ASYMPTOTIC SERIES ASSOCIATED WITH EPSTEIN ZETA-FUNCTIONS AND THEIR INTEGRAL TRANSFORMS

Masanori Katsurada

Mathematics, Hiyoshi Campus, Keio University

1. INTRODUCTION

Throughout the following, $s = \sigma + it$ denotes the complex variable, and $z = x + iy$ the complex parameter in the upper-half plane. The main object of this article is the Epstein zeta-function (attached to the positive-definite quadratic form $|u+ vz|^2$) defined by

$\zeta_{Z^2}(s; z) = \sum_{(m,n) \in Z^2 \setminus \{(0,0)\}} |m+nz|^{-2s} \quad (\text{Re } s > 1),$

and its meromorphic continuation over the whole $s$-plane (cf. [Si Chap. I]).

Let $\alpha, \beta$ be complex numbers which will be fixed later, and let $\Gamma(s)$ denote the gamma function. We introduce the Laplace-Mellin and the Riemann-Liouville (or the Erdélyi-Kober) transforms of $\zeta_{Z^2}(s; x+iy)$ (with the normalization multiples) as

$\mathcal{L}\mathcal{M}_{y; Y}^{\alpha} \zeta_{Z^2}(s; x+iy) = \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} \zeta_{Z^2}(s; x+iyY) y^{\alpha-1} e^{-y} dy,$

$\mathcal{R}\mathcal{L}_{y; Y}^{\alpha, \beta} \zeta_{Z^2}(s; x+iy) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_{0}^{1} \zeta_{Z^2}(s; x+iyY) y^{\alpha-1} (1-y)^{\beta-1} dy$

for $Y > 0$. These can be regarded as weighted mean values of $\zeta_{Z^2}(s; x+iy)$; the factor $y^{\alpha-1}$ is inserted to secure the convergence of the integrals as $y \to +0$, while the functions $e^{-y}$ and $(1-y)^{\beta-1}$ have effects to extract the parts corresponding to $y = O(Y)$ from $\zeta_{Z^2}(s; z)$ with their respective weights. Note that the confluence operation

$\mathcal{R}\mathcal{L}_{y; Y}^{\alpha, \beta} \zeta_{Z^2}(s; x+iy) \xrightarrow{\beta \to +\infty} \mathcal{L}\mathcal{M}_{y; Y}^{\alpha} \zeta_{Z^2}(s; x+iy)$

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is valid by the definitions (1.2) and (1.3), since \( \zeta_{\mathbb{Z}^2}(s; x + iy) = O(y^{\max(0, 1-2\sigma)}) \) as \( y \to +\infty \) (see Theorem 1 below).

It is of importance from both theoretical and applicational point of view to study asymptotic aspects of \( \zeta_{\mathbb{Z}^2}(s; z) \) when \( y = \Im z \) is large (cf. [CS1–CS2]). We have established in [Ka10] a complete asymptotic expansion of \( \zeta_{\mathbb{Z}^2}(s; z) \) when \( \Im z \to +\infty \), and that of the Laplace-Mellin transform (1.2) when \( Y \to +\infty \). The subsequent paper [Ka11] proceeds to this direction by showing that a similar asymptotic series still exists for the Riemann-Liouville transform (1.3) when \( Y \to +\infty \). It is the aim of this article to present these asymptotic expansions, together with their several consequences.

We first present a complete asymptotic expansion of \( \zeta_{\mathbb{Z}^2}(s; z) \) when \( \Im z \to +\infty \) (Theorem 1 below) upon giving an explicit (vertical) \( t \)-estimate for the remainder term. This theorem in particular clarifies the key ingredients by which the functional equatioan of \( \zeta_{\mathbb{Z}^2}(s; z) \) is to be valid (Corollary 1.1). Moreover, several specific cases of Theorem 1 naturally reduce to the Kronecker limit formula for \( \zeta_{\mathbb{Z}^2}(s; z) \) when \( s \to 1 \), and to its variants for \( \zeta_{\mathbb{Z}^2}(m; z) \) \( (m = 2, 3, \ldots) \) and \( \zeta_{\mathbb{Z}^2}(-n; z) \) \( (n = 0, 1, \ldots) \), where \( \zeta_{\mathbb{Z}^2}(s; z) = (\partial/\partial s)\zeta_{\mathbb{Z}^2}(s; z) \) (Corollaries 1.2 and 1.3). In connection with Theorem 1, Matsumoto [Ma] obtained asymptotic expansions (with respect to \( z \)) of holomorphic Eisenstein series, while Noda [No] derived an asymptotic formula (as \( t \to +\infty \)) for the non-holomorphic Eisenstein series on the line \( \sigma = 1/2 \). We next present complete asymptotic expansions of the Laplace-Mellin transform (1.2) and of the Riemann-Liouville transform (1.3) both when \( Y \to +\infty \) (Theorems 2 and 3 in Section 3). One can observe that the asymptotic expansion of (1.3) precisely reduces to that of (1.2) through the confluence operation (1.4). It should be noted that various hypergeometric functions appear and work in the proofs of these expansions; especially their summation and transformation properties play crucial rôles in the analysis of the remainder terms.

