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ON QUANTUM TELEPORTATION

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ABSTRACT. Mathematical foundation of quantum teleportation is discussed based on some works of the present author.

1. INTRODUCTION

In quantum communication theory, one looks for the most efficient way to code information and construct a physical device (channel) in order to send information as completely as possible. There "quantum" means that we code information by quantum states and send it through quantum device properly designed. If one can send any quantum state from an input system to an output system as it is, then it will be an ultimate way of information transmission. Such an ultimate method is not only ultimate for information transmission but also considered to enable sending matter existing in real world to other place without destroying itself which was a dream in science fiction, although it needs an assumption that quantum mechanics can describe all aspects of existence in our world. It is in quantum teleportation that we can discuss such an ultimate communication.

More precisely, quantum teleportation is to send a quantum state itself containing all information of a certain system from one place to another. The problem of quantum teleportation is whether there exist a physical device and a key (or a set of keys) by which a quantum state attached to a sender (Alice) is completely transmitted and a receiver (Bob) can reconstruct the state sent. Bennett and others [3] showed such a teleportation is possible through a device (channel) made from proper (EPR) entangled states of Bell basis. The basic idea behind their discussion is to divide the information encoded in the state into two parts, classical and quantum, and send them through different channels, a classical channel and an EPR channel. The classical channel is nothing but a simple coorespondecе between sender and receiver, and the EPR channel is constructed by using a certain entangled state. However the EPR channel is not so stable due to decoherence. Fichtner and Ohya [5, 6] studied the quantum teleprotation by means of general beam splitting processes so that it contains the EPR channel, and they constructed a more stable teleportation process with coherent entangled states.

In Section 2 of this paper, we discuss the channel expression of quantum teleportation. In Section 3, we explain original treatment of quantum teleportation done by Bennett et al within the channel expression. In Section 4, the weak type of quantum teleportation and the uniquness of the set of keys given to Bob are briefly discussed based on the paper [1]. In Sections 5 and 6, we discuss a general treatment of quantum teleportation process in Boson Fock space with basical techniques of beam splitting. In Section 7, a new scheme of quantum teleportation [8] is explained.

2. Channel Expression of Quantum Teleportation

Two dynamical systems, input and output, should be set for information transmission. Since every system can be described by a state, it is important to know the relation between input state and output state for the study of information transmission. Such a relation is described by a channel bridging between two systems, namely, providing the state change in the course of information transmission.

In the classical communication theory a state of input or output system is described by a probability distribution (or measure), so that a channel causes a change of this probability distribution. On the other hand, in quantum communication theory, a state of input or output system should be described by a certain noncommutative state (quantum state) such as a density operator or a positive normalized linear functional, more generally. Here we restrict our discussion to the former case, the case of density operators.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the separable complex Hilbert spaces describing an input system and an output system, respectively. Let $B(\mathcal{H}_k)$ $(k = 1, 2)$ be the set of all bounded linear operators on $\mathcal{H}_k$. Then the set $\mathcal{S}(\mathcal{H}_k)$ of all states (density operators) on the Hilbert spaces $\mathcal{H}_k$ is

$$\mathcal{S}(\mathcal{H}_k) = \{\rho \in B(\mathcal{H}_k)\ ;\ \rho^* = \rho,\ \rho \geq 0,\ \mathrm{tr}\rho = 1\}.$$

A quantum channel sends an input quantum state to an output quantum state, that is, a mapping from $\mathcal{S}(\mathcal{H}_1)$ to $\mathcal{S}(\mathcal{H}_2)$.

The quantum teleportation can be expressed in the following steps:

- *STEP0* : Alice has an unknown quantum state $\rho^{(1)}$ (on Hilbert space $\mathcal{H}_1$) and she is asked to send (teleport) it to Bob.
- *STEP1* : For this purpose, we need two other Hilbert spaces $\mathcal{H}_2$ and $\mathcal{H}_3$. $\mathcal{H}_2$ is attached to Alice and $\mathcal{H}_3$ is to Bob. Prearrange a state $\sigma^{(23)} \in \mathcal{S}(\mathcal{H}_2 \otimes \mathcal{H}_3)$ having a certain correlation between Alice and Bob, which is called an entangled state between two systems $\mathcal{H}_2$ and $\mathcal{H}_3$.
- *STEP2* : Prepare the set of projections $\left\{F_k^{(12)}\right\}$ and an observable $F^{(12)} := \sum_k \lambda_k F_k^{(12)}$ on a tensor product Hilbert space (system) $\mathcal{H}_1 \otimes \mathcal{H}_2$. Alice performs the joint measurement of the observable $F^{(12)}$.
- *STEP3* : Bob obtained a state $\rho^{(3)}$ due to the reduction of wave packet and he is informed which outcome was obtained by Alice. The result of this measurement can be classically informed from Alice to Bob without diturbance (for instance by telephone).
- *STEP4* : $\rho^{(1)}$ is reconstructed from $\rho^{(3)}$ by using the key which corresponds to the outcome as Bob is informed from Alice in the above *STEP3*.

The above steps can be exhibited by a channel $\Lambda^{[k]} : \mathcal{S}(\mathcal{H}_1) \longrightarrow \mathcal{S}(\mathcal{H}_3)$ constructed by the following three maps (channels):

(1) $\gamma : \mathcal{S}(\mathcal{H}_1) \longrightarrow \mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3)$,

$$\gamma\left(\rho^{(1)}\right) = \rho^{(1)} \otimes \sigma^{(23)} \quad \forall \rho^{(1)} \in \mathcal{S}(\mathcal{H}_1).$$

This channel γ expresses a coupling of an initial state $\rho^{(1)}$ with the entangled state $\sigma^{(23)}$.

(2) $\pi_k : \mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3) \longrightarrow \mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3)$ is a state change describing the nonclassical effect of the teleportation determined by Alice's measurement.

Since $F_k^{(12)}$ is a projection on $\mathcal{H}_1 \otimes \mathcal{H}_2$, the map $\pi_k : \mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3) \longrightarrow \mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3)$ is given by :

$$\pi_k\left(\rho^{(123)}\right) = \frac{1}{L}\left(F_k^{(12)} \otimes I^{(3)}\right)\rho^{(123)}\left(F_k^{(12)} \otimes I^{(3)}\right), \ \forall\rho^{(123)} \in \mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3)$$

Here $L = \mathrm{tr}_{123}\left(F_k^{(12)} \otimes I^{(3)}\right)\rho^{(123)}\left(F_k^{(12)} \otimes I^{(3)}\right)$ (von Neumann-Luder's projection hypothesis)

(3) $a : \mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3) \longrightarrow \mathcal{S}(\mathcal{H}_3)$,

$$\rho^{(3)} = a\left(\rho^{(123)}\right) = \mathrm{tr}_{12}\rho^{(123)}, \quad \forall\rho^{(123)} \in \mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3).$$

This channel a represents a reduction from the state $\rho^{(123)}$ to Bob's state $\rho^{(3)}$ due to Alice's measurement, where tr_{12} is the partial trace with respect to $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Therefore we obtain the channel $\Lambda_k : \mathcal{S}(\mathcal{H}_1) \longrightarrow \mathcal{S}(\mathcal{H}_3)$

$$\Lambda_k = a \circ \pi_k \circ \gamma$$

or more concretely

$$\Lambda_k\rho^{(1)} = \mathrm{tr}_{12}\pi_k\left(\rho^{(1)} \otimes \sigma^{(23)}\right), \quad \forall\rho^{(1)} \in \mathcal{S}(\mathcal{H}_1).$$

where the subscript "k" means that the channels Λ_k depends on the choice of Alice's measurement $F_k^{(12)}$.

