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# Transcendence of certain infinite products

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## 1 Introduction and the results

Duverney [1] introduced an inductive method to prove the transcendence of the number

$$\sum_{k=1}^{\infty}\frac{1}{a^{2^k}+b_k},$$

where $a\,(|a|\geq 2)$ is an integer, $\{b_k\}_{k\geq 1}$ is a sequence of integers satisfying $\log|b_k|=o(2^k)$, and $a^{2^k}+b_k\neq 0$ for every $k\geq 1$. Recently, Duverney and Nishioka [2] developed this method and gave a transcendence criterion for general series. As applications, they established necessary and sufficient conditions for transcendence of the following numbers

$$\sum_{k=0}^{\infty}\frac{a_k}{F_{r^k}+b_k},\quad \sum_{k=0}^{\infty}\frac{a_k}{L_{r^k}+b_k},$$

where $\{a_k\}_{k\geq 0}$ and $\{b_k\}_{k\geq 0}$ are suitable sequences of algebraic numbers, and $F_n$ and $L_n$ are Fibonacci numbers and Lucas numbers defined by $F_{n+2}=F_{n+1}+F_n\,(n\geq 0)$, $F_0=0$, $F_1=1$ and $L_{n+2}=L_{n+1}+L_n\,(n\geq 0)$, $L_0=2$, $L_1=1$, respectively. The purpose of this article is to prove the transcendence of the values of infinite products of the form (1) by modifying the method in [2].

For an algebraic number $\alpha$, we denote by $\overline{|\alpha|}$ the maximum of the absolute values of its conjugates and by $\mathrm{den}(\alpha)$ the least positive integer such that $\mathrm{den}(\alpha)\alpha$ is an algebraic integer, and define $||\alpha||=\max\{\overline{|\alpha|},\ \mathrm{den}(\alpha)\}$. Then we have the fundamental inequalities

$$|\alpha|\geq ||\alpha||^{-2[\mathbb{Q}(\alpha):\mathbb{Q}]}\quad\text{and}\quad ||\alpha^{-1}||\leq ||\alpha||^{2[\mathbb{Q}(\alpha):\mathbb{Q}]}$$

for nonzero algebraic $\alpha$ (cf. Lemma 2.10.2 in [5]).

Let $K$ be an algebraic number field, $O_K$ be the ring of integer in $K$. Let $r\geq 2$ and $L\geq 1$ be integers, and let

$$\Phi_0(x)=\prod_{k=0}^{\infty}\frac{E_k(x^{r^k})}{F_k(x^{r^k})}\tag{1}$$

with

$$E_k(x) = 1 + a_{k1}x + a_{k2}x^2 + \cdots + a_{kL}x^L \in K[x],$$

$$F_k(x) = 1 + b_{k1}x + b_{k2}x^2 + \cdots + b_{kL}x^L \in K[x],$$

where $\log ||a_{kl}||$, $\log ||b_{kl}|| = o(r^k)$, $1 \le l \le L$. We suppose that there exists a positive integer $D$ such that $DF_k(x) \in O_K[x]$ $(k \ge 0)$. Then for algebraic number $\alpha$ satisfying $0 < |\alpha| < 1$ and $E_k(\alpha^{r^k})F_k(\alpha^{r^k}) \ne 0$ $(k \ge 0)$, we prove the following:

**Theorem 1.** *$\Phi_0(\alpha)$ is algebraic if and only if $\Phi_0(x)$ is a rational function with coefficients in $K$.*

It should be noticed that in [2] they proved a similar result for infinite sums. The tools to prove Theorem 1 are also similar to those in [2], however we need some different techniques.

As applications, we have the following results.

