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Research problems in number theory III.

By
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Abstract

Some open problems in the field of arithmetic functions are presented.

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§1. Introduction

In this paper we shall formulate some open problems, conjectures in the field of number theory. Some of them were formulated earlier in one of my papers [1], [2], [3], [4].

Notations. As usual $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ denote the set of natural numbers, integers, rational, real, or complex numbers, respectively. Let $\mathbb{Q}_x$, $\mathbb{R}_x$ be the multiplicative group of positive elements of $\mathbb{Q}, \mathbb{R}$, resp. . Let $\mathcal{P}$ be the set of prime numbers. Let $P(n)$ be the largest prime factor of n.

§2. Continuous homomorphisms as arithmetical functions

2.1. For some additively written commutative group G let $\mathcal{A}_G^*$ be the set of those functions $f : \mathbb{N} \to G$, for which $f(mn) = f(m) + f(n)$ holds for every couple of $m, n \in \mathbb{N}$.
We say that $\mathcal{A}_G^*$ is the class of completely additive functions.

Let G be multiplicatively written, commutative group, $\mathcal{M}_G^*$ be the class of those $g : \mathbb{N} \to G$, for which $g(mn) = g(m) \cdot g(n)$ for every pair of $m, n \in \mathbb{N}$. We say that $\mathcal{M}_G^*$ is the set of completely multiplicative functions.

If $f \in \mathcal{A}_G^*$, then its domain $\mathbb{N}$ can be extended to $\mathbb{Q}_x$ by

$$f\left(\frac{m}{n}\right) := f(m) - f(n),$$

and the equation

$$f(r_1 r_2) = f(r_1) + f(r_2)$$

remains valid for every $r_1, r_2 \in \mathbb{Q}_x$.

Let G be a topological group and $f : \mathbb{Q}_x \to G$, $f \in \mathcal{A}_G^*$ be continuous at 1. Then, for each $\alpha \in \mathbb{R}_x$ there exists the limit

$$\lim_{r \to \alpha} f(r) =: \Phi(\alpha),$$

Φ is continuous everywhere in $\mathbb{R}_x$, furthermore $\Phi(\alpha\beta) = \Phi(\alpha) + \Phi(\beta)$ is valid for all $\alpha, \beta \in \mathbb{R}_x$, i.e. ϕ is a continuous homomorphism of $\mathbb{R}_x$ into G.

The following conjectures 1,2 are proposed by M.V. Subbarao and myself.

Conjecture 1. *Let G be a compact Abelian topological group, $f \in \mathcal{A}_G^*$, and let the closure of $f(\mathbb{N})$ be G (closure $f(\mathbb{N})$ is a closed subgroup in G). Let U be the set of those u for which there exists an infinite sequence of integers $n_\nu \nearrow$, such that $f(n_\nu + 1) - f(n_\nu) \to u$.*

Then U is a subspace in G, furthermore $f(n) := \Phi(n) + V(n)$, where Φ is a continuous homomorphism, $\phi : \mathbb{R}_x \to G$, $V(\mathbb{N}) \subseteq U$, clos $V(\mathbb{N}) = U$.

The next conjecture is a special case of Conjecture 1.

Conjecture 2. *Let $f : \mathbb{N} \to \mathbb{C}$ be a completely multiplicative function, $|f(n)| = 1$ $(n \in \mathbb{N})$, $\delta_f(n) = f(n+1)\overline{f}(n)$.*

Let $A_k = \{\alpha_1, \ldots, \alpha_k\}$ be the set of limit points of $\{\delta_f(n) \mid n \in \mathbb{N}\}$. Then $A_k = S_k$, where S_k is the set of k'th complex units, i.e. $S_k = \{w \mid w^k = 1\}$, furthermore *$f(n) = n^{i\tau} F(n)$ with a suitable $\tau \in \mathbb{R}$, and $F(\mathbb{N}) = S_k$, and for every $w \in S_k$ there exists a sequence $n_\nu \nearrow \infty$ such that $F(n_\nu + 1)\overline{F}(n_\nu) = w$ $(\nu = 1, 2, \ldots)$.*

The motivation of the problems, and partial results can be read in [7], [8].

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A weakened form of Conjecture 2 has been proved by E. Wirsing recently [6]: *Under the conditions of* Conjecture 2, $\mathcal{A}_k \subseteq S_l$ *for a suitable* l, *and* $f(n) = n^{i\tau}F(n)$, $\mathbb{F}(\mathbb{N}) \subseteq S_l$.

