) = \emptyset$, $S(PN)$ is a prefix code over $A$.

A submonoid $M$ of $A^*$ is called pure [7] if for all $x \in A^*$ and $n \geq 1$,

$$
x^n \in M \implies x \in M.
$$

A subsemigroup $H$ of a semigroup $S$ is extractable in $S$ [9, p.191] if

$$
x, y \in S, z \in H, xyz \in H \implies xyz \in H.
$$

Proposition 2.1. If $Stab(PN) \neq \emptyset$, then $Stab(PN)$ is a biunitary extractable pure monoid.
Definition 2.1. Let $PN = (P, A, F, W, \mu)$ be a Petri net with a positive marking $\mu$. Define the subset $D(PN)$ as a set of all positive sequence $w$ of $S(PN)$.

Since $D(PN)$ is a subset of $S(PN)$, $D(PN)$ is a bifix code over $A$.

Proposition 2.2. If $D(PN) \neq \emptyset$, then $D(PN)^*$ is a biunitary extractable pure monoid.

Example 2.1. Let $PN = (\{p, q\}, \{a, b\}, F, W, \mu_0)$ be a Petri net defined by $W(a, p) = W(p, b) = W(q, a) = W(b, q) = 1$, $\mu_0(p) = \mu_0(q) = 2$. Then $D(PN) = \{ab, ba\}$, therefore $\{ab, ba\}^*$ is pure [1, p.324, Ex.1.3].

Proposition 2.3. If $z, xzy \in D(PN), x, y \in A^+$, then $xz^*y \in D(PN)$.

A code $D$ is infix if $w, xwy \in D$ implies $x = y = 1$ [8, p.129].

Proposition 2.4. If $D(PN)$ is a non-empty finite set, then $D(PN)$ is an infix code.

3. Limited code

A submonoid $M$ of $A^*$ is very pure if for all $u, v \in A^*$,

$$u, v \in A^*, uv, vu \in M \Rightarrow u, v \in M.$$  

The base of a very pure monoid is called a circular code.

Let $p, q \geq 0$ be two integers. If for any sequence $u_0, u_1, \ldots, u_{p+q}$ of words in $A^*$, the assumptions $u_{i-1}u_i \in M$ ($1 \leq i \leq p + q$) imply $u_p \in M$, then a submonoid $M$ is said to satisfy condition $C(p,q)$. If a submonoid $M$ of $A^*$ satisfies condition $C(p,q)$, then $M$ is very pure [1, p.329, Proposition 2.1], and its base is called a $(p,q)$-limited code.

If a subset $D$ of $A^*$ is a bifix $(1,1)$-limited code, then for any $u_0, u_1, u_2 \in A^*$ such that $u_0u_1, u_1u_2 \in D$ we have $u_1 \in D$. Thus $u_0u_1, u_1, u_2 \in D$. This implies that $u_0, u_1, u_2 \in D$, since $D$ is bifix. Therefore $D$ is $(2,0)$-, $(1,1)$- and $(0,2)$-limited.

Let $PN_0 = (\{p\}, \{a, b\}, F, W, \mu_0)$ be a Petri net such that $W(a, p) = \alpha, W(p, b) = \beta, \mu_0 = (\lambda_p), \lambda_p > 0$. 

Consider the set \( \Omega \) of positive markings in \( PN_0 \):

\[
\Omega = \{ \mu | \mu = \mu_0 + \Delta(w), w \in PSeq(\mu_0) \}.
\]

\( \alpha \) and \( \beta \), and let \( N = \{0, 1, 2, \ldots\} \) be a set of non-negative integers. Then we have

(0) \( D(PN_0) \) is dense.

(1) If \( \lambda_p < g \), then \( \Omega = \{ \lambda_p + ng | n \in N \} \).

(2) If \( \lambda_p = sg, s \geq 0, s \in N \), then \( \Omega = \{ng | n \geq 1, n \in N \} \).

