数理解析研究所講究録
1509巻 2006年 100-112

超函数論におけるエネルギー法

片岡 清臣（東京大学 数理科学研究科）

What is the microlocal energy methods ?

An L^2-like method for <u>microfunctions with C^ω-parameters</u>

Ex. $\begin{cases} \partial f/\partial t = \Delta_x f, & t>0,\ x\in\Omega \\ f|_{x\in\partial\Omega} = 0, & t>0 \end{cases}$

- $f \in \mathcal{B}((0,\infty)\times\Omega)$, $\Omega \subset \mathbb{R}^n$; bdd, C^ω-bdry.
- $f|_{x\in\partial\Omega}$ is well-def. $SS(f) \subset \{(t, x; i\tau dt + i\xi dx); \xi = 0\}$

  $\Downarrow$ x is a C^ω-parameter.

$\Rightarrow$ $\underline{f \in \mathcal{A}((0,\infty)\times\overline{\Omega})}$

$\underline{E(t,s) = \int_{\mathbb{R}^n} f(t,x)\overline{f(s,x)}\cdot\chi_\Omega(x)\,dx} \in \mathcal{B}((0,\infty)^2)$

is well-def. because x is a C^ω-parameter for f up to $x \in \partial\Omega$ (mildness).

Note : Parabolic B.V.P. $\Rightarrow$ $(\partial_t - \partial_s)E(t,s) = 0$

$\therefore$ $SS(E(t,s)) \not\ni (\mathring{t}, \mathring{t}; \pm(dt - ds))$ $(\forall \mathring{t} > 0)$.

$\Downarrow$ Fundamental theorem

$SS(f(t,x)\cdot\chi_\Omega(x)) \not\ni (\mathring{t}, x; \pm i\,dt + i\xi\,dx)$

$(\forall \mathring{t} > 0, \forall x, \forall \xi)$

Hermitian Positivity for analytic Kernels.

S : a set

Def. $K(x,u)$: a $\mathbb{C}$-valued function on $S\times S$.

K is a hermitian kernel $\Leftrightarrow$ $\overline{K(u,x)} = K(x,u)$ on $S\times S$

" a positive kernel $\Leftrightarrow$ $\forall N$, $\forall x_1, \dots x_N \in S$

$\quad (K(x_j, x_\ell))_{j,\ell} \gg 0$ (positive semi-df.)

(Simply we write $K \gg 0$ on $S\times S$)

Def. $\Omega \subset \mathbb{C}^n$: open

$\mathcal{H}(\Omega) := \{K(z,w) \mid$ hermitian kernels on $\Omega\times\Omega$ & $K(z,\overline{w}) \in \mathcal{O}(\Omega\times\Omega^*)\}$

analytic h. kernels — Ω^* : complex conjugate.

$\mathcal{H}^+(\Omega) := \{K \in \mathcal{H}(\Omega) \mid K(z,w) \gg 0$ on $\Omega\times\Omega\}$

$\Rightarrow$ · $\mathcal{H}(\Omega)$: $\mathbb{R}$-algebra (as functions on $\Omega\times\Omega$) with order rel. " $\gg$ ".

$K_1 \gg 0,\ K_2 \gg 0 \Rightarrow K_1 + K_2,\ K_1\cdot K_2 \gg 0$

Facts
- $K_\Omega(z,w)$ (the Bergman kernel) $\in \mathcal{H}^+(\Omega)$
- $\Omega_1 \Subset \Omega_2 \Rightarrow 0 \ll K_{\Omega_2}|_{\Omega_1\times\Omega_1} \ll K_{\Omega_1}$
- $^\forall K \in \mathcal{H}(\Omega) \cap L^2(\Omega\times\Omega) \Rightarrow -\|K\|_{L^2}\cdot K_\Omega \ll K \ll \|K\|_{L^2} K_\Omega$
- (Analytic continuation) $\Omega_1 \subset \Omega_2$: domains
  $K_1, K_3 \in \mathcal{H}(\Omega_2)$, $K_2 \in \mathcal{H}(\Omega_1)$ s.t. $K_1 \ll K_2 \ll K_3$ on $\Omega_1\times\Omega_1$ $\Rightarrow$ $K_2 \in \mathcal{H}(\Omega_2)$ & $K_1 \ll K_2 \ll K_3$ on $\Omega_2\times\Omega_2$

Fundamental Theorem for microlocal energy methods.