Thus, the whole teleportation channel $\Lambda_k : \mathcal{S}(\mathcal{H}_1) \longrightarrow \mathcal{S}(\mathcal{H}_3)$ is written as

$$\Lambda_k\rho^{(1)} \equiv \mathrm{tr}_{12}\left[\frac{\left(F_k^{(12)} \otimes I^{(3)}\right)\left(\rho^{(1)} \otimes \sigma^{(23)}\right)\left(F_k^{(12)} \otimes I^{(3)}\right)}{\mathrm{tr}_{123}\left(F_k^{(12)} \otimes I^{(3)}\right)\left(\rho^{(1)} \otimes \sigma^{(23)}\right)\left(F_k^{(12)} \otimes I^{(3)}\right)}\right], \quad \forall\rho^{(1)} \in \mathcal{S}(\mathcal{H}_1).$$

Note that the teleportation channel Λ_k is generally nonlinear. Then the problem of quantum teleportation is stated as follows:

Definition 1. *Quantum teleportation is realized if there exist the set of operators* $\left\{F_k^{(12)}\right\}$ *on* $\mathcal{H}_1 \otimes \mathcal{H}_2$ *, an entangled state* $\sigma^{(23)}$ *on* $\mathcal{H}_2 \otimes \mathcal{H}_3$ *and the set of keys* $\{U_k\}$ *such that* $\Lambda_k\rho^{(1)} = U_k\rho^{(1)}U_k^*$ *for any state* $\rho^{(1)}$ *in* $\mathcal{H}_1$ *and for each* k.

When such teleportation is realized, the state $\rho^{(3)}$ transfered to Bob from Alice is unitarily equivalent to the original state $\rho^{(1)}$ sent by Alice, so that all information stored in $\rho^{(1)}$ is completely transmitted to Bob for any k.

3. BBCJPW Model of Teleportation

We in this section notice that the BBCJPW (Bennett, Brassard, Crepeau, Jozsa, Peres and Wootters) model provides an example to the above described framework. In their model, $\sigma^{(23)}$ is given by the EPR spin pair in a singlet state such as $\sigma^{(23)} = |\psi\rangle\langle\psi|$, where $|\psi\rangle = \sqrt{\frac{1}{2}}|\uparrow^{(2)}\rangle \otimes |\downarrow^{(3)}\rangle - \sqrt{\frac{1}{2}}|\downarrow^{(2)}\rangle \otimes |\uparrow^{(3)}\rangle$ with the spin up vector $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the spin down vector $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The projection $F_k^{(12)}$ is one of the Bell CONS ;

$F_1^{(12)} = |\xi^{(-)}\rangle\langle\xi^{(-)}|$, $F_2^{(12)} = |\xi^{(+)}\rangle\langle\xi^{(+)}|$, $F_3^{(12)} = |\zeta^{(-)}\rangle\langle\zeta^{(-)}|$, $F_4^{(12)} = |\zeta^{(+)}\rangle\langle\zeta^{(+)}|$

with

$$|\xi^{(-)}\rangle = \sqrt{\frac{1}{2}}\left(|\uparrow^{(1)}\rangle\otimes|\downarrow^{(2)}\rangle - |\downarrow^{(1)}\rangle\otimes|\uparrow^{(2)}\rangle\right), |\xi^{(+)}\rangle = \sqrt{\frac{1}{2}}\left(|\uparrow^{(1)}\rangle\otimes|\downarrow^{(2)}\rangle + |\downarrow^{(1)}\rangle\otimes|\uparrow^{(2)}\rangle\right)$$
$$|\zeta^{(-)}\rangle = \sqrt{\frac{1}{2}}\left(|\uparrow^{(1)}\rangle\otimes|\uparrow^{(2)}\rangle - |\downarrow^{(1)}\rangle\otimes|\downarrow^{(2)}\rangle\right), |\zeta^{(+)}\rangle = \sqrt{\frac{1}{2}}\left(|\uparrow^{(1)}\rangle\otimes|\uparrow^{(2)}\rangle + |\downarrow^{(1)}\rangle\otimes|\downarrow^{(2)}\rangle\right).$$

The unitary (key) operators U_k ($k = 1,2,3,4$) are given as

$$U_1 \equiv \left|\uparrow^{(1)}\right\rangle\left\langle\uparrow^{(3)}\right| + \left|\downarrow^{(1)}\right\rangle\left\langle\downarrow^{(3)}\right|, U_2 \equiv \left|\uparrow^{(1)}\right\rangle\left\langle\uparrow^{(3)}\right| - \left|\downarrow^{(1)}\right\rangle\left\langle\downarrow^{(3)}\right|$$
$$U_3 \equiv \left|\uparrow^{(1)}\right\rangle\left\langle\downarrow^{(3)}\right| + \left|\downarrow^{(1)}\right\rangle\left\langle\uparrow^{(3)}\right|, U_4 \equiv \left|\uparrow^{(1)}\right\rangle\left\langle\downarrow^{(3)}\right| - \left|\downarrow^{(1)}\right\rangle\left\langle\uparrow^{(3)}\right|.$$

The channel constructed by the above quantities became linear because the entangled state $\sigma^{(23)}$ is specially chosen as maximal.

4. Weak Teleportation and Uniqueness of Key

As we discussed in Section 1, the teleportation problem is to proceed along STEPS 1˜4. We here consider a bit simpler form in the steps, which is called a weaker form of quantum teleportation [1].

In the weak form, we consider; (1')an entangled state $\sigma^{(23)}$acting on $\mathcal{H}_2\otimes\mathcal{H}_3$; (2')a single projection $F^{(12)}$ acting on $\mathcal{H}_1\otimes\mathcal{H}_2$. Then we ask whether (3')a single unitary operator U such that the identity $\Lambda\rho^{(1)} = U^*\rho^{(1)}U$ holds for any state $\rho^{(1)} \in \mathcal{S}(\mathcal{H}_1)$.

The connection between the weak and the general teleportation problem is the following. Given a family $\{\sigma^{(23)}, F_k^{(12)}, U_k\}$ of solutions of the weak teleportation problem for each k such that the projections $F_k^{(12)}$ are mutually orthogonal, then this family provides a solution of the general teleportation problem. We shall solve the weak teleportation problem, and then we show the uniqueness of the key.

In the notations above, let assume that $N = \dim\mathcal{H}_1 = \dim\mathcal{H}_2 = \dim\mathcal{H}_3 < +\infty$ and $\sigma^{(23)} = |\psi\rangle\langle\psi|$ and $F := |\xi\rangle\langle\xi|$, where $\psi \in \mathcal{H}_2\otimes\mathcal{H}_3$ and $\xi \in \mathcal{H}_1\otimes\mathcal{H}_2$ are unit vectors. We look for a unitary operator $U : \mathcal{H}_3 \to \mathcal{H}_1$ such that for any density matrix $\rho \in \mathcal{S}(\mathcal{H}_1)$ one has $U(\Lambda\rho)U^* = \rho$, where $\Lambda\rho = \frac{1}{L}tr_{12}(F\otimes I^{(3)}(\rho\otimes|\psi\rangle\langle\psi|)F\otimes I^{(3)})$ with $L = tr_{123}(F\otimes I^{(3)}(\rho\otimes|\psi\rangle\langle\psi|)F\otimes I^{(3)})$.