Prior to the proof of Theorem 1, we have prepared the analytic continuation of \( \zeta_{\mathbb{Z}^2}(s; z) \) by means of Mellin-Barnes integral transformations (cf. [Ka10, Propositions 1 and 2]). This procedure was recently developed, independently of each other, by Kanemitsu-Tanigawa-Yoshimoto [KTY] (in a more general setting), and by the author [Ka10] for \( \zeta_{\mathbb{Z}^2}(s; z) \); the procedure, differs slightly from other previously known method of the analytic continuation, gives a new alternative proof of the Fourier expansion of \( \zeta_{\mathbb{Z}^2}(s; z) \), due to Chowla-Selberg [CS1–CS2]. We remark that Mellin-Barnes transformation technique was extensively utilized by Motohashi to investigate higher power moments of zeta and allied functions (see for e.g., [Mo1–Mo3]). The technique was also applied by the author [Ka1–Ka9] to study certain asymptotic aspects and transformation properties of zeta and theta functions.

2. RESULTS ON \( \zeta_{\mathbb{Z}^2}(s; z) \)

We write \( \sigma_{w}(l) = \sum_{0<h \mid l} h^{w} \), and use the notations \( e(z) = e^{2\pi i z} \) and

\[
e^{*}(z) = e(z) + \overline{e(z)} = e(z) + e(-\overline{z}),
\]
where $\overline{w}$ denotes the complex conjugate of $w$. We further introduce the function

$$\Phi_{r,s}^{*}(e(z)) = \sum_{h,k=1}^{\infty} h^{r}k^{s} e^{*}(hkz) = \sum_{l=1}^{\infty} \sigma_{r-s}(l) l^{s} e^{*}(lz),$$

which converges absolutely for all complex $r$, $s$ if $\text{Im} z > 0$, and for $\text{Re} r < -1$, $\text{Re} s < -1$ if $\text{Im} z = 0$; in each case it defines a holomorphic function of $r$ and $s$ in the region of absolute convergence.

Let $\zeta(s)$ be the Riemann zeta-function, and $(s)_n = \Gamma(s + n)/\Gamma(s)$ for any integer $n$ Pochhammer's symbol. Further let $U(\lambda; \nu; Z)$ denote the confluent hypergeometric function defined by

$$U(\lambda; \nu; Z) = \frac{1}{\Gamma(\lambda)} \int_{0}^{\infty} e^{-Zw} w^{\lambda-1} (1+w)^{\nu-\lambda-1} dw$$

for $\text{Re} \lambda > 0$ and $|\arg Z| < \pi/2$ (cf. [Sl, p.5, 1.3]). Then our first main result asserts

**Theorem 1.** ([Ka10, Theorem 1]). Let $\zeta_{Z^2}(s;z)$ be defined by (1.1). Then for any complex $z = x + iy$ with $y > 0$ and any integer $N \geq 0$ the formula

$$\zeta_{Z^2}(s;z) = 2\zeta(2s) + \frac{2\sqrt{\pi} \Gamma(s-1/2)}{\Gamma(s)} \zeta(2s-1)y^{1-2\epsilon} + \frac{2(2\pi)^{2\epsilon}}{\Gamma(s)} \{S_{N}(s,x;y) + R_{N}(s,x;y)\}$$