$\overset{\circ}{p} = (\overset{\circ}{x}, \overset{\circ}{x} ; i\overset{\circ}{\xi}\cdot(dx-du)) \in iT^*(\mathbb{R}^n_x \times \mathbb{R}^n_u)$ $(|\overset{\circ}{\xi}|=1)$

$z = x+iy$. $w = u+iv$ (the copy of z)

$\underline{t_j \in T_j\ (\subset\subset U_j) \subset \mathbb{R}^{m_j}}$: bdd. open, $\underline{C^\omega\text{-bdry}}$ $(j=1\ldots N)$

$$E(x,u) := \sum_{j=1}^{N} \int_{\mathbb{R}^{m_j}} (Q_j(t_j, x, D_x) + Q_j^*(t_j, u, D_u))$$
$$\times \left(f_j(t_j, x)\cdot \overline{f_j(t_j, u)}\, \chi_{T_j}(t_j) \right) \rho_j(t_j)\, dt_j$$

$Q_j(t_j, z, \xi)$: holomorphic Ψ.D.O. with C^ω-parameter t_j.

s.t. ① Defined on $X_j = \{(t_j, z; \xi) \in \mathbb{C}^{m_j+n+n} \mid t_j \in U_j,$
$|z-\overset{\circ}{x}| < r,\ \left|\frac{\xi}{|\xi|} - i\overset{\circ}{\xi}\right| < r,\ |\xi| > r^{-1}\}$

② $\underline{\exists l_j > \frac{7}{8}}$, $\exists C$,

$$\underline{C^{-1}|\xi|^{l_j} \le \operatorname{Re} Q_j \le |Q_j| \le C|\xi|^{l_j} \text{ on } X_j}$$

$$\underline{Q_j^*(t_j, z, \xi) = \overline{Q_j(\bar{t}_j, \bar{z}, \bar{\xi})}}$$

Th. (K ; 1985) $\rho_j(t_j) \in C^\omega(U_j)$, $\rho_j(t_j) \not\equiv 0$, $\rho_j(t_j) \ge 0$ $(t_j \in T_j)$

$f_j(t_j, x) \in \mathcal{B}(T_j \times \{|x-\overset{\circ}{x}| < r\})$ depending C^ω-ly on t_j up to $\partial T_j \times \{\overset{\circ}{x}\}$ (mild sense). Then,

$\underline{E(x,u) = 0 \text{ at } \overset{\circ}{p}} \Rightarrow \underline{SS(f_j(t_j,x)\chi_{T_j}(x)) \not\ni (t_j, \overset{\circ}{x}; i\tau_j dt_j + i\overset{\circ}{\xi}dx)}$.
$(\forall t_j, \forall \tau_j)$

Positivity for microkernels.

$\mathring{p} = (\mathring{x}, \mathring{x}\,;\, i\mathring{\xi}\cdot(dx-du)) \in iT^*(\mathbb{R}^n_x \times \mathbb{R}^n_u)$

<u>Def</u> $k(x,u) \in \mathcal{C}_{\mathbb{R}^n\times\mathbb{R}^n}|_{\mathring{p}}$

a <u>hermitian microkernel</u> $\Leftrightarrow$ $\overline{k(u,x)} = k(x,u)$ at $\mathring{p}$

a <u>positive microkernel</u> $\Leftrightarrow$ $\left\{\begin{array}{l} {}^\exists \Gamma \cap \{y\cdot\mathring{\xi}>0\} \neq \phi \quad \text{open cone} \\ {}^\exists K(z,w) \in \mathcal{H}^+(\{|z-\mathring{x}|<r,\ \mathrm{Im}\,z \in \Gamma\}) \\ \text{s.t. } k(x,u) = [K(x+i0\Gamma, \overline{u-i0\Gamma})] \text{ at } \mathring{p}. \end{array}\right.$

(simply we write $k(x,u) \gg 0$ at $\mathring{p}$)

<u>Th.</u> " $\gg 0$ at $\mathring{p}$ " is an <u>order relation</u> for <u>hermitian microkernels</u> ; that is,

$k(x,u) \gg 0,\ -k(x,u) \gg 0$ at $\mathring{p} \Leftrightarrow k(x,u) = 0$ at $\mathring{p}$

$k_1(x,u) \gg 0,\ k_2(x,u) \gg 0$ at $\mathring{p} \Rightarrow k_1 + k_2 \gg 0$ at $\mathring{p}$.