**Theorem 2.** *Let $K$ be an algebraic number field, $r \ge 2$ be an integer, and*

$$\Phi(x) = \prod_{k=0}^{\infty} \left(1 + a_k x^{r^k}\right),$$

*where $a_k \in K$ $(k \ge 0)$ and $\log||a_k|| = o(r^k)$. Let $\alpha \in K$ with $0 < |\alpha| < 1$ and $1 + a_k\alpha^{r^k} \ne 0$ $(k \ge 0)$. Then $\Phi(\alpha)$ is algebraic if and only if at least one of the following conditions holds:*
(i) *$a_n = 0$ for every large $n$.*
(ii) *$r = 2$ and there exists a root of unity $\omega$ such that $a_n = \omega^{2^n}$ for every large $n$.*

Nishioka [4] proved that the numbers $\prod_{k=0}^{\infty}(1-\alpha^{r^k})$, $r = 2, 3, 4, \dots$ are algebraically independent for any fixed algebraic number $\alpha$ with $0 < |\alpha| < 1$. Furthermore, Tanaka [6] proved the algebraic independence of the numbers $\prod_{k=0}^{\infty}(1-\alpha_i^{a_k})$, $i = 1, 2, \dots, n$, for a linear recurrence $\{a_k\}_{k\ge0}$ and algebraic numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ under some suitable conditions.

In the following, we consider the binary recurrences $\{R_n\}_{n\ge0}$ defined by

$$R_{n+2} = A_1R_{n+1} + A_2R_n, \quad A_1, A_2 \in \mathbb{Z} \setminus \{0\},\ R_0,\ R_1 \in \mathbb{Z}.$$

We suppose that $|A_2| = 1$ and $\Delta = A_1^2 + 4A_2 > 0$. Then $R_n$ is written as

$$R_n = g_1\rho_1^n + g_2\rho_2^n, \quad g_1, g_2 \in \mathbb{Q}(\rho_1)^\times, \tag{2}$$

where $\rho_1$ and $\rho_2$ are the roots of $g(x) = x^2 - A_1x - A_2$. By the assumption, $\rho_1\rho_2 = \pm1$. We may assume $|\rho_1| > |\rho_2|$, since $A_1 \ne 0$ and $\Delta > 0$. For a negative integer $n$, we define $R_n$ by (2).

**Theorem 3.** *Let $R_n$ be a binary recurrence given by (2) and $r$, $c$, and $d$ be integers such that $r \geq 2$ and $c \geq 1$. Let $K$ be an algebraic number field and $a_k \in K$ satisfy $a_k \neq -R_{cr^k+d}$ $(k \geq 0)$ and $\log||a_k|| = o(r^k)$. Then*

$$\prod_{\substack{k=0 \\ R_{cr^k+d}\neq 0}}^{\infty} \left(1+\frac{a_k}{R_{cr^k+d}}\right)$$

*is algebraic if and only if at least one of the following conditions is satisfied:*
(i) $a_n = 0$ *for every large* $n$.
(ii) $r = 2$ *and* $a_n = R_d$ *for every large* $n$.
(iii) $r = 2$, $g_1\rho_1^d = g_2\rho_2^d$, *and there exists a root of unity* $\omega$ *such that* $a_n = g_1\rho_1^d(\omega^{2^n} + \omega^{-2^n})$ *for every large* $n$.

In the following examples, let $\{a_k\}_{k\geq 0}$, $r$, $c$, and $d$ be as in Theorem 3.

**Example 1.** *Let $F_n$ be Fibonacci numbers defined above. Then*

$$\prod_{\substack{k=0 \\ cr^k+d\neq 0}}^{\infty} \left(1+\frac{a_k}{F_{cr^k+d}}\right)$$

*is algebraic if and only if at least one of the following conditions holds:*
(i) $a_n = 0$ *for every large* $n$.
(ii) $r = 2$ *and* $a_n = F_d$ *for every large* $n$.

In particular,

$$\prod_{k=0}^{\infty} \left(1+\frac{a_k}{F_{r^k}}\right)$$

is algebraic if and only if $a_n = 0$ for all large $n$.

**Example 2.** *Let $L_n$ be Lucas numbers defined above. Then*

$$\prod_{k=0}^{\infty} \left(1+\frac{a_k}{L_{cr^k+d}}\right)$$

*is algebraic if and only if at least one of the following conditions is satisfied:*
(i) $a_n = 0$ *for every large* $n$.
(ii) $r = 2$ *and* $a_n = L_d$ *for every large* $n$.
(iii) $r = 2$, $d = 0$, *and there exists a root of unity* $\omega$ *such that* $a_n = \omega^{2^n} + \omega^{-2^n}$ *for every large* $n$.

