Some Remarks on Generalized Inverse $*$-Semigroups

Teruo Imaoka (Takashi Iigai)
Department of Mathematics, Shimane University
Matsue, Shimane 690-8504, Japan

1 Preliminaries

A semigroup $S$ with a unary operation $*: S \to S$ is called a regular $*$-semigroup if it satisfies the conditions:

1. $(a^*)^* = a$,
2. $(ab)^* = b^*a^*$
3. $aa^*a = a$

for any $a, b \in S$.

Let $S$ be a regular $*$-semigroup. An idempotent $e$ in $S$ is called a projection if $e^* = e$. For a subset $A$ of $S$, denote the set of projections of $A$ by $P(A)$.

A regular $*$-semigroup $S$ is called a generalized inverse $*$-semigroup if $E(S)$, the set of idempotents of $S$, satisfies the identity:

$$x_1x_2x_3x_4 = x_1x_3x_2x_4$$

(1.1)

Such a semigroup is orthodox in the usual sense that $E(S)E(S) \subseteq E(S)$.

Result 1.1 ([12]) A regular $*$-semigroup $S$ is a generalized inverse $*$-semigroup if, and only if, $P(S)$ satisfies the identity (1.1).

Let $S$ be a regular $*$-semigroup. For $a, b \in S$, define a relation $\leq$ on $S$ by

$$a \leq b \iff a = eb = bf \text{ for some } e, f \in P(S).$$

Result 1.2 ([5]) Let $a$ and $b$ be elements of a regular $*$-semigroup $S$. Then the following are equivalent:

(i) $a \leq b$.
(ii) $aa^* = ba^*$ and $a^*a = b^*a$.
(iii) $aa^* = ab^*$ and $a^*a = a^*b$.
(iv) $a = aa^*b = ba^*a$.

Result 1.3 ([4]) Let $S$ be a regular $*$-semigroup. Then

(i) $E(S) = P(S)^2$. In fact, for any $e \in E(S)$, there exist $f, g \in P(S)$ such that $fR eL g$ and $e = fg$.
(ii) For any $a \in S$ and $e \in P(S)$, $a^*ea \in P(S)$.
(iii) For $e, f \in P(S)$, $ef \in P(S)$ if and only if $ef = fe$.
(iv) Each $L$-class and each $R$-class in $S$ contains one and only one projection.

Result 1.4 ([8]) Let $S$ be an orthodox semigroup. Then

$$\sigma = \{(a, b) \in S \times S : eae = ebe \text{ for some } e \in E(S)\}$$

is the minimum group congruence on $S$. 
2 E-unitary generalized inverse *-semigroups

PG*-semigroups

Let \((G, X, Y)\) be a McAlister triple, and let \(\{P_\alpha : \alpha \in Y\}\) be a family of disjoint non-empty sets indexed by the elements of \(Y\). Put \(P = \bigcup_{\alpha \in Y} P_\alpha\). For each pair \(\alpha, \beta\) of elements of \(Y\) where \(\alpha \geq \beta\), let \(\rho_{\alpha, \beta} : P_\alpha \to P_\beta\) be a mapping such that the following two axioms hold:

(PG*1) \(\rho_{\alpha, \alpha}\) is the identity mapping on \(P_\alpha\).

(PG*2) If \(\alpha \geq \beta \geq \gamma\) then \(\rho_{\alpha, \beta} \rho_{\beta, \gamma} = \rho_{\alpha, \gamma}\).

We call such a quintet \((G, X, Y, P, \{\rho_{\alpha, \beta}\})\) a PG*-quintet.