<u>Ex.</u> • $\delta(x-u) \gg 0$ at $(\mathring{x}, \mathring{x}\,;\, i\mathring{\xi}\cdot(dx-du))$ ${}^\forall\mathring{x}, {}^\forall\mathring{\xi}$.

• $\displaystyle\int_{\mathbb{R}^m} p(t,x,u,D_x,D_u)\big(f(t,x)\cdot\overline{f(t,u)}\,\chi_T(t)\big)\,dt \gg 0$

if " $p(t,x,u,D_x,D_u)$ " is a positive Ψ.D.O. with C^ω-parameter t.

Aoki's exp. cal. & Ψ.D.O. of restricted hermitian type

<u>Aoki's theory</u> : P, P_1, P_2, P_3 are Ψ.D.O $(\mathcal{E}^{R}_{\mathbb{C}^n})$ of growth order <1.

$$\Rightarrow \begin{cases} \forall P_1, P_2, \ \exists P_3 \ \text{ s.t. } :e^{P_1(z,\zeta)}: :e^{P_2(z,\zeta)}: = :e^{P_3(z,\zeta)}: \\ \forall p, \exists \tilde{p} \ (\text{or } \forall \tilde{p} \ \exists p) \text{ s.t. } :e^{p(z,\zeta)}: = e^{:\tilde{p}(z,\zeta):} \end{cases}$$

However, for product h. type Ψ.D.O of order <1 Aoki's exp. cal. <u>does not work</u> because $|\zeta|^{\sigma} \cdot \frac{1}{|\eta|}$ is not bounded on $V \times V^*$.

<u>Def</u> For a product h. type symbol $P(z,w,\zeta,\eta)$ at $\mathring{p}$, $:e^{P(z,w,\zeta,\eta)}:$ is said to be <u>of restricted hermitian type at $\mathring{p}$</u>

artificial restr.

$$\Leftrightarrow \begin{cases} 0 < \exists \sigma < \frac{1}{2}, \ \exists C, r > 0 \\ |\mathrm{grad}_{(z,w)} P| \le C \min\{|\zeta|^{\sigma}, |\eta|^{\sigma}\} \\ |\mathrm{grad}_{(\zeta,\eta)} P| \le C \min\{|\zeta|^{\sigma-1}, |\eta|^{\sigma-1}\} \end{cases}$$

on $V_r \times V_r^*$ $\quad \left(\Rightarrow \le C(|\zeta|+|\eta|)^{\sigma-1} \right)$

Indeed, $P = \log(Q_\lambda(z,\zeta) + Q_\lambda^*(w,\eta))$ o.K.

Examples of Ψ.D.O and Quasi-Positivity

Ex. $P(z, D_z) \in \mathcal{E}^{R}_{\mathbb{C}^n}|_{(\mathring{z}; \mathring{\zeta})}$

$$\Rightarrow \underbrace{P(z, D_z) + P^*(w, D_w)}_{\text{product } h.}, \quad \underbrace{P(z, D_z) \cdot P^*(w, D_w)}_{\text{positive } h.}$$

Ex. $P(z, D_z)$ as above & $\exists m, C > 0$.

$$\underbrace{C|\zeta|^m \leq \operatorname{Re} P(z, \zeta) \leq |P(z, \zeta)| \leq C^{-1}|\zeta|^m}$$

$$\Rightarrow (P(z, D_z) + P^*(w, D_w))^{-1} \left(= \int_0^\infty e^{-t(P(z, D_z) + P^*(w, D_w))} dt\right)$$

$$\gg 0 \quad \text{at } (\mathring{z}, \overline{\mathring{z}}; \mathring{\zeta}, \overline{\mathring{\zeta}})$$