In particular, for any integer $a \neq 0$ and $r \geq 2$ the number $\prod_{\substack{k=1 \\ L_{r^k}\neq -a}}^{\infty} \left(1+\frac{a}{L_{r^k}}\right)$ is transcendental, except for two algebraic cases

$$\prod_{k=1}^{\infty} \left(1+\frac{-1}{L_{2^k}}\right) = \frac{\sqrt{5}}{4}, \quad \prod_{k=1}^{\infty} \left(1+\frac{2}{L_{2^k}}\right) = \sqrt{5},$$

which are obtained from the case (iii) with $\omega = \frac{1 \pm \sqrt{-3}}{2}$ and $\omega = \pm 1$, respectively. These examples of algebraic infinite products involving Lucas numbers seems to be new.

# 2 Transcendence of $\Phi_0(\alpha)$

For a formal power series $f(x) \in K[[x]]$ such that $f(x) = \sum_{l \le n} a_n x^n$ with $a_l \neq 0$, we define $\mathrm{ord} f(x) = l$.

**Lemma 1.** *Let $\Phi_0(x)$ and $\alpha$ be given in Section 1. For every positive integer $m$, suppose that there is a positive constant $c(m)$ such that*

$$\mathrm{ord}\ (A_0(x) + A_1(x)\Phi_0(x)^m) \le c(m)M \tag{3}$$

*for any $M \ge 1$ and any polynomials $A_0, A_1 \in K[x]$, not both zero, satisfying* $\deg A_0(x)$, $\deg A_1(x) \le M$. *Then $\Phi_0(\alpha)$ is transcendental.*

Lemma 1 will be used in the proof of Theorem 1 in the next section. For the proof of Lemma 1, we apply the following criterion of Loxton and van der Poorten [3]. We put

$$\Phi_n(x) = \prod_{k=0}^{\infty} \frac{E_{n+k}(x^{r^k})}{F_{n+k}(x^{r^k})}, \quad n \ge 0.$$

**Lemma 2** (cf. Theorem 2.9.1 in [5]). *Let $K$ be an algebraic number field, $r \ge 2$ be an integer, $\{\Phi_n(x)\}_{n \ge 0}$ be a sequence in the ring of formal power series $K[[x]]$, and $\alpha \in K$ with $0 < |\alpha| < 1$. If the following three properties are satisfied, then $\Phi_0(\alpha)$ is transcendental.*

(I) $\Phi_n(\alpha^{r^n}) = a_n \Phi_0(\alpha) + b_n$ *with $a_n, b_n \in K$, and* $\log ||a_n||, \log ||b_n|| = O(r^n)$.

(II) *If $\Phi_n(x) = \sum_{l=0}^{\infty} \sigma_l^{(n)} x^l$, then for any $\varepsilon > 0$ there is a positive integer $n_0$ such that*

$$\log ||\sigma_l^{(n)}|| \le \varepsilon r^n (1 + l)$$

*for any $n \ge n_0$ and $l \ge 0$.*

(III) *Let $\{s_l\}_{l \ge 0}$ be variables and*

$$F(x; s) = F(x; \{s_l\}_{l \ge 0}) = \sum_{l=0}^{\infty} s_l x^l,$$

*in such a way that*

$$F(x; \sigma^{(n)}) = F(x; \{\sigma_l^{(n)}\}_{l \ge 0}) = \Phi_n(x).$$

*Then for any polynomials* $P_0(x,s),\dots,P_d(x,s) \in K[x,\{s_l\}_{l\geq 0}]$ *and*

$$E(x,s) = \sum_{j=0}^{d} P_j(x,s)F(x,s)^j,$$

*there is positive integer* $I$ *with the following property: if* $n$ *is sufficiently large and* $P_0(x,\sigma^{(n)}),\dots,P_d(x,\sigma^{(n)})$ *are not all zero, then* $\mathrm{ord}E(x,\sigma^{(n)}) \leq I$.

