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RELATIVE E-RINGS

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1. Introduction

In [4, Problem 45], L. Fuchs posed the following problem:

Which rings R satisfy $R \cong End(R^+)$? The author presents

In 1973, P. Schultz [5] gave a partial solution to this problem. In particular, he studied commutative rings R satisfying $R \cong End(R^+)$ and he called such rings E-rings. For E-rings, see the book"Additive Groups of Rings I ([3])" by S. Feigelstock.. Recently, in [2] R. Göbel, S. Shelah and L. Strñgmann constructed noncommutative rings R satisfying $R \cong End(R^+)$.

2. Relative E-rings

Let R be a ring with identity. By R^+ we denote the additive group of the ring R. For an element $a \in R$, we have the mapping $a_l : R \to R$ defined by $x \to ax$. a_l is called the left multiplication induced by a. Similarly we have the right multiplication induced by a. Obviously the sets $\{a_l \mid a \in R\}$ and $\{a_r \mid a \in R\}$ form rings. We denote thses rings by R_l and R_r, respectively.

Definition 2.1. A ring R is called an E-ring if $R_l = End(R^+)$.

The detailed version of this paper has been submitted for publication elsewhere.

This notion is generalized as follows.

Definition 2.2. Let S be a ring and let R be a ring such that R is a right S-module. A ring R is called a left E-ring relative to S if $R_l = End_S(R_S)$.

Let $\mathbf{Z}$ denote the ring of rational integers. Then a left E-ring relative $\mathbf{Z}$ is nothing else but an E-ring. Let S be a ring and let R be a ring such that R is a right S-module. Then $End_S(R_S)$ always contains R_l. Hence we can say that left E-rings relative to S are those rings R such that $End_S(R_S)$ is small as possible.

From the definition of a relative E-ring, the following is obvious.

Proposition 2.3. *Let S be a ring and let R be a ring such that R is a right S-module. If R is a left E-ring relative to S and if $f \in End(R_S)$, then $f(R)$ is a principal right ideal of R.*

Also we can easily see the following:

Proposition 2.4. *Let S be a ring and let R be a ring such that R is a right S-module.*

(1) The ring R is a left E-ring relative to S.

(2) Every element of R_r commutes with any element of $End_S(R_S)$.

As a corollary, we have the following charactrizations of an E-algebra relative to a commutative ring.

Corollary 2.5. *Let S be a commutative ring and R be an S-algebra. Then the following are equivalent:*

(1) R is an E-ring relative to S.

(2) $R_r = End_S(R_S)$.

(3) R is a commutative ring and $R \cong End_S(R_S)$.

(4) $End_S(R_S)$ is a commutative ring.

Example 2.6. Let R be a commutative ring and let $S = R[x, y]$. Consider the ring $A = S/(x) \oplus S/(y)$. Then A is a S-algebra, but A is not a cyclic S-module. Clearly $End_S(A) \cong A$ and so $End_S(A)$ is commutative. Therefore A is a relative E-algebra over S.

Example 2.7. Let R be a commutative ring and let S be a multiplicatively closed subset of R. Then $S^{-1}R$ is a relative E-algebra over S.

Let R be a commutative ring and let $\{I_n\}_{n\geq 0}$ be a fimily of ideals of R satisfying the condition that $I_n \subseteq I_m$ whenever $n \geq m$. We can then define a topology on the set R with an open basis $\{a + I_n \mid a \in R,\ n \geq 0\}$. This topology is called the linear topology defined by a family of ideals $\{I_n\}_{n\geq 0}$. Then we can construct the completion $\hat{R}$ of R. It is well-known that

$$\hat{R} \cong \lim_{\leftarrow n} R/I_n.$$

Example 2.8. Let R be a commutative ring and consider the linear topology defined by a family of ideals $\{I_n\}_{n\geq 0}$. Then the completion $\hat{R}$ of R is a relative E-algebra over R.

References

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