Let us fix three arbitrary orthonormal bases: $(x_\alpha")$ of $\mathcal{H}_3$, (x'_h) of $\mathcal{H}_2$, (x_μ) of $\mathcal{H}_1$

Thus we have [1]:

Theorem 1. *Fix an arbitrary $N\times N$ complex unitary matrix $(\lambda_{h\alpha})$ and let*

$$\psi := \frac{1}{\sqrt{N}}\sum\lambda_{h\alpha}|x'_h\rangle\otimes|x_\alpha"\rangle \in \mathcal{H}_2\otimes\mathcal{H}_3,\ \xi = \frac{1}{\sqrt{N}}\sum_\mu x_\mu\otimes x'_\mu \in \mathcal{H}_1\otimes\mathcal{H}_2$$

Define a unitary operator $U : \mathcal{H}_3 \to \mathcal{H}_1$ such that $\sum_h \overline{\lambda}_{h\alpha}x_h = Ux_\alpha"$, then the triple (ψ,ξ,U) satisfies

$$tr_{12}(F\otimes I^{(3)}(\rho\otimes|\psi\rangle\langle\psi|)F\otimes I^{(3)}) = U^*\rho U$$

for any density operator $\rho \in \mathcal{S}(\mathcal{H}_1)$.

The uniqueness of the key is guaranteed by the next theorem.

Theorem 2. *Let $\rho = \sum p_\gamma P_\gamma$ be the spectral decomposition of $\rho \in \mathcal{H}_1$. If U_1 and U_2 satisfy the condition (3) of key with the above ρ, then there exists a unitary operator V from $\mathcal{H}_1$ to $\mathcal{H}_1$ such that $U_2 U_1^* = \Sigma V_\gamma$ with $V_\gamma \equiv P_\gamma V P_\gamma$. Moreover, the equality $V_\gamma V_\gamma^* = \delta_{\gamma\gamma'} P_\gamma$ is satisfied.*

In the notations and assumptions above, let us suppose that the normalized state vector $\widetilde{\psi}$ has the form with some constants $\{t_k\}$

$$\widetilde{\psi} = \sum_k t_k x_k'' \otimes x_k'.$$

and let us look for the conditions under which the yeleportation map Λ becomes linear.

Theorem 3. *Given ξ and $\widetilde{\psi}$ as above, the teleportation channel Λ,so*

$$tr_{123} F \otimes I^{(3)}(\rho \otimes |\psi\rangle\langle\psi|) F \otimes I^{(3)}$$

is independent of ρ if and only if the coefficients t_k have the following form

$$t_k = e^{i\theta_k}/\sqrt{N}$$

which means that the entangled state σ is maximal.

5. Perfect Teleportation in Bose Fock Space

Bennett and others used EPR spin pair to construct a teleportation model. In order to have a more realistic model, we briefly discuss a model based on the works by Fichtner and Ohya [5, 6]. One of the main points for such models is how to prepare the entangled state. The EPR entangled state used by Bennett et al can be identified with the splitting of a one particle state, and FO teleportation model is described by Fock spaces and the splittings, so that it is possible to work the whole teleportation process in general beam splitting scheme. Moreover to work with beams having a fixed number of particles seems to be not realistic, especially in the case of large distance between Alice and Bob, because we have to take into account that the beams will lose particles (or energy). For that reason one should use a class of beams being insensitive to this loss of particles. That and other arguments lead to superpositions of coherent beams.

In FO teleportation scheme, the teleportation scheme mentioned in Sec2. is slightly modified, namely, E1 and E2 below. Remark that we abbreviate the indices (1), (12), (23) for $\rho^{(1)}, F_k^{(12)}, \sigma^{(23)}$ and others for notational simplicity.

Step 2:: One then fixes a family of mutually orthogonal projections $(F_{nm})_{n,m=1}^N$ on the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ corresponding to an observable $F := \sum_{n,m} z_{nm} F_{nm}$, and for a fixed one pair of indices n, m, Alice performs a measurement of the observable F, involving only the $\mathcal{H}_1 \otimes \mathcal{H}_2$ part of the system in the state $\rho \otimes \sigma$. When Alice obtains an outcome z_{nm}, the state becomes

$$\rho_{nm}^{(123)} := \frac{1}{L}(F_{nm} \otimes I)\rho \otimes \sigma(F_{nm} \otimes I)$$

where $L = \mathrm{tr}_{123}(F_{nm} \otimes \mathbf{1})\rho \otimes \sigma(F_{nm} \otimes \mathbf{1})$ and tr_{123} is the full trace on the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$.

Step 3:: Bob is informed which outcome was obtained by Alice. This is equivalent to transmit the information that the eigenvalue z_{nm} was detected. This information is transmitted from Alice to Bob without disturbance and by means of classical tools.

Step 4:: Making only partial measurements on the third part on the system in the state $\rho_{nm}^{(123)}$ means that Bob will control a state $\Lambda_{nm}^{*}(\rho)$ on $\mathcal{H}_3$ given by the partial trace on $\mathcal{H}_1 \otimes \mathcal{H}_2$ of the state $\rho_{nm}^{(123)}$ (after Alice's measurement)

$$\begin{aligned} \Lambda_{nm}(\rho) &= \operatorname{tr}_{12} \rho_{nm}^{(123)} \\ &= \operatorname{tr}_{12} \frac{1}{L}(F_{nm} \otimes \mathbf{1})\rho \otimes \sigma(F_{nm} \otimes \mathbf{1}) \end{aligned}$$

where $L = \operatorname{tr}_{123}(F_{nm} \otimes \mathbf{1})\rho \otimes \sigma(F_{nm} \otimes \mathbf{1})$.

Thus the whole teleportation scheme given by the family (F_{nm}) and the entangled state σ can be characterized by the family (Λ_{nm}) of channels from the set of states on $\mathcal{H}_1$ into the set of states on $\mathcal{H}_3$ and the family (p_{nm}) given by

$$p_{nm}(\rho) := \operatorname{tr}_{123}(F_{nm} \otimes \mathbf{1})\rho \otimes \sigma(F_{nm} \otimes \mathbf{1})$$

of the probabilities that Alice's measurement according to the observable F will show the value z_{nm}.

The teleportation scheme works perfectly with respect to a certain class $\mathcal{S}$ of states ρ on $\mathcal{H}_1$ if the following conditions are fulfilled.

(E1): For each n, m there exists a unitary operator $W_{nm} : \mathcal{H}_1 \to \mathcal{H}_3$ such that

$$\Lambda_{nm}(\rho) = W_{nm}\, \rho\, W_{nm}^{*}, \ \forall \rho \in \mathcal{S}$$

(E2):

$$\sum_{nm} p_{nm}(\rho) = 1, \quad \forall \rho \in \mathcal{S}$$

(E1) means that Bob can reconstruct the original state ρ by unitary keys $\{W_{nm}\}$ provided to him.

(E2) means that Bob will succeed to find a proper key with certainty.

In this section, we construct a teleportation model being perfect in the sense of conditions (E1) and (E2) above, where we take the Boson Fock space $\Gamma(L^2(G)) := \mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_3$ over a configuration space G with a certain class ρ of states on this Fock space. Before stating the main results on perfect teleportation, we review basic notations and facts on Bose Fock space.