The property (I) follows from the functional equation

$$\Phi_n(x^{r^n}) = \Phi_0(x)\prod_{k=0}^{n-1}\frac{F_k(x^{r^k})}{E_k(x^{r^k})}. \tag{4}$$

It is not difficult to see that the property (II) is satisfied. The crucial point in applying Lemma 2 is to check property (III), which is done via Lemma 3.

**Lemma 3.** *Suppose that* $\Phi_0(x)$ *satisfy the assumption 3. Then for every positive integer* $d$*, there exists a positive constant* $c_d$ *such that*

$$\mathrm{ord}(A_0(x) + A_1(x)\Phi_0(x) + \cdots + A_d(x)\Phi_0(x)^d) \leq c_d M$$

*for any* $M \geq 1$ *and any polynomials* $A_0(x),\dots,A_d(x) \in K[x]$*, not all zero, with* $\deg A_i(x) \leq M$ $(0 \leq i \leq d)$.

# 3 Proof of Theorem 1

We use the following lemma.

**Lemma 4** (Theorem 5 in [2]). *Let* $r \geq 2$ *be an integer,* $K$ *be a commutative field, and* $f \in K[[x]]$. *Let* $\{m(n)\}_{n\geq 0}$ *be an increasing sequence of nonnegative integers with* $(m(n+1) - m(n)) \leq c_1$ *for some* $c_1 \geq 1$. *Suppose that there exists a sequence* $\{(P_n(x), Q_n(x))\}_{n\geq 0}$ *in* $K[x]^2$ *such that*

$$P_n(x)Q_{n+1}(x) - P_{n+1}(x)Q_n(x) \neq 0 \tag{5}$$

$$\deg Q_n(x), \deg P_n(x) \leq c_2 r^{m(n)} \tag{6}$$

$$\mathrm{ord}(Q_n(x)f(x) - P_n(x)) \geq c_3 r^{m(n)} \tag{7}$$

*for every* $n \geq 0$, *where* $0 < c_2 < c_3$. *Then we have*

$$\mathrm{ord}(A_0(x) + A_1(x)f(x)) \leq \left(c_2 r^{m(0)+2C}\left(1 + \frac{1}{c_3 - c_2}\right) + 1\right) M$$

*for any* $M \geq 1$ *and for any polynomials* $A_0(x), A_1(x) \in K[x]$*, not both zero, satisfying* $\deg A_0(x), \deg A_1(x) \leq M$.

For each $f(x) = \Phi_0(x)^m$ ($m = 1, 2, \dots$), we construct a sequence $\{(P_{m,n}, Q_{m,n})\}_{n\geq 0}$ satisfying the hypotheses of Lemma 4. Consider the $(mL, mL)$ Pade-approximants to $\Phi_n(x)^m$, that is, polynomials $A_{m,n}(x)$ and $B_{m,n}(x)$ satisfying $\deg A_{m,n}(x)$, $\deg B_{m,n}(x) \leq mL$ and

$$A_{m,n}(x)\Phi_n(x)^m - B_{m,n}(x) = O(x^{2mL+1}). \tag{8}$$

By Siegel's lemma (cf. Lemma 1.4.2 in [5]), we may assume that $\log\|\ \ \|$ of the coefficients of $A_{m,n}(x)$ and $B_{m,n}(x)$ are $o(r^n)$. Define

$$D_{m,n}(x) = \begin{vmatrix} A_{m,n}(x) & B_{m,n}(x) \\ A_{m,n+1}(x^r)F_n(x)^m & B_{m,n+1}(x^r)E_n(x)^m \end{vmatrix}.$$