2.2. Let $T = \mathbb{R}/\mathbb{Z}$ be the one-dimensional torus. Let $\mathcal{A}_T$ be the set of additive functions taking values from T, i.e. $F : \mathbb{N} \to T$ belongs to $\mathcal{A}_T$ if $F(mn) = F(m) + F(n)$ holds for all coprime pairs of m, n. We say that F is of finite support, if $F(p^\alpha) = 0$ holds for every large prime p, and every $\alpha \in \mathbb{N}$.

For $F_\nu \in \mathcal{A}_T(\nu = 0, \ldots, k-1)$ let

$$L_n(F_0, \ldots, F_{k-1}) := F_0(n) + \ldots + F_{k-1}(n + k - 1).$$

Let $\mathcal{L}_0^{(k)}$ be the space of those k-tuples $(F_0, \ldots, F_{k-1})$ of $F_\nu \in \mathcal{A}_T$ for which

$$L_n(F_0, \ldots, F_{k-1}) = 0 \quad (n \in \mathbb{N})$$

holds.

Conjecture 3. *$\mathcal{L}_0^{(k)}$ is a finite dimensional $\mathbb{Z}$ module, and each F_j is of finite support.*

Conjecture 4. *If $F_\nu \in \mathcal{A}_T$ $(\nu = 0, 1, \ldots, k-1)$, and*

$$L_n(F_0, \ldots, F_{k-1}) \to 0 \quad (n \to \infty)$$

then there exist suitable real numbers $\tau_0, \ldots, \tau_{k-1}$ such that $\tau_0 + \ldots + \tau_{k-1} = 0$, and for $H_j(n) := = F_j(n) - \tau_j \log n$ $(j = 0, \ldots, k-1)$ we have

$$L_n(H_0, \ldots, H_{k-1}) = 0 \quad (n = 1, 2, \ldots).$$

Conjecture 5. *For every integer $k(\geq 1)$ there exists a constant c_k such that for every prime p greater than c_k,*

$$\min_{j=1,\ldots,p-1} \max_{\substack{l \in [-k,k] \\ l \neq 0}} P(jp + l) < p.$$

The conjecture is open even in the case $k = 2$.

Let Q_x^l be the l-fold direct product of Q_x. Let furthermore O_l be its subgroup, generated by the elements $(n, n+1, \ldots, n+l-1)$ $(n \in \mathbb{N})$.

The following assertions are true:

(1) Let $\mathcal{L}_0^{*(l)}$ be the space of those l-tuples $(F_0, \ldots, F_{l-1})$ of $F_\nu \in \mathcal{A}_T^*$ for which $L_n(F_0, \ldots, F_{l-1}) = = 0$ $(n \in \mathbb{N})$. Assume that Conjecture 5 is true for $k = l$. Then $\mathcal{L}_0^{*(l)}$ is a finite dimensional space.

(2) $\mathcal{L}_0^{*(l)}$ (defined in (1)) is of finite dimensional, if and only if the factor group Q_x^l/O_l is finite. $\mathcal{L}_0^{*(l)}$ is trivial (it contains only $(0, \ldots, 0)$) if and only if $O_l = Q_x^l$.

2.3. Let $\mathcal{A}^* = \mathcal{A}_{\mathbb{R}}^*$.

Definition 1. (Set of uniqueness). We say that $E \subseteq \mathbb{N}$ is a set of uniqueness for the class of completely additive functions, if $f \in \mathcal{A}^*$, $f(E) = 0$ implies that $f(\mathbb{N}) = 0$.

Definition 2. (Set of uniqueness mod 1). We say that $E \subseteq \mathbb{N}$ is a set of uniqueness mod 1, if $f \in \mathcal{A}_T^*$, $f(E) = 0$ implies that $f(\mathbb{N}) = 0$.

I introduced the notion "set of uniqueness" in [10] and proved [11] that the set of "primes +1" can be extended by finitely many integers so that the resulting set is a set of uniqueness. My guess that the set of shifted primes itself is a set of uniqueness, was proved by Elliott [12]. It was proved by Wolke [13] that E is a set of uniqueness if and only if for every $n \in \mathbb{N}$ there exists a suitable $k \in \mathbb{N}$, such that

$$n^k = \prod_{i=1}^{h} a_i^{\varepsilon_i}, \quad \text{where} \quad a_i \in E,\ \varepsilon_i = \pm 1.$$

It was proved (by Meyer, Indlekofer, Dress and Volkman, Hoffman, Elliott, independently) that E is a set of uniqueness mod 1, if every $n \in \mathbb{N}$ can be written as

$$n = \prod_{j=1}^{s} a_j^{d_j}, \quad a_j \in E,\ d_j \in \mathbb{Z}, \quad (j = 1, \dots, s).$$

Conjecture 6. *The set of "prime +1" s is a set of uniqueness* mod 1.