Here $P(z, D_z) + P^*(w, D_w)$ is not of positive h. type but close to of positive h. type in the following sense :

For a family $P_\lambda + P_\lambda^*$ as above

$\exists Q(z, w, D_z, D_w) \gg 0$, & elliptic

s.t. $\quad \underline{Q \cdot (P_\lambda + P_\lambda^*)} \gg 0 \quad (\forall \lambda)$ --- ☆

$\Downarrow$ **Quasi-Positivity**

Write $\quad Q = e^q, \quad P_\lambda + P_\lambda^* = e^{\log(P_\lambda + P_\lambda^*)}$

Then ☆ is almost equivalent to

$$\underline{q \gg 0, \ \& \quad q \gg -\log(P_\lambda + P_\lambda^*)} \quad (\forall \lambda)$$

Hermitian Positivity for Ψ.D.O. at $\mathring{p} = (\mathring{z}, \overline{\mathring{z}}; i\mathring{\xi}dz + \overline{\mathring{\xi}}dw) \in T^*(\mathbb{C}^n_z \times \mathbb{C}^n_w)$ $(|\mathring{\xi}|=1)$

Def $P(z, w, D_z, D_w) \in \mathcal{E}^{\mathbb{R}}_{\mathbb{C}^n_z \times \mathbb{C}^n_w | \mathring{p}}$

is <u>hermitian</u> $\Leftrightarrow$ $P(z, w, \xi, \eta) - \overline{P(\overline{w}, \overline{z}, \overline{\eta}, \overline{\xi})} \sim 0$ at $\mathring{p}$

of <u>Product hermitian type</u>

$\Leftrightarrow$
- $P(z, w, \xi, \eta) \in \mathcal{O}(V_r \times V_r^*)$ with
  $V_r := \{(z, \xi) \in \mathbb{C}^{n+n} \mid |z - \mathring{z}| < r, |\frac{\xi}{|\xi|} - \mathring{\xi}| < r, |\xi| > r^{-1}\}$
- $|P(z, w, \xi, \eta)| \le C_\varepsilon e^{\varepsilon(|\xi| + |\eta|)}$ on $V_r \times V_r^*$ $(\forall \varepsilon > 0)$
- $P(z, w, \xi, \eta) = \overline{P(\overline{w}, \overline{z}, \overline{\eta}, \overline{\xi})}$ on $V_r \times V_r^*$.

of Positive hermitian type

$\Leftrightarrow$
- of product h. type &
- $P(z, \overline{w}, \xi, \overline{\eta}) \in \mathcal{H}^+(V_{r'})$

<u>**Facts**</u>
- $P, Q \gg 0$ at $\mathring{p}$ $\Rightarrow$ $P+Q$, $P \cdot Q \gg 0$ at $\mathring{p}$
  operator product.
- $\mathring{p} = (\mathring{x}, \mathring{x}; i\mathring{\xi}(dx - du))$, $P \gg 0$, $k(x, u) \gg 0$ at $\mathring{p}$
  $\Rightarrow P(x, u, D_x, D_u) k(x, u) \gg 0$ at $\mathring{p}$

※ <u>0-classes</u> in Product h. type operators

$\overset{\text{def}}{\Leftrightarrow}$ $\exists \delta > 0, \exists r' > 0$ s.t.
$|P(z, w, \xi, \eta)| \le C_\varepsilon e^{\varepsilon|\xi| - \delta|\eta|}$ on $V_r \times V_{r'}^* \cap \{|\xi| \ge |\eta|\}$

Quasi-positivity of medium restricted h. Ψ.D.O.