5.1. **Symmetric (Bose) Fock space.** Let us recall the definition of symmetric Fock space (see Ch.) Let $\mathcal{L}_1$ be one-particle Hilbert space. Then the n-particles Hilbert space is defined by $\mathcal{L}_n \equiv S_+ \mathcal{L}_1^{\otimes^n}$

where S_+ is the symmertizing operator on n-tuple tensor product Hilbert space $\mathcal{L}_1^{\otimes^n}$;

$$S_+ \equiv \frac{1}{n} \sum_P S_P,\ S_P(f_1 \otimes f_2 \otimes \cdots \otimes f_n) \equiv f_{P(1)} \otimes f_{P(2)} \otimes \cdots \otimes f_{P(n)}$$

The above P indicates a permutation of indices, and note that $\mathcal{L}_0 \equiv \mathbb{C}\Omega$ describes the zero-particle Hilbert space with the vacuum vector Ω. Indistinguishable Bose particles are described in the symmetric (Boson) Fock space

$$\Gamma_+(\mathcal{L}_1) \equiv \oplus_{n=0}^{\infty} \mathcal{L}_n.$$

Note that we have an important equality $\Gamma_+(\mathcal{L}_1 \oplus \mathcal{L}_1') = \Gamma_+(\mathcal{L}_1) \otimes \Gamma_+(\mathcal{L}_1')$ for two Hilbert spaces $\mathcal{L}_1$ and $\mathcal{L}_1'$.

5.1.1. *Fichtner-Freudenberg expression of Fock space.* We discuss Fichtner-Freudenberg expression of Fock space in a way adapted to the language of counting measures. Their expression looks difficult, but it is very useful to prove some thesis.

Let G be an arbitrary complete separable metric space. Further, let μ be a locally finite diffuse measure on G, i.e. $\mu(B) < +\infty$ for bounded measurable subsets of G and $\mu(\{x\}) = 0$ for all singletons $x \in G$. In order to describe the teleportation of states on a finite dimensional Hilbert space through the k–dimensional space $\mathbb{R}^k$, especially we are concerned with the case

$$\begin{aligned} G &= \mathbb{R}^k \times \{1, \ldots, N\} \\ \mu &= l \times \# \end{aligned}$$

where l is the k–dimensional Lebesgue measure and $\#$ denotes the counting measure on $\{1, \ldots, N\}$.

We denote the set of all finite counting measures on G by $M = M(G)$. Since $\varphi \in M$ can be written in the form $\varphi = \sum_{j=1}^{n} \delta_{x_j}$ for some $n = 0, 1, 2, \ldots$ and $x_j \in G$ with the Dirac measure δ_x corresponding to $x \in G$, the elements of M can be interpreted as finite (symmetric) point configurations in G. We equip M with its canonical σ–algebra$\mathfrak{M}$ and we consider the σ–finite measure F by setting

$$F(Y) := \mathcal{X}_Y(O) + \sum_{n \geq 1} \frac{1}{n!} \int_G 1_Y \left(\sum_{j=1}^{n} \delta_{x_j} \right) \mu^n(d[x_1, \ldots, x_n])(Y \in \mathfrak{M}),$$

where 1_Y denotes the indicator function of a set Y and O represents the empty configuration, i. e., $O(G) = 0$.

Since μ was assumed to be diffuse one easily checks that F is concentrated on the set of a simple configurations (i.e., without multiple points)

$$\hat{M} := \{\varphi \in M | \varphi(\{x\}) \leq 1 \text{ for all } x \in G\}$$

Definition 2. $\mathcal{M} = \mathcal{M}(G) := L^2(M, \mathfrak{M}, F)$ *is called the (symmetric) Fock space over* G.

It was proved by Fichtner and Freudenberg that $\mathcal{M}$ and the Boson Fock space $\Gamma_+(L^2(G))$ in the usual definition are isomorphic.

5.1.2. *Basic facts in symmetric Fock space.* For each vector Φ in symmetric Fock space$\mathcal{M}$ with $\Phi \neq 0$ we denote by $|\Phi>$ the corresponding normalized vector

$$|\Phi> := \frac{\Phi}{||\Phi||}.$$

Further, $|\Phi> < \Phi|$ denotes the corresponding one–dimensional projection describing a pure state given by the normalized vector $|\Phi>$. Now, for each $n \geq 1$ let $\mathcal{M}^{\otimes n}$

be the n–fold tensor product of the Hilbert space $\mathcal{M}$, which can be identified with $L^2(M^n, F^n)$.

Definition 3. *For a given function* $g : G \to \mathbb{C}$ *the function* $\exp(g) : M \to \mathbb{C}$ *defined by*

$$\exp(g)(\varphi) := \begin{cases} 1 & \text{if} \quad \varphi = 0 \\ \prod_{x \in G, \varphi(\{x\}) > 0} g(x) & \text{otherwise} \end{cases}$$

is called exponential vector generated by g.

This vector g is written as

$$\exp(\mathrm{g}) \equiv \oplus_{\mathrm{n}=0}^{\infty} \frac{1}{\sqrt{n!}} g^{\otimes^n} \in \Gamma_+(\mathcal{L}_1).$$

We note that $g^{\otimes^0} \equiv \Omega$. Observe that $\exp(g) \in \mathcal{M}$ if and only if $g \in L^2(G)$ and one has in this case $||\exp(g)||^2 = e^{\|g\|^2}$, where $\|\cdot\|$ is the norm deduced from the inner product $(\cdot, \cdot)$ of $L^2(G)$, so that the normalized vector is $|\exp(g)> = \frac{\exp(\mathrm{g})}{\|\exp(\mathrm{g})\|} = e^{-\frac{1}{2}\|g\|^2} \exp(g)$. The projection $|\exp(g)><\exp(g)|$ is called the coherent state corresponding to $g \in L^2(G)$. In the special case $g \equiv 0$ we get the vacuum state $|\exp(0)> = 1_{\{0\}} = \Omega$.The linear span of the exponential vectors of $\mathcal{M}$ is dense in $\mathcal{M}$, so that bounded operators and certain unbounded operators can be characterized by their actions on exponential vectors.

Definition 4. *The operator* $D : \mathrm{dom}(D) \to \mathcal{M}^{\otimes 2}$ *given on a dense domain* $\mathrm{dom}(D) \subset \mathcal{M}$ *containing the exponential vectors from* $\mathcal{M}$ *by*

$$D\psi(\varphi_1, \varphi_2) := \psi(\varphi_1 + \varphi_2) \quad (\psi \in \mathrm{dom}(D),\ \varphi_1, \varphi_2 \in M)$$

is called compound Hida-Malliavin derivative.

On exponential vectors $\exp(g)$ with $g \in L^2(G)$, one gets immediately

$$D \exp(g) = \exp(g) \otimes \exp(g)$$

Definition 5. *The operator* $S : \mathrm{dom}(S) \to \mathcal{M}$ *given on a dense domain* $\mathrm{dom}(S) \subset \mathcal{M}^{\otimes 2}$ *containing tensor products of exponential vectors by*

$$S\Phi(\varphi) := \sum_{\tilde{\varphi} \leq \varphi} \Phi(\tilde{\varphi}, \varphi - \tilde{\varphi}) \quad (\Phi \in \mathrm{dom}(S),\ \varphi \in M)$$

is called compound Skorohod integral.

After some calculation one gets

$$\langle D\psi, \Phi \rangle_{\mathcal{M}^{\otimes 2}} = \langle \psi, S\Phi \rangle_{\mathcal{M}} \quad (\psi \in \mathrm{dom}(D),\ \Phi \in \mathrm{dom}(S))$$

$$S(\exp(g) \otimes \exp(h)) = \exp(g + h) \quad (g, h \in L^2(G)).$$

Definition 6. *Let* T *be a linear operator on* $L^2(G)$ *with* $\|T\| \leq 1$. *Then the operator* $\Gamma(T)$ *called second quantization of* T *is the (uniquely determined) bounded operator on* $\mathcal{M}$ *fulfilling*

$$\Gamma(T)\exp(g) = \exp(Tg) \quad (g \in L^2(G))$$

Clearly, it holds

$$\Gamma(T_1)\Gamma(T_2) = \Gamma(T_1T_2), \Gamma(T^*) = \Gamma(T)^*$$

It follows that $\Gamma(T)$ is an unitary operator on $\mathcal{M}$ if T is an unitary operator on $L^2(G)$. This second quantization is expressed as unitary operator: $\Gamma(T) \equiv \oplus_n T^{\otimes^n}$.