Conjecture 6 is proposed by several mathematicians independently.
A quite detailed treatment of this topic is given by Elliott [14].
Indlekofer and Timofeev proved that $\{u^2 + v^2 + a \mid u, v \in \mathbb{Z}\}$ is a set of uniqueness mod 1, if $a \neq 0$.
The same result is proved by De Koninck and Kátai.

§3. On q-additive and q-multiplicative functions

Let $q \geq 2$ be an arbitrary integer, $\mathcal{E} = \{0, 1, \dots, q-1\}$ and let $\varepsilon_0(n),\ \varepsilon_1(n), \dots$ be the digits in the q-ary expansion of $n : n = \varepsilon_0(n) + \varepsilon_1(n)q + \dots$. This is a finite expansion, since $\varepsilon_j(n) = 0$ if $q^j > n$.
Let $f : \mathbb{N}_0 \to \mathbb{R}$ be such a function for which $f(0) = 0$, and

$$f(n) = \sum_{j=0}^{\infty} f(\varepsilon_j(n)q^j)$$

holds for every n. We say that f is q-additive, and the set of q-additive functions is denoted by $\mathcal{A}_q$.

We say that $g : \mathbb{N}_0 \to \mathbb{C}$ is q-multiplicative if $g(0) = 1$, and $g(n) = \prod_{j=0}^{\infty} g(\varepsilon_j(n)q^j)$ holds for every n. Let $\mathcal{M}_q$ be the set of q-multiplicative functions, and $\overline{\mathcal{M}}_q$ be those of $\mathcal{M}_q$ for the elements $g \in \overline{\mathcal{M}}_q$ additionally $|g(n)| = 1$ $(n \in \mathbb{N}_0)$ holds as well.

Let $g \in \overline{\mathcal{M}}_q$,

$$P(x) = \sum_{p \leq x} g(p), \quad S(x \mid \alpha) = \sum_{\substack{l < x \\ (l,q)=1}} g(l)e(\alpha l)$$

where $e(y) := e^{2\pi i y}$.

We are interested in to give necessary and sufficient conditions for g to satisfy

$$\lim_{x \to \infty} \frac{P(x)}{\pi(x)} = 0. \tag{3.1}$$

Conjecture 7. *Let $g \in \overline{\mathcal{M}}_q$. Then* (3.1) *holds if and only if*

$$x^{-1} S(x, r) \to 0 \tag{3.2}$$

for every $r \in \mathbb{Q}$.

The necessity of (7.2) is quite obvious, since if it does not hold, then

$$\sum_{j=0}^{\infty} \sum_{c \in E} \text{Re } (1 - g(cq^j)e(cq^j r)) < \infty,$$

whence one can prove easily that (3.2) cannot hold. The difficulty is in the sufficienty.

Let $T^M_{l_1,l_2} = T_{l_1,l_2} =$

$$= \#\{p_1, p_2 \in \mathcal{P},\ p_2 - p_1 = l_2 - l_1,\ p_1 \equiv l_1 \pmod{q^M},\ p_2 \leq x\},$$

$$H(d) := \prod_{\substack{p|d \\ p\nmid 2q}} \left(1 + \frac{1}{p-2}\right).$$

Conjecture 8. *There exists a constant $\delta \in (0, 1/2)$, such that for $M = [\delta N]$, $N = \left[\frac{\log x}{\log q}\right]$,*

$$\sum_{\substack{l_1, l_2 < q^M \\ (l_1 l_2, q) = 1 \\ l_1 \neq l_2}} \left| T_{l_1, l_2}^{(M)} - \frac{x}{\varphi(q^M)(\log x)^2} H(l_2 - l_1) \right| < \frac{\varepsilon(x) x \cdot q^M}{(\log x)^2}$$

with a suitable function $\varepsilon(x) \to 0$ $(x \to \infty)$.

In [15] we proved that Conjecture 8 implies the fulfilment of Conjecture 1.

Furthermore in [15] we proved the following assertion: Let $Y(x) \nearrow \infty$, so that $\frac{\log Y(x)}{\log x} \to 0$. Let $\mathcal{N}_x := \{n \le x,\ p(n) > Y(x)\}$, where $p(n)$ is the smallest prime factor of n.