We add the following conditions to restricted h.-Ψ.D.O. $:e^{P(z,w,\xi,\eta)}:$

Def. <u>Medium</u> $|P(z,w,\xi,\eta)| \le C(|\xi|\cdot|\eta|)^{\frac{\sigma}{2}}$ on $V\times V^*$

$\cup$

<u>Minimum</u> $|P(z,w,\xi,\eta)| \le C\min\{|\xi|^\sigma, |\eta|^\sigma\}$ on $V\times V^*$

Indeed, $P = \log(Q_\lambda + Q_\lambda^*)$ falls in {Medium} \ {Minimum}

(In general, {restricted} $\Rightarrow |P| \le C(|\xi|+|\eta|)^\sigma$)

Th (K: 1985)

$:e^{P_\lambda(z,w,\xi,\eta)}:$ one of medium, restricted h. type of order σ $(<\frac{1}{2})$ with a common constant $C>0$

$\Rightarrow$ $\sigma < {}^\forall\sigma' < \frac{1}{2}$, $\exists\, p \gg 0$ & e^p is of minimum & restricted h. type of order σ'

s.t. $:e^p::e^{P_\lambda}: \gg 0$ $({}^\forall\lambda)$

Ex. $0<\sigma<\frac{1}{2}$, $N = 2, 3, 4, \dots$

$$\boxed{P = \frac{(\xi\eta)^{N+\frac{\sigma}{2}}}{(\xi^N+\eta^N)^2}} \gg 0$$ & e^P is of minimum & restricted h. type

Our Conjecture.

We want to remove the order restriction ($0<\sigma<\frac{1}{2}$) in the preceding theorem. To do so, we extend the conditions on restricted h. type:

Def (J. Kumagai, 2000; Master thesis)

For a product h. type symbol $P(z,w,\xi,\eta)$ at $\mathring{p}$

$:e^{P(z,w,\xi,\eta)}:$ is said to be of <u>m-restricted h. type</u> ($m=1,2,\cdots$)

$$\Longleftrightarrow \begin{cases} 0<{}^{\exists}\sigma<\frac{m}{m+1}, \ {}^{\exists}C, {}^{\exists}r>0 \\ |\mathrm{grad}_{(z,w)}P| \le C\min\{|\xi|^{\sigma}, |\eta|^{\sigma}\} \\ |\partial_\xi^\alpha \partial_\eta^\beta P| \le C\min\{\underline{|\xi|^{\sigma-|\alpha|-|\beta|}, |\eta|^{\sigma-|\alpha|-|\beta|}}\} \end{cases}$$

$$({}^{\forall}\alpha,\beta \text{ s.t. } 0<|\alpha|+|\beta|\le m) \text{ on } V\times V^*$$

<u>Conjecture</u> ; $:e^{P_\lambda(z,w,\xi,\eta)}:$ of <u>medium</u>, <u>m-rest.</u> h. type of order σ ($<\frac{m}{m+1}$)

$\Rightarrow$ $\sigma<{}^{\forall}\sigma'<\frac{m}{m+1}$, ${}^{\exists}p\gg 0$ & e^{p} is of minimum & m-restricted, order σ'

s.t. $:e^{p}:\,:e^{P_\lambda}:\ \gg 0$ (${}^{\forall}\lambda$)

C. H. Lee's results

He generalized minimum type Ψ.D.O to any product space $X \times Y$ and proved that Aoki calculus works well for such $:e^{p(z,w,\xi,\eta)}:$ i.e.

$$\boxed{|p(z,w,\xi,\eta)| \leq C \min\{\Lambda_1(|\xi|), \Lambda_2(|\eta|)\}}$$

where $\Lambda_1(t)$, $\Lambda_2(t)$ are infra-linear weight functions.

Th (C.H. Lee, 2003)

For minimum type Ψ.D.O. on $X \times Y$, we have

① $\forall :e^{p_1(z,w,\xi,\eta)}: \, \forall :e^{p_2(z,w,\xi,\eta)}: \; = \; :e^{\exists p_3(z,w,\xi,\eta)}:$