holds in the region $-N < \sigma < 1 + N$ except at $s = 1$. Here

$$S_{N}(s;z) = \sum_{n=0}^{N-1} \frac{(-1)^n (s)_n (1-s)_n}{n!} \Phi_{s-n-1,-\epsilon-n}^{*}(e(z))(4\pi y)^{-\epsilon-n}$$

is the asymptotic series in the descending order of $y$, and $R_{N}$ is the remainder term, which is expressed as

$$R_{N}(s;z) = \frac{(-1)^N (s)_N (1-s)_N}{(N-1)!} \sum_{h,k=1}^{\infty} e^{*}(hkz)h^{2s-1} \times \int_{0}^{1} \xi^{s-N}(1-\xi)^{N-1} U(s+N;2s;4\pi hky/\xi) d\xi$$

for $N \geq 0$ (the case $N = 0$ should read without the factor $(-1)!$ and the $\xi$-integration), satisfying the estimate

$$R_{N}(s;z) = O\{(|t| + 1)^{2N} e^{-2\pi y y^{-\sigma-N}}\}$$
for any $y \geq y_0 > 0$, in the region $-N < \sigma < 1 + N$, where the $O$-constant depends on $N$, $\sigma$ and $y_0$.

Remark. We see that

$$\Phi_{r,s}^*(e(z)) = e^*(z) + O\left\{ \sum_{l=2}^{\infty} l^{\max(\Re r, \Re s)+\epsilon} |e^*(lz)| \right\} = e^*(z) + O(e^{-4\pi y})$$

as $y \to +\infty$, and hence

$$\Phi_{r,s}^*(e(z)) \ll e^{-2\pi y} \quad (y \geq y_0 > 0).$$

Therefore the term with the index $n$ in $S_N(s;z)$ is estimated as $\ll (|t|+1)^{2n} e^{-2\pi y} y^{-\sigma-n}$; this shows that the presence of the bound above for $R_N(s;z)$ is reasonable.

Let $\zeta_{\mathbb{Z}^2}(s;z)$ be defined by

$$\zeta_{\mathbb{Z}^2}(s;z) = 2\zeta(2s) + \frac{2\sqrt{\pi}\Gamma(s-1/2)}{\Gamma(s)} \zeta(2s-1)y^{1-2\epsilon} + 2\zeta_{\mathbb{Z}^2}^*(s;z).$$

Then the proof of Theorem 1 show that the following functional equation of $\zeta_{\mathbb{Z}^2}(s;z)$ reduces eventually to the simple property

$$\Phi_{r,s}^*(e(z)) = \Phi_{s,r}^*(e(z)).$$

Corollary 1.1. ([Ka10, Corollary 1.1]). For any real $x$, $y$ with $y > 0$ the functional equation

$$(y/\pi)^{s} \Gamma(s) \zeta_{\mathbb{Z}^2}(s;z) = (y/\pi)^{1-s} \Gamma(1-s) \zeta_{\mathbb{Z}^2}(1-s;z)$$

follows, and this with the functional equation of $\zeta(s)$ implies that

$$(y/\pi)^{s} \Gamma(s) \zeta_{\mathbb{Z}^2}(s;z) = (y/\pi)^{1-s} \Gamma(1-s) \zeta_{\mathbb{Z}^2}(1-s;z).$$

We next state the Kronecker limit formula for $\zeta_{\mathbb{Z}^2}(s;z)$ and its variants. Let $\eta(z) = e(z/24) \prod_{n=1}^{\infty}(1-e(nz))$ be the Dedekind eta function, $\gamma_0 = -\Gamma'(1)$ Euler's constant, and $B_n$ the $n$-th Bernoulli number (cf. [Er, p.35, 1.13(1)]). Then

Corollary 1.2. ([Ka10, Corollary 1.2]). For any complex $z = x + iy$ with $y > 0$ the following formulæ hold:

$$\lim_{s \to 1} \left\{ \zeta_{\mathbb{Z}^2}(s;z) - \frac{\pi/y}{s-1} \right\} = \frac{\pi^2}{3} + \frac{2\pi}{y} \{ \gamma_0 - \log(2y) + \Phi_{0,-1}^*(e(z)) \}$$