Proposition 2.1 Let \((G, X, Y, P, \{\rho_{\alpha, \beta}\})\) be a PG*-quintet. Then

\[ S = \{ (\alpha, g, x_1, x_2) \in Y \times G \times P \times P : g^{-1}\alpha \in Y, x_1 \in P_\alpha, x_2 \in P_{g^{-1}\alpha}\} \]

with multiplication and a unary operation given by

\[ (\alpha, g, x_1, x_2)(\beta, h, y_1, y_2) = (\alpha \land g \beta, gh, x_1 \rho_{\alpha, \alpha \land g \beta} y_2 \rho_{h^{-1} \beta, \beta} y_1 \rho_{h^{-1} \beta, g}^{-1} (\alpha \land g \beta)) \]

\[ (\alpha, g, x_1, x_2)^* = (g^{-1} \alpha, g^{-1}, x_2, x_1) \]

is an E-unitary generalized inverse *-semigroup.

We say that \(S\) is a PG*-semigroup and denoted by \(PG^*(G, X, Y, P, \{\rho_{\alpha, \beta}\})\), or simply by \(PG^*(G, X, Y, P)\).

We now characterise the Green's relations \(\mathcal{L}, \mathcal{R}\), the minimum group congruence \(\sigma\) and the natural order \(\leq\) on \(PG^*(G, X, Y, P, \{\rho_{\alpha, \beta}\})\).

Proposition 2.2 Let \((\alpha, g, x_1, x_2), (\beta, h, y_1, y_2)\) be elements of \(S = PG^*(G, X, Y, P, \{\rho_{\alpha, \beta}\})\).

(i) \((\alpha, g, x_1, x_2) \leq (\beta, h, y_1, y_2)\) if, and only if, \(\alpha \leq \beta, g = h, y_1 \rho_{\alpha, \alpha} x_1 = x_2, y_2 \rho_{h^{-1} \beta, \beta}^{-1} = x_2\).

(ii) \((\alpha, g, x_1, x_2) \sigma (\beta, h, y_1, y_2)\) if, and only if, \(g = h\).

(iii) \((\alpha, g, x_1, x_2) \mathcal{L} (\beta, h, y_1, y_2)\) if, and only if, \(g^{-1} \alpha = h^{-1} \beta\) and \(x_2 = y_2\).

(iv) \((\alpha, g, x_1, x_2) \mathcal{R} (\beta, h, y_1, y_2)\) if, and only if, \(\alpha = \beta\) and \(x_1 = y_1\).

Now we have the following theorem.

Theorem 2.3 The semigroup \(PG^*(G, X, Y, P, \{\rho_{\alpha, \beta}\})\) is an E-unitary generalized inverse *-semigroup and maximum group homomorphic image isomorphic to \(G\).

Construction of E-unitary generalized inverse *-semigroups

Let \(S\) be an E-unitary generalized inverse *-semigroup. Put \(G = S/\sigma\), and denoted its identity by \(1\). Since \(E(S)\) is a minimum group congruence class of \(S\), \(E(S)\) is the identity of \(G\). Let \(E(S) \sim \sum \{E_\alpha : \alpha \in Y\}\) be the structure decomposition of \(E(S)\), that is \(E(S)\) is a semilattice \(Y\) of rectangular bands \(E_\alpha (\alpha \in Y)\). Put \(\mathcal{E} = \{E_\alpha : \alpha \in Y\}\). We shall construct \(PG^*-\)quintet.

First, we define a relation \(\rho\) on \(\mathcal{E} \times G\) by

\[ (E_\alpha, g)(E_\beta, h) \iff xx^* \in E_\alpha \text{ and } x^*x \in E_\beta \text{ for some } x \in g^{-1}h. \]

Lemma 2.4 The relation \(\rho\) is an equivalence relation on \(\mathcal{E} \times G\).

We shall write \(\mathcal{E}^*\) for \((\mathcal{E} \times G)/\rho\), and denote the \(\rho\)-class of \(\mathcal{E} \times G\) which contains \((E_\alpha, g)\) by \((E_\alpha, g)\rho\). The following lemmas are immediate.
Lemma 2.5 For any element $x \in S$, $xx^* \in E_{\alpha}, x^*x \in E_{\beta}$ for some $\alpha, \beta \in Y$. Then

$$(E_{\alpha}, 1)\rho (E_{\beta}, x\sigma) \text{ and } (E_{\beta}, 1)\rho (E_{\alpha}, (x\sigma)^{-1}).$$

Lemma 2.6 Let $\alpha, \beta \in Y$ and $g \in G$ such that $(E_{\alpha}, g)\rho (E_{\beta}, g)$. Then $\alpha = \beta$.