(3) If \( \lambda_p = sg + t_p, s \geq 0, s \in N, 0 < t_p < g \), then \( \Omega = \{t_p + ng | n \geq 0, n \in N \} \).

Proposition 3.1. If \( \lambda_p > gcd(\alpha, \beta) \), then \( D(PN_0) \) is not circular.

Proposition 3.2. \( D(PN_0) \) is circular if and only if \( \lambda_p \leq gcd(\alpha, \beta) \).

Let \( PN_1 = (\{p, q\}, \{a, b\}, F, W, \mu_0) \) be a Petri net such that \( W(a, p) = \alpha, W(p, b) = \alpha', W(q, a) = \beta, W(b, q) = \beta' \), \( \mu_0(p) = \lambda_p, \mu_0(q) = \lambda_q \).

Suppose that \( D(PN_1) \neq \emptyset \) and \( w \in D(PN_1) \). Let \( n = |w|_a \) and \( m = |w|_b \), then \( \Delta(w) = n\Delta(a) + m\Delta(b) = 0 \) (zero vector). Consequently, the linear equation

\[
\begin{pmatrix}
\alpha & -\alpha' \\
-\beta & \beta'
\end{pmatrix}
\begin{pmatrix}
n \\
m
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

has a non-trivial solution in \( N \). Thus \( \alpha\beta' = \alpha'\beta \). Therefore, if \( D(PN_1) \neq \emptyset \), then \( PN_1 = (\{p, q\}, \{a, b\}, F, W, \mu_0) \) has the following forms:

\( W(a, p) = \alpha, W(p, b) = k\alpha, W(q, a) = \beta, W(b, q) = k\beta \), \( k > 0 \).

Here we assume that \( k \) is an integer. That is, we define a Petri net \( PN_1 = (\{p, q\}, \{a, b\}, F, W, \mu_0) \) as follows

\[
\Delta(a) = \begin{pmatrix}
\alpha \\
-\beta
\end{pmatrix}, \quad \Delta(b) = \begin{pmatrix}
-k\alpha \\
k\beta
\end{pmatrix},
\]

where \( k \) is a positive integer.

We define an integer \( M_p \) as follows

\[
M_p = \begin{cases}
\frac{\lambda_p}{\alpha} - 1, & \text{if } \frac{\lambda_p}{\alpha} \text{ is an integer}, \\
\lfloor \frac{\lambda_p}{\alpha} \rfloor, & \text{if } \frac{\lambda_p}{\alpha} \text{ is not an integer}.
\end{cases}
\]
where $[ ]$ is the symbol of Gauss. Similarly we define an integer $M_q$ as follows, $M_q = \frac{q}{\beta} - 1$ if $\frac{q}{\beta}$ is an integer, and $M_q = [\frac{q}{\beta}]$ if $\frac{q}{\beta}$ is not an integer.

**Proposition 3.3.** We have

1. If $M_p + M_q > k,$ $M_p \geq k$ and $M_q \geq 1,$ then $D(PN_1)$ is not circular.
2. If $M_p + M_q > k,$ $M_p > 1, M_q > 1,$ then $D(PN_1)$ is not circular.
3. If $M_p + M_q = k,$ $M_p \geq 1,$ then $D(PN_1)$ is a singleton.
4. If $M_p + M_q \geq k,$ $M_p = 0,$ $M_q \geq k,$ then $D(PN_1)$ is $(1,1)$-limited.
5. If $M_p + M_q \geq k,$ $M_p \geq k,$ $M_q = 0,$ then $D(PN_1)$ is $(1,1)$-limited.

**Corollary 3.1.** Let $n$ and $k$ be arbitrary integers such that $n > k > 1$. Define the automaton $A_{(n,k)} = (\{1, 2, \ldots, n\}, \{a, b\}, f, 1, \{1\})$ by $f(i, a) = i + 1, 1 \leq i \leq n - 1,$ $f(j, b) = j - k,$ $k + 1 \leq j \leq n.$ Then the base of language $L(A_{(n,k)})$ recognized by $A_{(n,k)}$ is a $(1,1)$-limited code.