$$= \frac{2\pi}{y} \{ \gamma_0 - \log(3y|\eta(z)|^2) \},$$

and for any integer $m \geq 2$,

$$\zeta_{\mathbb{Z}^2}(m;z) = \frac{(-1)^{m+1}(2\pi)^{2m} B_{2m}}{(2m)!} + \frac{2\pi(2m-1)!}{(2m-1)(m-1)!!^2} \zeta(2m-1)y^{1-2m}$$

$$+ \frac{(2\pi)^{2m}}{(m-1)!!^2} \sum_{n=0}^{m-1} \binom{m-1}{n} (m+n-1)!$$

$$\times \Phi_{m-n-1,-m-n}^*(e(z))(4\pi y)^{-m-n}. $$
Corollary 1.3. ([Ka10, Corollary 1.3]). Let $\zeta'_{Z^2}(s;z) = (\partial/\partial s)\zeta_{Z^2}(s;z)$. Then for any complex $z = x + iy$ with $y > 0$ the following formulae hold:

$$
\zeta'_{Z^2}(0;z) = -2\log 2\pi + \frac{\pi y}{3} + 2\Phi_{-1,0}^{*}(e(z)) = -2\log(2\pi|\eta(z)|^2),
$$

and for any integer $m \geq 1$,

$$
\zeta'_{Z^2}(-m;z) = \frac{2(-1)^m(2m)!}{(2\pi)^{2m}} \zeta(2m + 1) + \frac{2\pi(2^m m!)^2 B_{2m+2}}{(2m+1)!(m+1)} y^{2m+1} + \frac{2(-1)^m}{(2\pi)^{2m}} \sum_{n=0}^{m} \binom{m}{n} (m+n) ! \Phi_{-m-n-1,m-n}^{*}(e(z))(4\pi y)^{m-n}.
$$

3. Results on $L_{\alpha,Y} \zeta_{Z^2}(s;z)$ and $R_{\alpha,Y} \zeta_{Z^2}(s;z)$

We write

$$
\Gamma_{\alpha_1,\ldots,\alpha_m}^{\beta_1,\ldots,\beta_n} = \frac{\prod_{h=1}^{m} \Gamma(\alpha_h)}{\prod_{k=1}^{n} \Gamma(\beta_k)}
$$

for complex numbers $\alpha_h, \beta_k (1 \leq h \leq m; 1 \leq k \leq n)$, and denote the generalized hypergeometric function by $mF_n(\alpha_1,\ldots,\alpha_m; \beta_1,\ldots,\beta_n; z)$ for $m \leq n + 1$. Then our second main result can be stated as

Theorem 2. ([Ka10, Theorem 2]). Let $\alpha$ be fixed with $\mathrm{Re}\alpha > 1$. Then for any integer $N \geq 0$ and any real $x, Y$ with $Y > 0$ the formula

$$
\begin{align*}
L_{\alpha,Y} \zeta_{Z^2}(s;x+iy) &= 2\zeta(2s) + 2\sqrt{\pi} \Gamma\left(\frac{s-1/2}{s}, \frac{\alpha + 1 - 2s}{\alpha}\right) \zeta(2s - 1) Y^{1-2s} \\
&+ \frac{2\pi^{2s}}{\Gamma(s)} \{S_{\alpha,N}(s,x;Y) + R_{\alpha,N}(s,x;Y)\}
\end{align*}
$$

holds in the region $\sigma < \Re\alpha/2$. Here

$$
S_{\alpha,N}(s,x;Y) = \sum_{n=0}^{N-1} \frac{(-1)^n(\alpha)_n}{n!} \Gamma\left(\frac{\alpha + n + 1/2}{\alpha + n + 1/2}\right) \Phi_{a-\alpha-n,\alpha}^{*}(e(x))(2\pi Y)^{-\alpha-n}
$$

is the asymptotic series in the descending order of $Y$, and $R_{\alpha,N}$ is the remainder term, which is expressed as

$$
R_{\alpha,N}(s,x;Y) = \frac{(-1)^N}{(N-1)!} \Gamma\left(\frac{\alpha + 1 - 2s}{\alpha + 1 - s}\right) \sum_{h,k=1}^{\infty} e^{*}(hkx)h^{2s-1} \\
\times \int_{0}^{1} \xi^{-\alpha-N}(1-\xi)^{N-1}(1+2\pi hkY/\xi)^{-\alpha-N} \\
\times 2F_1\left(\alpha+N, s; 1-2\pi hkY/\xi, 1+2\pi hkY/\xi\right)d\xi
$$