Let $N(x) = \text{card}\ (\mathcal{N}_x)$. Let $L(x)$ be strongly multiplicative, $(L(p^h) =) L(p) = \frac{1}{p-2}$ if $p \nmid 2q$, and $L(p) = 0$ otherwise. Let

$$U(x) := \sum_{n \in \mathcal{N}_x} g(n).$$

Then

$$\left| \frac{U(x)}{N(x)} \right|^2 \le \sum_{d < D} \frac{L(d)}{d} \sum_{a=0}^{d-1} \left| q^{-M} S\left(q^M \;\middle|\; \frac{a}{d}\right) \right|^2 + \frac{c_1}{D} + o_x(1),$$

where $M = \left[\frac{1}{4} \frac{\log x}{\log q}\right]$, c_1 is a positive constant which depends only on q, $o_x(1)$ does depend on $Y(x)$, and D is an arbitrary real numbers.

§4. The distribution of q-ary digits on some subsets of integers

4.1. Let $\mathcal{B} (\subseteq \mathbb{N}_0)$ be infinite, $B(x) = \#\{b \le x,\ b \in \mathcal{B}\}$. For $0 \le l_1 < l_2 < \ldots < l_h$, $b_1, \ldots, b_h \in E$, let $A_{\mathcal{B}}\left(x \;\middle|\; \begin{matrix} l \\ b \end{matrix}\right)$ be the size of those integers $n \in \mathcal{B}$, $n \le x$, for which $\varepsilon_{l_j}(n) = b_j$ $(j = 1, \ldots, h)$ simultaneously hold.

Conjecture 9. *For every $h \left(\le \frac{N}{3}\right)$, $1 \le l_1 < \ldots < l_h (\le N)$, and $b_1, \ldots, b_h \in \mathcal{E}$ denote*

$$\left(\Delta_h \begin{pmatrix} l \\ b \end{pmatrix} =\right) \Delta_h \begin{pmatrix} l_1, \ldots, l_h \\ b_1, \ldots, b_h \end{pmatrix} := \frac{q^h A_{\mathcal{P}} \left(q^n \;\middle|\; \begin{matrix} l \\ b \end{matrix}\right)}{\pi(q^N)} - 1.$$

Then

$$\sup_{1 \le h \le \frac{N}{3}} \sup_{\substack{l_1, \ldots, l_h \\ b_1, \ldots, b_h}} \left| \Delta_h \begin{pmatrix} l \\ b \end{pmatrix} \right| \to 0 \quad \text{as} \quad N \to \infty. \tag{4.1}$$

Here $\mathcal{P}$ is the set of primes.

Remarks. 1. Inequality, similar but much weaker than (4.1) was proved in [16].

2. These type of inequalities would be interesting for other sets $\mathcal{B}$ instead of $\mathcal{P}$, like $\mathcal{B} = \{$fixed polynomial $(n) \mid n \in \mathbb{N}\}$, or $=\{$fixed polynomial $(p) \mid p \in \mathcal{P}\}$. We would be able to use them in proving

central limit theorems with remainder terms for $f(P(n))$, or $f(P(p))$, where $f \in \mathcal{A}_q$, $P =$ polynomial. (See [17], [18], [19], [20], [21]).

4.2.

Conjecture 10. *If* $g \in \overline{\mathcal{M}}_q$, $g(p) = 1$ *for every* $p \in \mathcal{P}$, *then* $g(nq) = 1$ $(n \in \mathbb{N})$.

See [22], where it is proved that there exists an absolute constant $c\,(> 0)$ such that $g \in \overline{\mathcal{M}}_q$, $g(p) = 1$ implies that there exists an integer k, $1 \le k \le c$ for which $g^k(nq) = 1$ $(n \in \mathbb{N})$.

§5. On a theorem of H. Daboussi

H. Daboussi proved several years ago that for $f \in \mathcal{A}$, $|f(n)| \le 1$, and for every irrational α, in the notation

$$m(f,\alpha,x) := \frac{1}{x}\left|\sum_{n\le x} f(n)e(n\alpha)\right|,$$

we have

$$\sup_{f\in\mathcal{A},\ |f|\le 1} m(f,\alpha,x) \to 0 \quad (x\to\infty).$$

This theorem has been generalized in different directions.