Let K_1, K_2 be linear operators on $L^2(G)$ with property

$$K_1^*K_1 + K_2^*K_2 = \mathbf{1}\,. \tag{5.1}$$

Then there exists exactly one isometry ν_{K_1,K_2} from $\mathcal{M}$ to $\mathcal{M}^{\otimes 2} \equiv \mathcal{M}\otimes\mathcal{M}$ with

$$\nu_{K_1,K_2}\exp\,(g) = \exp(K_1g)\otimes\exp(K_2g) \quad (g\in L^2(G)).$$

Thus defined isometry ν_{K_1,K_2} is called generalized beam splitting. Further it holds

$$\nu_{K_1,K_2} = (\Gamma(K_1)\otimes\Gamma(K_2))D$$

at least on dom(D) but one has the unique extension. The adjoint $\nu^*_{K_1,K_2}$ of ν_{K_1,K_2} is characterized by

$$\nu^*_{K_1,K_2}(\exp\,(h)\otimes\exp\,(g)) = \exp(K_1^*h + K_2^*g) \quad (g,h\in L^2(G))$$

and it holds

$$\nu^*_{K_1,K_2} = S(\Gamma(K_1^*)\otimes\Gamma(K_2^*)),$$

which is a certain conditional expectation: $\mathcal{M}\otimes\mathcal{M}\to\mathcal{M}$ [9].

Here we explain the ordinary scheme of beam splitting is one of the example of the above generalized one. Let $K_1 := \alpha\mathbf{1}$ and $K_2 := \beta\mathbf{1}$ with $|\,\alpha\,|^2 + |\,\beta\,|^2 = 1$. Then we obtain $\nu_{K_1,K_2}exp(g) = exp(\alpha g)\otimes exp(\beta g)$, which is the well-known beam splitting used in optical communication and quantum measurements [2].

5.2. **A perfect teleportation.** The state of Alice asked to teleport is of the type

$$\rho = \sum_{s=1}^{N}\lambda_s|\Phi_s\rangle\langle\Phi_s|, \tag{5.2}$$

where $|\Phi_s\rangle$ is

$$|\Phi_s\rangle = \sum_{j=1}^{N}c_{sj}|\exp\,(aK_1g_j) - \exp\,(0)\rangle \quad \left(\sum_j |c_{sj}|^2 = 1; s = 1,\dots,N\right) \tag{5.3}$$

with an ONS $\{g_j\}_{j=1,2,\cdots,N}$ (i.e., $(g_i,g_j) = \delta_{ij}$ holds) of the one-particle space $\mathcal{L}_1$ and $a = \sqrt{d}$. One easily checks that $(|\exp\,(aK_1g_j) - \exp\,(0)\rangle)_{j=1}^N$ and $(|\exp\,aK_2g_j) - \exp\,(0)\rangle)_{j=1}^N$ are ONS in $\mathcal{M}$. The set $\{\Phi_s; s=1,\dots,N\}$ makes the N-dimensional Hilbert space $\mathcal{H}_1$ defining an input state teleported by Alice. Although we may include the vaccum state $|\exp\,(0)\rangle$ to define $\mathcal{H}_1$, here we take the N-dimensional Hilbert space $\mathcal{H}_1$ as above because of computational simplicity.

In order to achieve that $(|\Phi_s\rangle)_{s=1}^N$ is still an ONS in $\mathcal{M}$ we assume

$$\sum_{j=1}^{N}\bar{c}_{sj}c_{kj} = 0 \quad (j\neq k;\ j,k = 1,\dots,N)\,. \tag{5.4}$$

Denote $c_s = [c_{s1,\dots,}c_{sN}] \in \mathbb{C}^N$, then $(c_s)_{s=1}^N$ is an CONS in $\mathbb{C}^N$.

Let $(b_n)_{n=1}^N$ be a sequence in $\mathbb{C}^N$; $b_n = [b_{n1,\dots,}b_{nN}]$ with properties

$$|b_{nk}| = 1 \quad (n,k = 1,\dots,N), \langle b_n\,,\ b_j\rangle = 0 \quad (n\neq j;\ n,j = 1,\dots,N). \tag{5.5}$$

Now, for each $m, n\,(=1,\ldots,N)$, we have unitary operators U_m, B_n on $\mathcal{M}$ given by

$$B_n|\exp\,(aK_1g_j)-\exp\,(0)\rangle = b_{nj}|\exp\,(aK_1g_j)-\exp\,(0)\rangle \quad (j=1,\ldots,N)$$

$$U_m|\exp\,(aK_1g_j)-\exp\,(0)\rangle = |\exp\,(aK_1g_{j\oplus m})-\exp\,(0)\rangle \quad (j=1,\ldots,N) \tag{5.6}$$

where $j\oplus m := j+m(\mathrm{mod}\,N)$.

Then Alice's measurements are performed with projection

$$F_{nm} = |\xi_{nm}\rangle\langle\xi_{nm}| \quad (n,m=1,\ldots,N)$$

given by

$$|\xi_{nm}\rangle = \frac{1}{\sqrt{N}}\sum_{j=1}^{N} b_{nj}|\exp\,(aK_1g_j)-\exp\,(0) > \otimes|\exp\,(aK_1g_{j\oplus m})-\exp\,(0)\rangle.$$

One easily checks that $(|\xi_{nm}\rangle)_{n,m=1}^N$ is an ONS in $\mathcal{M}^{\otimes 2}$. Further, the state vector $|\psi\rangle$ of the entangled state $\sigma=|\psi\rangle\langle\psi|$ is given by

$$|\psi\rangle = \frac{1}{\sqrt{N}}\sum_k |\exp\,(aK_1g_k)-\exp\,(0)\rangle\otimes|\exp\,(aK_2g_k)-\exp\,(0)\rangle\,.$$

By using the above facts, it can be easily seen that a special case of the above model is unitary equivalent with the original perfect teleportation model proposed by Bennet et al.. The following theorem is proved.

Theorem 4. *For each $n,m=1,\ldots,N$, define a channel Λ_{nm} by*

$$\Lambda_{nm}(\rho) := \mathrm{tr}_{12}\frac{(F_{nm}\otimes 1)\,(\rho\otimes\sigma)\,(F_{nm}\otimes 1)}{\mathrm{tr}_{123}\,(F_{nm}\otimes 1)\,(\rho\otimes\sigma)\,(F_{nm}\otimes 1)} \quad (\rho \text{ normal state on } \mathcal{M}) \tag{5.7}$$

Then we have for all states ρ on M

$$\Lambda_{nm}(\rho) = (\Gamma(T)U_mB_n^*)\,\rho\,(\Gamma(T)U_mB_n^*)^*$$

If Alice performs a measurement according to the following selfadjoint operator $F=\sum_{n,m=1}^{N} z_{nm}F_{nm}$ with $\{z_{nm}|n,m=1,\ldots,N\}\subseteq\mathbb{R}-\{0\}$, then she will obtain the value z_{nm} with probability $1/N^2$. The sum over all this probabilities is 1, so that the teleportation model works perfectly.