for $N \geq 0$ (the case $N = 0$ should read without the factor $(-1)!$ and the $\xi$-integration), satisfying the estimate
\[
R_{\alpha,N}(s, x; Y) = O(Y^{-\operatorname{Re} \alpha - N})
\]
for any $Y \geq Y_0 > 0$, in the region $\sigma < \operatorname{Re} \alpha / 2$, where the $O$-constant depends at most on $\alpha$, $N$, $\sigma$, $t$ and $Y_0$. In particular when $\alpha \in \mathbb{R}$, more explicitly
\[
R_{\alpha,N}(s, x; Y) = O\left\{e^{-\pi|t|/2}(|t| + 1)^{(\alpha + N)/2 - \sigma}Y^{-\alpha - N}\right\}
\]
for any $Y \geq Y_0 > 0$ in the region $\sigma < \alpha/2$, where the $O$-constant depends at most on $\alpha$, $N$ and $\sigma$.

Remark 2.1. The condition $\operatorname{Re} \alpha > 1$ is crucial for the convergence of $\mathcal{L}\mathcal{M}_{y;Y}^\alpha \zeta_{\mathbb{R}}^2(s; x + iy)$, especially for that of $\mathcal{L}\mathcal{M}_{y;Y}^\alpha \zeta_{\mathbb{R}}^2(s; x + iy)$.

Remark 2.2. It is seen that
\[
|f_{r, s}(e(x))| \leq 2\zeta(-\operatorname{Re} r)\zeta(-\operatorname{Re} s) < +\infty
\]
for $\operatorname{Re} r < -1$, $\operatorname{Re} s < -1$, and hence when $\alpha \in \mathbb{R}$ the term with the index $n$ in $S_{\alpha,N}(s, x; Y)$ is estimated as $\ll e^{-\pi|t|/2}(|t| + 1)^{(\alpha + n)/2 - \sigma}Y^{-\alpha - n}$; this shows that the presence of the bound for $R_{\alpha,N}(s, x; Y)$ above is reasonable.

It is in fact shown that $\lim_{N \to \infty} R_{\alpha,N}(s, x; Y) = 0$ for $\sigma < \operatorname{Re} \alpha/2$ and $Y > 1/2\pi$. The limiting case $N \to \infty$ of Theorem 2 therefore gives

Corollary 2.1. ([Kal10, Corollary 2.1]). For any real $x$, $Y$ with $Y > 1/2\pi$ the formula
\[
\mathcal{L}\mathcal{M}_{y;Y}^\alpha \zeta_{\mathbb{R}}(s; x + iy) = 2\zeta(2s) + 2\sqrt{\pi}\Gamma\left(s - 1/2, \alpha + 1 - 2s\right)\zeta(2s - 1)Y^{1 - 2s}
+ \frac{2\pi^{2s}}{\Gamma(s)}S_{\alpha}(s, x; Y)
\]
holds in the region $\sigma < \operatorname{Re} \alpha/2$, where
\[
S_{\alpha}(s, x; Y) = \sum_{n=0}^{\infty} \frac{(-1)^n(\alpha)_n}{n!} \Gamma\left(\frac{\alpha + n + 1}{2} - s\right) \times f_{2\epsilon - 1 - \alpha - n, -\alpha - n}(e(x))(2\pi Y)^{-\alpha - n}.
\]

We next proceed to state our third main result.

Theorem 3. ([Kal11, Theorem 2]). Let $\alpha, \beta$ be fixed with $\operatorname{Re} \alpha > 1$, $\operatorname{Re} \beta > 1$. Then for any integer $N \geq 0$ and any real $x$, $Y$ with $Y > 0$ the formula
\[
\mathcal{R}\mathcal{M}_{y;Y}^{\alpha,\beta} \zeta_{\mathbb{R}}^2(s; x + iy) = 2\zeta(2s) + 2\sqrt{\pi}\Gamma\left(s - 1/2, \alpha + \beta, \alpha + 1 - 2s\right)\zeta(2s - 1)Y^{1 - 2s}
+ 2\pi^{s}\Gamma\left(\frac{\alpha + \beta}{s}\right)\{S_{\alpha,\beta}(s, x; Y) + R_{\alpha,\beta,N}(s, x; Y)\}
\]
holds in the region \( \sigma < \Re \alpha/2 \). Here