Let $\mathcal{P}_k$ be the set of square-free numbers n with exactly n prime-factors: $n = p_1p_2\dots p_k$. Let α be an irrational number. Let $q_1 < q_2 < \dots < q_r$ be the whole set of primes less than x. Let X_{q_j} $(j = 1,\dots,r)$ be complex numbers,

$$Q_k\left(X_{q_1},\dots,X_{q_r}\right) := \left|\sum_{\substack{n=p_1\dots p_k<x\\ n\in\mathcal{P}_k}} X_{p_1}\dots X_{p_k}e(n\alpha)\right|.$$

Let

$$\delta_k(x) := \max_{|X_{q_1}|\le 1,\dots,|X_{q_r}|\le 1} \frac{Q_k(X_{q_1},\dots,X_{q_r})}{\tilde{\pi}_k(x)},$$

$$\delta_k := \limsup_{x\to\infty} \delta_k(x),$$

where $\tilde{\pi}_k(x)$ is the number of $n \le x$, $n \in \mathcal{P}_k$.

Conjecture 11. *We have* $\delta_k < 1$, *if* $k \ge 2$. *Furthermore* $\delta_k \to 0$ *(if* $k\to\infty$*).*

Remark. Recently I could prove that $\delta_2 = 0$ for almost all α.

§6. Some problems originated from Rényi-Parry expansions

See our papers written jointly with Daróczy [23 - 26].

Let $\mathbb{C}^\infty$ denote the space of sequences $\underline{c} = (c_0, c_1, \dots)$ the coordinates c_ν of which $\in \mathbb{C}$. This shift operator $\sigma : \mathbb{C}^\infty \to \mathbb{C}^\infty$ is defined by $\sigma((c_0,c_1,\dots)) = (c_1,c_2,\dots)$. Let $t_0 = 1$, $t_\nu \in \mathbb{C}$, t_ν be bounded, $\underline{t} = (t_0, t_1, \dots)$. Let

$$R(z) = t_0 + t_1 z + \dots . \tag{6.1}$$

Let l_1 be the linear space of those $\underline{b} \in \mathbb{C}^\infty$, for which $\sum |b_\nu| < \infty$.
The scalar product of an element $\underline{b} \in l_1$ and a bounded sequence $\underline{c}$ let:

$$\underline{c}\,\underline{b} = \underline{b}\,\underline{c} = \sum_{\nu=0}^{\infty} b_\nu c_\nu.$$

Let

$$\mathcal{H}_t := \{\underline{b} \in l_1 \mid \sigma^l(\underline{b})\underline{t} = 0,\ l = 0,1,2,\dots\}. \tag{6.2}$$

It is clear that $\mathcal{H}_t$ is a closed linear subspace of l_1, furthermore $\sigma(\mathcal{H}_t) \subseteq \mathcal{H}_t$.

Let $\mathcal{H}_t^{(0)} \subseteq \mathcal{H}_t$ be the set of those $\underline{b} \in \mathcal{H}_{\underline{t}}$ for which

$$|b_\nu| \le C(\varepsilon, \underline{b}) e^{-\varepsilon\nu} \quad (\nu \ge 0) \tag{6.3}$$

holds with some $\varepsilon > 0$ and finite $C(\varepsilon, \underline{b})$.

If ρ is a root of $R(z)$, $|\rho| < 1$, then $b_\nu := \rho^\nu$ satisfies $\sigma^l(\underline{b})\underline{t} = 0$ $(l = 0,1,\dots)$, and $|b_\nu| \le C \cdot e^{-\varepsilon\nu}$ with $C = 1$, and with ε counted from $e^{-\varepsilon} = |\rho|$. If the order of the multiplicity of the root ρ is m, then $\underline{b} \in \mathcal{H}_t$, if $b_\nu = \nu^j \rho^\nu$ $(\nu = 0,1,\dots)$, for every $j = 0,1,\dots,m-1$. The sequences $b_\nu = \nu^j\rho^\nu$ $(\nu = 0,1,\dots)$ are called elementary solutions. Let $\mathcal{H}_t^{(e)}$ be the space of finite linear combinations of the elementary solutions, and let $\overline{\mathcal{H}}_t^{(e)}$ be the closure of $\mathcal{H}_t^{(e)}$.

Conjecture 12. *We have:* $\overline{\mathcal{H}}_t^{(e)} = \mathcal{H}_t$.

Conjecture 13. *Assume that* $R(z) \ne 0$ *in* $|z| < 1$. *Then* $\mathcal{H}_{\underline{t}} = \{0\}$.

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