6. Non-perfect Teleportation in Bose Fock Space

In this section we consider a teleportation model where the entangled state σ is given by the splitting of a superposition of certain coherent states instead of subtracting the vacuum. Unfortunately this model doesn't work perfectly, that is, neither (E2) nor (E1) hold. However this model is more realistic than that in the previous section, and we show that this model provides a nice approximation to be perfect. To estimate the difference between the perfect teleportation and non-perfect teleportation, we add a further step in the teleportation scheme:

Step 5:: Bob will perform a measurement on his part of the system according to the projection

$$F_+ := \mathbf{1} - |\exp(0) >< \exp(0)|$$

where $|\exp(0) >< \exp(0)|$ denotes the vacuum state (the coherent state with density 0).

Then our new teleportation channels (we denote it again by Λ_{nm}) have the form

$$\Lambda_{nm}(\rho) := \mathrm{tr}_{12} \frac{(F_{nm} \otimes F_+)\rho \otimes \sigma(F_{nm} \otimes F_+)}{\mathrm{tr}_{123}(F_{nm} \otimes F_+)\rho \otimes \sigma(F_{nm} \otimes F_+)}$$

and the corresponding probabilities are

$$p_{nm}(\rho) := \mathrm{tr}_{123}(F_{nm} \otimes F_+)\, \rho \otimes \sigma(F_{nm} \otimes F_+)$$

For this teleportation scheme, we consider when (E1) is fulfilled.

We discuss the above facts and explain such a teleportation scheme can be understood as "quantum teleportation with test".

Take the normalized vector which is a superposition of coherent states,

$$|\eta\rangle := \frac{\gamma}{\sqrt{N}} \sum_{k=1}^{N} |\exp\,(ag_k)\rangle \text{ with } \gamma := \left(\frac{1}{1+(N-1)e^{-a^2}}\right)^{\frac{1}{2}}$$

and we employ it as the imput of the beam splitter to obtain the entangled state,

$$\widetilde{\psi} := \nu_{K_1,K_2}(\eta) = \frac{\gamma}{\sqrt{N}} \sum_{k=1}^{N} |\exp\,(aK_1 g_k)\rangle \otimes |\exp\,(aK_2 g_k)\rangle.$$

We hence replace in the entangled state σ of the perfect teleportation by

$$\tilde{\sigma} := |\widetilde{\psi}\rangle\langle\widetilde{\psi}|.$$

Then for each $n, m = 1, \ldots, N$, we get the channels on any normal state ρ on $\mathcal{M}$ such as

$$\tilde{\Lambda}_{nm}(\rho) := \quad \mathrm{tr}_{12} \frac{(F_{nm} \otimes \mathbf{1})\,(\rho \otimes \tilde{\sigma})\,(F_{nm} \otimes \mathbf{1})}{\mathrm{tr}_{123}\,(F_{nm} \otimes \mathbf{1})\,(\rho \otimes \tilde{\sigma})\,(F_{nm} \otimes \mathbf{1})}$$

$$\Theta_{nm}(\rho) := \quad \mathrm{tr}_{12} \frac{(F_{nm} \otimes F_+)\,(\rho \otimes \tilde{\sigma})\,(F_{nm} \otimes F_+)}{\mathrm{tr}_{123}\,(F_{nm} \otimes F_+)\,(\rho \otimes \tilde{\sigma})\,(F_{nm} \otimes F_+)}, \tag{6.1}$$

where $F_+ = \mathbf{1} - |\exp\,(0)\rangle\langle\exp\,(0)|$, i.e.., F_+ is the projection onto the space $\mathcal{M}_+$ of configurations having no vacuum part;

$$\mathcal{M}_+ := \{\psi \in \mathcal{M} |\ \| |\exp\,(0)\rangle\langle\exp\,(0)|\psi\| = 0\}$$

One easily checks that

$$\Theta_{nm}(\rho) = \frac{F_+ \tilde{\Lambda}_{nm}(\rho) F_+}{\mathrm{tr}\left(F_+ \tilde{\Lambda}_{nm}(\rho) F_+\right)}$$

that is, after receiving the state $\tilde{\Lambda}_{nm}(\rho)$ from Alice, Bob has to omit the vacuum.

From Theorem 4 it follows that for all ρ with (5.2)

$$\Lambda_{nm}(\rho) = \frac{F_+ \Lambda_{nm}(\rho) F_+}{\mathrm{tr}\,(F_+ \Lambda_{nm}(\rho) F_+)}.$$

Theorem 5. *For all states ρ on $\mathcal{M}$ with (5.2) and each pair $n, m\,(= 1, \ldots, N)$, we have*

$$\Theta_{nm}(\rho) = (\Gamma\,(T)\,U_m B_n^*)\,\rho\,(\Gamma\,(T)\,U_m B_n^*)^* \quad \textit{or} \quad \Theta_{nm}(\rho) = \Lambda_{nm}(\rho) \tag{6.2}$$

and

$$(6.3)\quad \sum_{n,m} p_{nm}(\rho) = \sum_{n,m} \mathrm{tr}_{123}\left(F_{nm}\otimes F_+\right)(\rho\otimes\tilde{\sigma})\left(F_{nm}\otimes F_+\right) = \frac{\left(1-e^{-\frac{a^2}{2}}\right)^2}{1+(N-1)e^{-a^2}}.$$

That is, the model works only asymptotically perfect in the sense of condition (E2). In other words, the model works perfectly for the case of high density (or energy) of the considered beams.

We can further generalize the above teleportation schemes, namely, replacing the projectors F_{nm} by projectors $\tilde{F}_{nm}$ defined as $\tilde{F}_{nm} := (B_n \otimes U_m\Gamma(T)^*)\,\tilde{\sigma}\,(B_n \otimes U_m\Gamma(T)^*)^*$,for which see [6].

6.1. **Fidelity.** The non-perfect teleportation scheme can be understood as the perfect teleportation with "test" of Alice and Bob in the following sense: when Alice performs a measurement of the observable F, there is a possibility to obtain an outcome 0, that is, none of $\{z_{nm}\}$ is obtained. In such a case Alice quits the experiment and try again from the first procedure. And at the final step Bob performs a measurement with $F_+ = 1 - |exp(0)\rangle\langle exp(0)|$. If his result $= 0$, then he askes Alice to try again, and his result $= 1$, then he continues. The state he obtained is

$$\Theta_{nm}(\rho) := \frac{tr_{12}(F_{nm}\otimes F_+)(\rho\otimes\tilde{\sigma})(F_{nm}\otimes F_+)}{tr_{123}[(F_{nm}\otimes F_+)(\rho\otimes\tilde{\sigma})(F_{nm}\otimes F_+)]},$$

on which he applied the proper key provided to get the state alice sent.

To discuss the (non-)perfectness of channels, we need some proper quantity to measure how close two states are. We use a notion *fidelity* for the non-perfect teleportation model. The notion of fidelity is frequently used in the context of quantum information and quantum optics. The fidelity of a state ρ with respect to another state σ is defined by

$$F(\rho,\sigma) := tr[\sqrt{\sigma^{1/2}\rho\sigma^{1/2}}],$$

which possesses some nice properties.

$$0 \le F(\rho,\sigma) = F(\sigma,\rho) \le 1,\ F(\rho,\sigma) = 1 \Leftrightarrow \rho = \sigma.$$

Thus we can say two states ρ and σ are close when the fidelity between them is close to 1.