Proposition 2.7 Let $E_{\alpha}, E_{\beta}, E_{\gamma} \in \mathcal{E}$ and $g, h \in G$. If $\alpha \leq \beta$ and $(E_{\beta}, g)\rho (E_{\gamma}, h)$, then there exists $\delta \in Y$ such that $\delta \leq \gamma, (E_{\alpha}, g)\rho (E_{\delta}, h)$.

We define a relation $\leq$ on $\mathfrak{X}$ as follows:

$$A \leq B \iff \alpha \leq \beta, (E_{\alpha}, g) \in A, (E_{\beta}, g) \in B$$

for some $\alpha, \beta \in Y$ and $g \in G$. The proof of the following is straightforward from Proposition 2.7 and the definition of $\leq$.

Corollary 2.8 Let $A \leq B$, where $A, B \in \mathfrak{X}$. If $(E_{\gamma}, h) \in B$, then there exists $\delta \in Y$ such that $\delta \leq \gamma$ and $(E_{\delta}, h) \in A$.

Lemma 2.9 The relation $\leq$ is a partial order on $\mathfrak{X}$.

Let

$$\mathfrak{Y} = \{(E_{\alpha}, 1)\rho : \alpha \in Y\}.$$

We define an action of $G$ on $\mathfrak{Y}$ by order automorphisms. Suppose first that $(E_{\alpha}, g)\rho (E_{\beta}, h)$. This means that there exists $x \in g^{-1}h$ such that $xx^* \in E_{\alpha}, x^*x \in E_{\beta}$. Let $k \in G$. Then $x \in (kg)^{-1}(kh)$ and so $(E_{\alpha}, kg)\rho = (E_{\beta}, kh)\rho$. We can therefore define $\circ : G \times \mathfrak{X} \to \mathfrak{X}$ by

$$k \circ (E_{\alpha}, g)\rho = (E_{\beta}, kg)\rho.$$

We shall show that the triple $(G, \mathfrak{X}, \mathfrak{Y})$ form a McAlister triple.

Lemma 2.10 The mapping $\varphi : Y \to \mathfrak{Y}$ defined by $\alpha \varphi = (E_{\alpha}, 1)\rho$ is an order isomorphism.

Lemma 2.11 The mapping $\circ$ is an action of $G$ on $\mathfrak{X}$, on the left by order automorphisms.

Lemma 2.12 With the above notation:

(i) $\mathfrak{Y}$ is an order ideal of $\mathfrak{X}$.

(ii) $G \circ \mathfrak{Y} = \mathfrak{X}$.

(iii) $g \circ \mathfrak{Y} \cap \mathfrak{Y} \neq \emptyset$ for all $g \in G$.

By the lemma above, we have that the triple $(G, \mathfrak{X}, \mathfrak{Y})$ is a McAlister triple. We shall construct $PG^*$-quintet by making use of McAlister triple $(G, \mathfrak{X}, \mathfrak{Y})$ and form the $PG^*$-semigroup $PG^*(G, \mathfrak{X}, \mathfrak{Y}, P)$. Put $P_{\alpha} = P(E_{\alpha})$ for each $\alpha \in Y$ and let $P = \bigcup_{\alpha \in Y} P_{\alpha}$. For each pair $\alpha, \beta$ of elements of $Y$ where $\alpha \geq \beta$, define the mapping

$$\rho_{\alpha, \beta} : P_{\alpha} \to P_{\beta} \text{ by } e\rho_{\alpha, \beta} = efe \text{ where } f \in P_{\beta}.$$

Lemma 2.13 With the definition above, $\rho_{\alpha, \beta}$ is a mapping satifying the conditions (PG$^*$1) and (PG$^*$2).