\[
S_{\alpha,\beta,N}(s, x; Y) = \sum_{n=0}^{N-1} \frac{(-1)^n(\alpha)_n}{n!} \Gamma\left(\frac{(\alpha + n + 1)/2 - s}{\beta - n}\right) \times \Phi^*_{2s-1-\alpha-n,-\alpha-n}(e(x))(2\pi Y)^{-\alpha-n}
\]
is the asymptotic series in the descending order of \( Y \), and \( R_{\alpha,\beta,N} \) is the remainder term, which is expressed as

\[
R_{\alpha,\beta,N}(s, x; Y) = \frac{2^{2s}(-1)^N(\alpha)_N}{(N-1)!} \sum_{h,k=1}^{\infty} e^{*}(hkx) h^{2s-1} \times \int_{0}^{1} \xi^{-\alpha-N}(1-\xi)^{N-1} F_{\alpha+N-N}(s;2\pi hkY/\xi)d\xi
\]
for any \( N \geq 0 \) (the case \( N = 0 \) should read without the factor \((-1)!\) and the \( \xi \)-integration), where

\[
F_{\alpha,\beta}(s; Z) = \Gamma\left(\frac{1-2s}{1-s, \alpha+\beta}\right) _{2} F_{3}\left(\frac{\alpha/2, (\alpha+1)/2}{(\alpha+\beta)/2, (\alpha+\beta+1)/2, s+1/2}; Z^2/4\right)
\]

\[
+\Gamma\left(\frac{2s-1, \alpha+1-2s}{s, \alpha+\beta+1-2s}\right) _{2} F_{3}\left(\frac{(\alpha+1)/2-s, \alpha/2+1-s}{(\alpha+\beta+1)/2-s, (\alpha+\beta)/2+1-s, 3/2-s}; Z^2/4\right)
\]

with \( \alpha, \beta \) replaced by \( \alpha+n, \beta-N \), and it satisfies

\[
R_{\alpha,\beta,N}(s, x; Y) = O(Y^{-\Re \alpha-N})
\]
for any \( Y \geq Y_0 > 0 \) in the region \( \sigma < \Re \alpha/2 \), where the \( O \)-constant depends on \( \alpha, \beta, N, \sigma, t \) and \( Y_0 \). In particular when \( \alpha, \beta \in \mathbb{R} \), more explicitly

\[
R_{\alpha,\beta,N}(s, x; Y) = O\{e^{-\pi|t|/2}(|t|+1)^{\alpha+n}/(\alpha+N)/2-s Y^{-\alpha-n}\}
\]
for any \( Y \geq Y_0 > 0 \) in the region \( \sigma < \alpha/2 \), where the \( O \)-constant depends on \( \alpha, \beta, N, \sigma \) and \( Y_0 \).

**Corollary 3.1.** (\cite[Corollary 2.1]{Ka11}). The asymptotic expansion in Theorem 3 for \( RL_{\alpha,\beta}^\alpha Y \zeta^2(s; x+iy) \) precisely reduces to that in Theorem 2 for \( LM_{\alpha,\beta}^\alpha Y \zeta^2(s; x+iy) \) through the confluence operation (1.4).

**Remark 3.1.** The conditions \( \Re \alpha > 1 \) and \( \Re \beta > 1 \) are crucial for the convergence of \( RL_{\alpha,\beta}^\alpha Y \zeta^2(s; x+iy) \), especially for that of \( RL_{\alpha,\beta}^\alpha Y \zeta^2(s; x+iy) \).

**Remark 3.2.** Similarly to Remark 2.2, when \( \alpha, \beta \in \mathbb{R} \) the term with the index \( n \) in \( S_{\alpha,\beta,N} \) is estimated as \( \ll e^{-\pi|t|/2}(|t|+1)^{\alpha+n}/2-s Y^{-\alpha-n} \); this shows that the presence of the bound for \( R_{\alpha,\beta,N}(s, x; Y) \) above is reasonable.
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4-1-1 Hiyoshi, Kouhoku-ku, Yokohama 223-8521, Japan

E-mail address: katsurad@hc.cc.keio.ac.jp; masanori@math.hc.keio.ac.jp