Now let us begin with the teleportation model with the entangled state $\tilde{\sigma}$. Since in this section Bob is not allowed to subtract the vacuum, put

$$\Xi_{nm}(\rho) := \frac{tr_{1,2}[(F_{nm}\otimes W_{nm})(\rho\otimes\tilde{\sigma})(F_{nm}\otimes W_{nm}^*)]}{tr_{1,2,3}[(F_{nm}\otimes 1)(\rho\otimes\tilde{\sigma})(F_{nm}\otimes 1)]},$$

which takes place with probability $p_{nm} \equiv tr_{1,2,3}[(F_{nm}\otimes 1)(\rho\otimes\tilde{\sigma})(F_{nm}\otimes 1)]$.

In this section we do not perform any tests and therefore even if the outcome of Alice's measument is 0, the procedure is not stopped. We put the key of Bob corresponding to the outcome 0 as W_0. Then the nonlinear channel with the result 0 is

$$\Xi_0(\rho) = \frac{tr_{1,2}[(F_0\otimes W_0)(\rho\otimes\tilde{\sigma})(F_0\otimes W_0^*)]}{tr_{1,2,3}[(F_0\otimes 1)(\rho\otimes\tilde{\sigma})(F_0\otimes 1)]}$$

which takes place with probability

$$p_0 \equiv tr_{1,2,3}[(F_0 \otimes 1)(\rho \otimes \tilde{\sigma})(F_0 \otimes 1)]$$

Without knowing the result the expected state is due to a linear channel (=unital completely positive map)

$$\begin{aligned}\Lambda(\rho) &\equiv \sum_{nm} p_{nm}\Xi_{nm}(\rho) + p_0\Xi_0(\rho) \\ &= \sum_{nm} tr_{1,2}[(F_{nm} \otimes W_{nm})(\rho \otimes \tilde{\sigma})(F_{nm} \otimes W^*_{nm})] + tr_{1,2}[(F_0 \otimes W_0)(\rho \otimes \tilde{\sigma})(F_0 \otimes W_0^*)].\end{aligned}$$

Note that $\{F_{nm} \otimes W_{nm}\}_{nm}$ and $F_0 \otimes W_0$ forms a partition of unity.

To estimate $F(\rho, \Xi(\rho))$ it needs to compute $\Xi(\rho) = \sum_{nm} \Xi_{nm}(\rho) + \Xi_0(\rho)$. The result is

Theorem 6. *For any imput state ρ of type (5.2) and (5.3), it holds*

$$F(\rho, \Xi^*(\rho)) \geq \sqrt{\frac{(1-e^{-a^2/2})^2}{1+(N-1)e^{-a^2}}}.$$

Therefore the teleportation protocol approaches perfect one as the parameter $|a|$ goes to infinity.

With some additional conditions, one can strengthen the above estimate to an equality.

7. New Scheme of Quantum Teleportation

In most of models as we discussed, perfect teleportation can be occurred if the entangled state between Alice and Bob is maximal. Moreover teleportation channel is linear iff the entangled state is maximal. In [8], we reformulated the teleportation process and show in that model that the teleportation channel is always linear the perfect teleportation is possible even in the case for non-maximal entangled state. Here we discuss fumdamental points of our new treatment of [8].

7.1. **Basic Setting.** Let $\mathcal{H} = \mathbb{C}^n$ be a finite dimensional complex Hilbert space, in which the scalar product $< , >$ is defined as usual. Let e_n ($n = 1, \cdots, n$) be a fixed orthonormal basis (ONB) in $\mathcal{H}$, and let $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$, which is simply denoted by M_n.In M_n, the scalar product (,) is defined by

$$(A, B) := trA^*B = \sum_{i=1}^{n} < Ae_i, Be_i >$$

Note that $e_{ij} := |e_i\rangle \langle e_j|$ $(i, j = 1, \cdots, n)$ is a ONB in M_n with respect to the above scalar product. The mappings

$$A \in M_n \to A^L := \sum\nolimits_{i=1}^{n} Ae_i \otimes e_i \in \mathcal{H} \otimes \mathcal{H}, \ A \in M_n \to A^R := \sum\nolimits_{i=1}^{n} e_i \otimes Ae_i \in \mathcal{H} \otimes \mathcal{H}$$

define the inner product isomorphisms from M_n into $\mathcal{H} \otimes \mathcal{H}$ such that

$$(A, B) = << A^L, B^L >> = << A^R, B^R >>,$$

where the inner products in $\mathcal{H} \otimes \mathcal{H}$ is denoted by $<< \cdot, \cdot >>$.

Let $L(M_n, M_n)$ be the vector space of all linear maps $\Phi : M_n \to M_n$. $M_n \otimes M_n$ is the set of all linear maps from $\mathcal{H} \otimes \mathcal{H}$ to $\mathcal{H} \otimes \mathcal{H}$. By analogy between M_n and $\mathcal{H} \otimes \mathcal{H}$, one can construct the inner product isomorphisms between $L(M_n, M_n)$ and $M_n \otimes M_n$ such as

$$\Phi \in L(M_n, M_n) \to \Phi^L := \sum_{i,j=1}^{n} \Phi e_{ij} \otimes e_{ij} \in M_n \otimes M_n$$

$$\Phi \in L(M_n, M_n) \to \Phi^R := \sum_{i,j=1}^{n} e_{ij} \otimes \Phi e_{ij} \in M_n \otimes M_n$$

The inner products in $L(M_n, M_n)$ is defined as follows:

$$((\Phi, \Psi)) := tr\Phi^*\Psi = \sum_{i,j=1}^{n} (\Phi e_{ij}, \Psi e_{ij}).$$

Let $\{f_\alpha; \alpha = 1, \cdots, n^2\}$ be another ONB in M_n so that one has $tr f_\alpha^* f_\beta = \delta_{\alpha\beta}$. It is easy to check that the maps $\Phi_{\alpha\beta} \in L(M_n, M_n)$ defined by $\Phi_{\alpha\beta}(A) := f_\alpha A f_\beta^*$ for any $A \in M_n$ can be written as $\Phi_{\alpha\beta} = |f_\alpha)(f_\beta|$ and the set $\{\Phi_{\alpha\beta}\}$ is a ONB of $M_n \otimes M_n$. Moreover the corresponding elements $\Phi_{\alpha\beta}^L, \Phi_{\alpha\beta}^R \in M_n \otimes M_n$ form ONBs of $M_n \otimes M_n$.The explicit expression of $\Phi_{\alpha\beta}^L$ and $\Phi_{\alpha\beta}^R$ are

$$\Phi_{\alpha\beta}^L := \sum_{i,j=1}^{n} f_\alpha e_{ij} f_\beta^* \otimes e_{ij} \text{ and } \Phi_{\alpha\beta}^R := \sum_{i,j=1}^{n} e_{ij} \otimes f_\alpha e_{ij} f_\beta^*.$$

It is easily shown that for any ONB $\{f_\alpha\}$

$$P_\alpha := \Phi_{\alpha\alpha}^L = \sum_{i,j=1}^{n} f_\alpha e_{ij} f_\alpha^* \otimes e_{ij},\ Q_\alpha := \Phi_{\alpha\alpha}^R = \sum_{i,j=1}^{n} e_{ij} \otimes f_\alpha e_{ij} f_\alpha^*$$

are mutual orthogonal projections in $\mathcal{H} \otimes \mathcal{H}$ satisfying

$$\sum_{\alpha=1}^{n^2} P_\alpha = \sum_{\alpha=1}^{n^2} Q_\alpha = I \otimes I \ (I \text{ is unity of } M_n)$$

A any state (density operator) σ_{12} on $\mathcal{H} \otimes \mathcal{H}$ can be written in the form

$$\sigma_{12} = \sum_{\alpha=1}^{n^2} \lambda_\alpha Q_\alpha = \sum_{\alpha=1}^{n^2} \lambda_\alpha \sum_{i,j=1}^{n} e_{ij} \otimes f_\alpha e_{ij} f_\alpha^*$$

with $\sum_{\alpha=1}^{n^2} \lambda_\alpha = 1$ and $\lambda_\alpha \geq 0$. Put $\Theta(A) := \sum_{\alpha=1}^{n^2} \lambda_\alpha f_\alpha A f_\alpha^*$ for any $A \in M_n$. Then Θ is a completely positive (CP) map on M_n, and σ_{12} is written as

$$\sigma_{12} = \sum_{i,j=1}^{n} e_{ij} \otimes \Theta(e_{ij}).$$

Let take $A \in M_n$ with $trA^*A = 1$. Then A^L (A^R) is a normalized vector in $\mathcal{H} \otimes \mathcal{H}$ and it defines a state σ in $\mathcal{H} \otimes \mathcal{H}$ as $\sigma := |A^L >>< < A^L|$.