Thus $PG^*(G, \mathfrak{X}, \mathfrak{Y}, P, \{\rho_{\alpha, \beta}\})$, constructed above, forms a $PG^*$-semigroup.
Lemma 2.14 For any $xx^* \in P_\alpha$ and $e \in P_\beta$, $xx^* \in P_{\alpha \wedge (\lambda \sigma)\beta}$.

Lemma 2.15 The mapping $\theta : S \to PG^*(G, \mathcal{A}, \mathcal{Y}, P, \{\rho_\alpha, \beta\})$ defined by

$$x\theta = ((E_\alpha, 1)\rho, xx^*, x^*x),$$

where $xx^* \in E_\alpha$, is a $*$-isomorphism.

Now we have the structure of generalized inverse $*$-semigroups.

Proposition 2.16 A generalized inverse $*$-semigroup is $E$-unitary if, and only if, it is $*$-isomorphic to some $PG^*$-semigroup.

3 The compatibility relations

Let $S$ be a regular $*$-semigroup. For all $s, t \in S$, the left compatibility relation is defined by

$$s \sim_l t \iff st^* \in E(S),$$

the right compatibility relation is defined by

$$s \sim_r t \iff s^*t \in E(S),$$

and the compatibility relation, the intersection of the above two relations, is defined by

$$s \sim t \iff st^*, s^*t \in E(S).$$

It is clear that all three relations are reflexive and symmetric, but none of them need be transitive (see Theorem 3.2 for a characterisation of the generalized inverse $*$-semigroups having a transitive compatibility relation). The next lemma describe some of the basic property of these relations.

Lemma 3.1 Let $S$ be a generalized inverse $*$-semigroup and $\rho$ be any one of the three relations $\sim_l, \sim_r$, and $\sim$. Then the following two properties hold.

(i) $s \rho t$ and $u \rho v$ imply that $su \rho tv$.

(ii) $s \leq t, u \leq v$ and $t \rho v$ imply that $s \rho u$.

Theorem 3.2 Let $S$ be a generalized inverse $*$-semigroup. Then the compatibility relation is transitive if, and only if, $S$ is $E$-unitary.

Proof Suppose that $\sim$ is transitive. Let $es \in E(S)$, where $e$ is an idempotent. Then $s \sim es$ since elements $s(es)^*$ and $s^*es$ are idempotents. Clearly $es \sim s^*s$, and so, by our assumption that the compatibility relation is transitive, we have that $s \sim s^*s$. But $s(s^*s)^* = s$, so that $s$ is an idempotent.

Conversely, suppose that $S$ is $E$-unitary and $s \sim t$ and $t \sim u$. Clearly $(s^*t)(t^*u)$ is an idempotent and

$$(s^*t)(t^*u) = s^*u(t^*u)^*(t^*u)$$

But $S$ is $E$-unitary and so $s^*u$ is an idempotent. Similarly, $su^*$ is an idempotent. Hence $s \sim u$.

Proposition 3.3 Let $S$ be a regular $*$-semigroup. Then the following are equivalent:

(i) The left and right compatibility relations are equal.

(ii) For all $s, t \in S$, we have that $st \in E(S)$ if, and only if, $ts \in E(S)$. 


A congruence $\rho$ on an orthodox semigroup $S$ is said to be idempotent pure if $a \in S, e \in E(S)$ and $(a, e) \in \rho$ then $a$ is an idempotent.

**Proposition 3.4** Let $S$ be an $E$-unitary regular $\ast$-semigroup. Then a congruence $\rho$ is idempotent pure if, and only if, $\rho \subseteq \sim$.

**Proof** Let $\rho$ be idempotent pure and let $(a, b) \in \rho$. Then $(ab^*, bb^*) \in \rho$. But $\rho$ is idempotent pure and $bb^*$ is an idempotent. Thus $ab^*$ is an idempotent. Similarly, $a^*b$ is an idempotent. Thus $a \sim b$.