**Proposition 3.4.** Let $PN = (\{p_1, \ldots, p_n\}, \{a_1, \ldots, a_n\}, F, W, \mu_0), n \geq 2,$ be a Petri net such that $W(p_i, a_i) = \alpha_i,$ $W(a_i, p_{i+1}) = \beta_i, 1 \leq i \leq n - 1,$ and $W(p_n, a_n) = \alpha_n,$ $W(a_n, p_1) = \beta_n.$ $\mu_0 = (\lambda_1, \ldots, \lambda_n),$ $\mu_0(p_i) = \lambda_i, 1 \leq i \leq n.$ Furthermore let $g_j = gcd(\beta_{j-1}, \alpha_j), 2 \leq j \leq n.$ If $\lambda_1/\alpha_1 > 1$ and $\lambda_i \leq g_i$ for all $i = 2, \ldots, n,$ then $D(PN)$ is $(1,1)$-limited.

Let $PN_2 = (\{p_1, p_2\}, \{a, b, c\}, F, W, \mu_0)$ be a Petri net such that $W(a, p_1) = \alpha_1,$ $W(p_1, b) = \alpha_2,$ $W(b, p_2) = \beta_1,$ $W(p_1, c) = \alpha_3,$ $W(p_2, c) = \beta_2.$ $\mu_0(p_1) = \lambda_1,$ $\mu_0(p_2) = \lambda_2.$

**Lemma 3.1.** Let $PN_2$ be a Petri net mentioned above, and let $\alpha = gcd(\alpha_1, \alpha_2, \alpha_3), \beta = gcd(\beta_1, \beta_2).$ Suppose that $D(PN_2) \neq \emptyset$ and $\lambda_1 \leq \alpha, \lambda_2 \leq \beta.$ If $d \in D(PN_2)$ and $v$ is its proper suffix, then we have one of the following:

1. $\Delta(v)(p_1) \leq -\alpha,$ $\Delta(v)(p_2) \leq -\beta.$
2. $\Delta(v)(p_1) = 0,$ $\Delta(v)(p_2) \leq -\beta.$
3. $\Delta(v)(p_1) \leq -\alpha,$ $\Delta(v)(p_2) \leq 0.$

**Proposition 3.5.** If $D(PN_2) \neq \emptyset$ and $\lambda_1 \leq \alpha, \lambda_2 \leq \beta,$ then $D(PN_2)$ is $(1,1)$-limited.
\[ \alpha + \beta, W(b, q) = \alpha + \beta, W(c, p) = \beta, W(q, c) = \alpha, \mu_0(p) = \lambda_p, \mu_0(q) = \lambda_q. \]

**Lemma 3.2.** Let \( PN_3 \) be a Petri net mentioned above. Suppose that \( \beta < \lambda_p \leq \alpha + \beta \) and \( \beta < \lambda_q \leq \alpha \), then for any \( u \in PSeq(PN_3) \) we have one of the following.

1. \[ \Delta(u) = \begin{pmatrix} k(\alpha - \beta) \\ k(\alpha - \beta) \end{pmatrix}, k \geq 0, \]
2. \[ \Delta(u) = \begin{pmatrix} k(\alpha - \beta) + l\alpha \\ k(\alpha - \beta) - l\beta \end{pmatrix}, k \geq 0, l \geq 1, \]
3. \[ \Delta(u) = \begin{pmatrix} k(\alpha - \beta) - l\beta \\ k(\alpha - \beta) + l\alpha \end{pmatrix}, k \geq 0, l \geq 1. \]

**Proposition 3.6.** Suppose that \( D(PN_3) \neq \emptyset \). If \( \beta < \lambda_p \leq \alpha + \beta \) and \( \beta < \lambda_q \leq \alpha \), then \( D(PN_3) \) is \((1,1)\)-limited.

**References**