② $\forall \; e^{:p(z,w,\xi,\eta):} = \; :e^{\exists \tilde{p}(z,w,\xi,\eta)}:$

Conversely $\forall$

$$:e^{\tilde{p}(z,w,\xi,\eta)}: = e^{:\exists p(z,w,\xi,\eta):}$$

Proofs are written in formal symbols

Lee's Proof

$\left(\begin{array}{c}\text{notation change}\\ z, w, \xi, \eta \Rightarrow z, z^*, \xi, \xi^*\end{array}\right)$

$$① \;:e^{p\left(\begin{smallmatrix}t & ; & z, \xi\\ t^* & ; & z^*, \xi^*\end{smallmatrix}\right)} :: e^{q\left(\begin{smallmatrix}t & ; & z, \xi\\ t^* & ; & z^*, \xi^*\end{smallmatrix}\right)}:$$

$$= \underbrace{\exp\left(t_2\hbar\, \partial_\xi \cdot \partial_w + \hbar_2^* t^*\, \partial_{\xi^*} \cdot \partial_{w^*}\right)\left(\exp\left(p\left(\begin{smallmatrix}t & ; & z, \xi\\ t^* & ; & z^*, \xi^*\end{smallmatrix}\right) + q\left(\begin{smallmatrix}t & ; & w, \eta\\ t^* & ; & w^*, \eta^*\end{smallmatrix}\right)\right)\right)}_{\substack{|||\\ G}}\Bigg|_{\substack{t_2 = \hbar_2^* = 1 \quad w = z,\ \eta = \xi\\ w^* = z^*,\ \eta^* = \xi^*}}$$

Hence, it is sufficient to calculate G.

Further, express G as

$$\underline{G = \exp\left(\sum_{k, k^* = 0}^{\infty} \hbar_2^k \hbar_2^{*\,k^*}\, r_{k,k^*}\left(\begin{smallmatrix}t & ; & z, \xi, w, \eta\\ t^* & ; & z^*, \xi^*, w^*, \eta^*\end{smallmatrix}\right)\right)}$$

Then we have the following recurrence formulas

$$\begin{cases} r_{0,0} = p\left(\begin{smallmatrix}t & ; & z, \xi\\ t^* & ; & z^*, \xi^*\end{smallmatrix}\right) + q\left(\begin{smallmatrix}t & ; & w, \eta\\ t^* & ; & w^*, \eta^*\end{smallmatrix}\right) \\ r_{k+1, k^*} = \dfrac{\hbar}{k+1}\left\{\partial_\xi \cdot \partial_w r_{k,k^*} + \displaystyle\sum_{\substack{k' + k'' = k\\ k'^* + k''^* = k^*}} \partial_\xi r_{k', k'^*} \cdot \partial_w r_{k'', k''^*}\right\} \\ r_{k, k^*+1} = \dfrac{\hbar^*}{k^*+1}\left\{ \text{similar terms} \right\} \end{cases}$$

The difference from Aoki's estimate:

$$r_{k,k^*} = \sum_{j=k,\ j^*=k^*}^{\infty} t^j t^{*j^*} \gamma_{(j,k),(j^*,k^*)}$$

NO USE of (l, l^*) in the formulas

That is, we have the following estimates:

$$|\gamma_{(j,k),(j^*,k^*)}| \le \sum_{\substack{0\le l\le k \\ 0\le l^*\le k^*}} B^{k+k^*} A^{j-k+j^*-k^*} (j+1)^{-2} (j^*+1)^{-2}$$

$$\times (k+1)^{k-l-3} (k^*+1)^{k^*-l^*-3}\, \varepsilon^{-2k} \varepsilon^{*\,-2k^*} |\xi|^{-k} |\xi^*|^{-k^*}$$

$$\times \underline{(\Lambda(|\xi|)+\Lambda(|\eta|))^{l} \cdot (\Lambda^*(|\xi^*|)+\Lambda^*(|\eta^*|))^{l^*} (\tilde\Lambda(\xi,\xi^*)+\tilde\Lambda(\eta,\eta^*))}$$

Here $\underline{\tilde\Lambda(\xi,\xi^*) = \min\{\Lambda(|\xi|), \Lambda^*(|\xi^*|)\}}$

We need l, l^*, but "$\underline{\gamma^{(l,l^*)}_{(j,k),(j^*,k^*)}}$" does not work. Further, In the estimates of non-linear terms, Replace one factor of $(\tilde\Lambda(\xi,\xi^*)+\tilde\Lambda(\eta,\eta^*))^2$ by $\underline{\Lambda(|\xi|)+\Lambda(|\eta|)}$ or $\Lambda^*(|\xi^*|)+\Lambda^*(|\eta^*|)$