Conversely, let $\rho$ be a congruence contained in the compatibility relation. Let $(a, e) \in \rho$, where $e$ is an idempotent. Then $a \sim e$. Thus $ae^* \in E(S)$. But $e^*$ is an idempotent and so $a$ is an idempotent, since $S$ is $E$-unitary. 

## 4 Enlargements

We proved in Section 2, that $E$-unitary generalized inverse $\ast$-semigroups are essentially isomorphic to the generalized inverse $\ast$-subsemigroups of $PG^\ast$-semigroups. The point is that if $X$ is a meet semilattice, we can form the semigroup $PG^\ast(G, X, Y, P, \{\rho_{\alpha,\beta}\})$, which contains $PG^\ast(G, X, Y, P, \{\rho_{\alpha,\beta}\})$ as a generalized inverse $\ast$-subsemigroup. In the following proposition, we shall describe the abstract relationship between $PG^\ast(G, X, Y, P, \{\rho_{\alpha,\beta}\})$ and $PG^\ast(G, X, Y, P, \{\rho_{\alpha,\beta}\})$.

**Proposition 4.1** Let $(G, X, Y, P, \{\rho_{\alpha,\beta}\})$ be a $PG^\ast$-quintet. where $X$ is a meet semilattice.

(i) The idempotents of $PG^\ast(G, X, Y, P)$ form an order ideal of $PG^\ast(G, X, X, P)$.

(ii) If $(\alpha, g, x_1, x_2) \in PG^\ast(G, X, X, P)$ is such that

$$(\alpha, g, x_1, x_2)^\ast(\alpha, g, x_1, x_2)(\alpha, g, x_1, x_2)^\ast \in PG^\ast(G, X, Y, P)$$

then $(\alpha, g, x_1, x_2) \in PG^\ast(G, X, Y, P)$.

(iii) For each projection $(\alpha, 1, x, y) \in PG^\ast(G, X, X, P)$ there exists a projection $(\beta, 1, y, y) \in PG^\ast(G, X, Y, P)$ such that $(\alpha, 1, x, x) \mathcal{R} (\beta, 1, y, y)$.

On the basis of the above proposition, we make the following definition. Let $S$ be a generalized inverse $\ast$-subsemigroup of a generalized inverse $\ast$-semigroup $T$. We say that $T$ is an enlargement of $S$ if the following three axioms hold:

(E1) $E(S)$ is an order ideal of $E(T)$.

(E2) If $t \in T$ and $t^*t, tt^* \in S$ then $t \in S$.

(E3) For every projection $e \in T$ there exists a projection $f \in S$ such that $e \mathcal{R} f$.

The following is easy to prove.

**Lemma 4.2** Let $S$ be a generalized inverse $\ast$-subsemigroup of $T$. Then axiom (E1) holds if, and only if, $S$ is an order ideal of $T$.

We may find a $PG^\ast$-representation of an $E$-unitary generalized inverse $\ast$-semigroup.

**Theorem 4.3** Let $G$ be a group and $X$ a semilattice, and let $S$ be a generalized inverse $\ast$-subsemigroup of the generalized inverse $\ast$-semigroup $PG^\ast(G, X, X, P, \{\rho_{\alpha,\beta}\})$. Suppose that $PG^\ast(G, X, X, P, \{\rho_{\alpha,\beta}\})$ is an enlargement of $S$. Let

$$Y = \{\alpha \in X : (\alpha, 1, x, y) \in E(S)\} \text{ and } Q = \{x \in P : (\alpha, 1, x, y) \in E(S)\}.$$

Then $(G, X, Y, Q, \{\rho_{\alpha,\beta}\})$ is a $PG^\ast$-quintet and $S = PG^\ast(G, X, Y, Q, \{\rho_{\alpha,\beta}\})$. 