Definition 7. *The above state σ is maximal entangled if $A^*A = AA^* = \frac{I}{n}$, equivalently, $A = \frac{1}{\sqrt{n}} U$ with some unitary operator U in $\mathcal{H}$.*

Note that if one can construct an ONB $\{f_\alpha = U_\alpha / \sqrt{n}; \alpha = 1, \cdots, n^2\}$ with unitary U_α, then the corresponding projections P and Q given above are maximal entangled states.

Definition 8. *The map $\Phi \in L(M_n, M_n)$ is said to be normalized if $\Phi(I) = I$, base preserving if $tr\Phi(A) = trA$ for all $A \in M_n$, selfadjoint if $\Phi(A)^* = \Phi(A^*)$ for all $A \in M_n$, positive if $\Phi(A^*A) \geq 0$ for all $A \in M_n$ and completely positive if $\sum_{i,j=1}^{n} < x_i, \Phi(A_i^* A_j) x_j > \geq 0$ for any x_i $(i = 1, \cdots, n) \in \mathcal{H}$ and any A_i $(i = 1, \cdots, n) \in M_n$.*

Note that the canonical form of completely positive map is given by Θ above.

7.2. **New Scheme of Entanglement and Teleportation.** We propose a new protocol for quantum teleportation. Let us take the conditions that all three Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ are $\mathbb{C}^n$.Let the state σ in $\mathcal{H}_2 \otimes \mathcal{H}_3 = \mathbb{C}^n \otimes \mathbb{C}^n$ be

$$\sigma = \sum_{i,j=1}^{n} e_{ij} \otimes \Theta(e_{ij})$$

Here e_{ij}, Θ are those given in Section 2 with an ONB $\{f_\alpha; \alpha = 1, \cdots, n^2\}$ but are defined on $\mathcal{H}_2$ and $\mathcal{H}_3$. We set an observable F in $\mathcal{H}_1 \otimes \mathcal{H}_2$ to be measured by Alice as follows:

$$F = \sum_\alpha z_\alpha P_\alpha := \sum_\alpha z_\alpha \sum_{i,j=1}^{n} g_\alpha^* e_{ij} g_\alpha \otimes e_{ij},$$

where $\{g_\alpha; \alpha = 1, \cdots, n^2\}$ is another ONB of M_n. Then we define the teleportation map for an input state ρ in $\mathcal{H}_1$ and the measured value z_α of Alice by

$$T_\alpha(\rho) := tr_{12}(P_\alpha \otimes \mathbf{1})\rho \otimes \sigma(P_\alpha \otimes \mathbf{1}).$$

Theorem 7. *The teleportation map T_α has the form $T_\alpha(\rho) = \Theta(g_\alpha \rho g_\alpha^*)$ for any ρ in $\mathcal{H}_1$.*

It is easily seen that T_α is completely positive but not trace preserving. In order to consider the trace preserving map from T_α, let us consider the dual map $\widetilde{T}_\alpha$ of T_α, i.e., $trAT_\alpha(\rho) =: tr\widetilde{T_\alpha}(A)\rho$. Indeed it is

$$\widetilde{T}_\alpha(A) = g_\alpha^* \widetilde{\Theta}(A) g_\alpha, A \in M_n$$

where $\widetilde{\Theta}$ is the dual to Θ;

$$\widetilde{\Theta}(A) = \sum_{\alpha=1}^{n^2} \lambda_\alpha f_\alpha^* A f_\alpha.$$

The map $\widetilde{T}_\alpha$ is normalizable iff $rank\widetilde{T}_\alpha(I) = n$, that is, the operator $\widetilde{T}_\alpha(I)$ is invertible. Put

$$\kappa_\alpha := \widetilde{T}_\alpha(I).$$

In this case the teleportation map $\widetilde{T}_\alpha$ is normalized as

$$\widetilde{\Upsilon}_\alpha := \kappa_\alpha^{-\frac{1}{2}} \widetilde{T}_\alpha \kappa_\alpha^{-\frac{1}{2}}.$$

The dual map Υ_α of $\widetilde{\Upsilon}_\alpha$ is trace preserving and it has the form as

$$\Upsilon_\alpha(\rho) = \Theta\left(g_\alpha \kappa_\alpha^{-\frac{1}{2}} \rho \kappa_\alpha^{-\frac{1}{2}} g_\alpha^*\right) = \sum_{\beta=1}^{n^2} \lambda_\beta f_\beta g_\alpha \kappa_\alpha^{-\frac{1}{2}} \rho \kappa_\alpha^{-\frac{1}{2}} (f_\beta g_\alpha)^*.$$

It is important to note that this teleportation map is linear with respect to all initial states ρ. The above map provides a new example of Umegaki's conditional expectation [9].

Let us consider a special case of σ such that

$$\sigma = \sum_{i,j=1}^{n} e_{ij} \otimes \Theta(e_{ij}) \text{ with } \Theta(\bullet) := f \bullet f^* \text{ and } trf^*f = 1.$$

That is, σ is a pure state. In this case, one has

$$T_\alpha(\rho) = (g_\alpha f)\rho(g_\alpha f)^* \text{ and } \kappa_\alpha = (g_\alpha f)^*(g_\alpha f).$$

Remark 1. *If* $g_\alpha = U_\alpha/\sqrt{n}$ *and* $f = V/\sqrt{n}$*, where* U_α *and* V *are unitary operators, then* $\kappa_\alpha = 1/n^2$*,which corresponds to the usual discussion.*

Further, it follows that Υ_α is trace preserving iff $rank\,(g_\alpha) = rank\,(f) = n$, and in such a case one has

$$\Upsilon_\alpha\,(\rho) = (fg_\alpha)\,\kappa_\alpha^{-\frac{1}{2}}\rho\kappa_\alpha^{-\frac{1}{2}}\,(fg_\alpha)^*$$

Put

$$W_\alpha := fg_\alpha\kappa_\alpha^{-\frac{1}{2}},$$

which is easily seen to be unitary. Thus we proved the following theorem.

Theorem 8. *Given ONB* $\{g_\alpha; \alpha = 1, \cdots, n^2\}$ *and a vector* f *in* M_n *on the n-dimensional Hilbert space, if* $rank\,(g_\alpha) = rank\,(f) = n$ *is satisfied, then one can construct an entangled state* σ *and the set of keys* $\{W_\alpha\}$ *such that complete teleportation occurs.*

Note here that our teleportation protocol is not required that the entangled state is maximal for linear complete teleportation.

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