5 A Structure Theorem

We can now prove the uniqueness of the $PG^*$-representation of an $E$-unitary generalized inverse $*$-semigroup.

**Theorem 5.1** Let $(G, X, Y, P, \{\rho_{\alpha, \beta}\})$ and $(G', X', Y', P', \{\rho'_{\alpha', \beta'}\})$ be two $PG^*$-quintets. Let $\theta : G \to G'$ be a group isomorphism and let $\psi : X \to X'$ be an order isomorphism such that $\psi|_Y$ is an isomorphism from the semilattice $Y$ onto $Y'$; now let $\xi : P \to P'$ be a bijection. Suppose also that, for all $g$ in $G$, $a$ in $X$ and $x$ in $P_{\beta}$,

$$(ga)\psi = (g\theta)(a\psi),$$

$$(x\rho_{\beta, \gamma})\xi = (x\xi)\rho_{\beta, \gamma}\psi,$$

where $\beta, \gamma \in Y$ such that $\beta \geq \gamma$. Then the mapping $\phi : PG^*(G, X, Y, P) \to PG^*(G', X', Y', P')$ defined by

$$(\alpha, g, x, y)\phi = (\alpha\psi, g\theta, x\xi, y\xi)$$

is a $*$-isomorphism. Conversely, every $*$-isomorphism from $PG^*(G, X, Y, P)$ onto $PG^*(G', X', Y', P')$ is of this type.

6 The minimum group congruence

In this subsection, we shall first give an alternative characterization of the minimum group congruence on a generalized inverse $*$-semigroup.

**Theorem 6.1** If $S$ is a generalized inverse $*$-semigroup, then the relation

$$\sigma = \{(a, b) \in S \times S : ea = eb \text{ for some } e, f \in P(S)\}$$

is the minimum group congruence on $S$.

Idempotent pure congruences, the minimum group congruence and $E$-unitary generalized inverse $*$-semigroups are all linked by the following result.

**Theorem 6.2** Let $S$ be a generalized inverse $*$-semigroup. Then the following conditions are equivalent:

(i) $S$ is $E$-unitary.

(ii) $\sim = \sigma$.

(iii) $\sigma$ is idempotent pure.

(iv) $\sigma(e) = E(S)$ for any idempotent $e$.

**Proof**

(i) $\Leftrightarrow$ (iv). Immediate.

(i) $\Rightarrow$ (ii). Let $a \sim b$. Then $ab^*, a^*b \in E(S)$. Thus

$$(ab^*)(ab^*)^*a(a^*b)(a^*b)^* = ab^*ba^*bb^*a$$

$$= ab^*ba^*bb^*a$$

$$= ab^*(ba^*)(bb^*)a$$

$$= (ab^*)(ab^*)^*b(a^*b)(a^*b)^*.$$

Hence $a \sigma b$. 
Conversely, suppose $a \sigma b$. Then $eaf = ebf$ for some $e, f \in P(S)$ by Theorem 6.1. Thus we have

$$(eaf)(ebf)^* = eafb^*bb^*e = (eab^*)bfb^*e \in E(S).$$

But $bfb^*e$ is an idempotent. Thus, by (i), $eab^* \in E(S)$. By using (i) again, we obtain $ab^* \in E(S)$ since $e \in E(S)$. Similarly, $a^*b$ is an idempotent.

(ii) $\Rightarrow$ (iii). Let $(a, e) \in \sigma$, where $e$ is an idempotent. Clearly, $e \sim a^*a$. But $\sim = \sigma$ and so $a \sim a^*a$. Hence $a$ is an idempotent.

(iii) $\Rightarrow$ (i). Let $a \in S$ and $e \in E(S)$ such that $ea \in E(S)$. Then $ea = e(eae)e$. Thus, by Result 14. $(a, eae) \in \sigma$. But $eae = (ea)e \in E(S)$ and so $a$ is an idempotent since $\sigma$ is idempotent pure.

### References