**Lemma 5.** *Suppose that $D_{m,n}(x) \neq 0$. Then*

$$\operatorname{ord}(A_{m,n}(x)\Phi_n(x)^m - B_{m,n}(x)) < r(2mL+1).$$

**Proof.** This can be proved similarly as the proof of Lemma 4 in [2].

Replacing $x$ by $x^{r^n}$ in (8) and use the functional equation (4), we have

$$Q^*_{m,n}(x)\Phi_0(x)^m - P^*_{m,n}(x) = O(x^{(2mL+1)r^n}),$$

where

$$Q^*_{m,n}(x) = A_{m,n}(x^{r^n})\prod_{k=0}^{n-1} F_k(x^{r^k})^m, \quad P^*_{m,n}(x) = B_{m,n}(x^{r^n})\prod_{k=0}^{n-1} E_k(x^{r^k})^m.$$

Since $\deg Q^*_{m,n}(x)$, $\deg P^*_{m,n}(x) \leq 2mLr^n$, the sequence $\{(P_{m,n}, Q_{m,n})\}_{n\geq 0} = \{(P^*_{m,l(m,n)}, Q^*_{m,l(m,n)})\}_{n\geq 0}$ satisfies hypotheses (6) and (7) of Lemma 4 for every increasing sequence $\{l(m,n)\}_{n\geq 0}$. To study the condition (5) in Lemma 4, we need the following lemma. We put

$$\Delta_{m,n}(x) = \begin{vmatrix} Q^*_{m,n}(x) & P^*_{m,n}(x) \\ Q^*_{m,n+1}(x) & P^*_{m,n+1}(x) \end{vmatrix}.$$

**Lemma 6.** *$\Delta_{m,n}(x) = 0$ if and only if $D_{m,n}(x) = 0$, that is,*

$$\left(\frac{E_n(x)}{F_n(x)}\right)^m = \frac{B_{m,n}(x)A_{m,n+1}(x^r)}{A_{m,n}(x)B_{m,n+1}(x^r)}.$$

**Proof.** By definition, $\Delta_{m,n}(x) = 0$ if and only if

$$\begin{vmatrix} A_{m,n}(x^{r^n}) & B_{m,n}(x^{r^n}) \\ A_{m,n+1}(x^{r^{n+1}})F_n(x^{r^n})^m & B_{m,n+1}(x^{r^{n+1}})E_n(x^{r^n})^m \end{vmatrix} = 0,$$

which is equivalent to

$$D_{m,n}(x) = \begin{vmatrix} A_{m,n}(x) & B_{m,n}(x) \\ A_{m,n+1}(x^r)F_n(x)^m & B_{m,n+1}(x^r)E_n(x)^m \end{vmatrix} = 0.$$

**Lemma 7.** *For each positive integer $m$, we define $f_{m,n}(x)$ by*

$$f_{m,n}(x) = 1 - \frac{A_{m,n}(x)}{B_{m,n}(x)}\Phi_n(x)^m.$$

*Let $I$ be a positive integer and $\alpha$ be algebraic number with $0 < |\alpha| < 1$. Then there exists a positive number $\eta_m < 1$ such that*

$$0 < |f_{m,n}(\alpha^{r^n})| < \eta_m^{r^n \operatorname{ord} f_{m,n}(x)}$$

*for every large $n$ satisfying* $\operatorname{ord} f_{m,n}(x) \le I$.

**Proof.** We may assume $A_{m,n}(0) = B_{m,n}(0) = 1$ by (8). Let $\theta > 1$ and $A_{m,n}(x)/B_{m,n}(x) = \sum_{l=0}^{\infty} \tau_l^{(m,n)} x^l$. Then we obtain $||\tau_l^{(m,n)}|| \le (\theta^{2mL})^{lr^n}$ for any $n \ge n_0$ and $l \ge 0$. Let $\Phi_n(x)^m = (\sum_{l=0}^{\infty} \sigma_l^{(n)} x^l)^m = \sum_{l=0}^{\infty} \mu_l^{(m,n)} x^l$, then we have

$$f_{m,n}(x) = 1 - \left(\sum_{l=0}^{\infty} \tau_l^{(m,n)} x^l\right)\left(\sum_{l=0}^{\infty} \mu_l^{(m,n)} x^l\right) = \sum_{l=1}^{\infty}\left(\sum_{s+t=l} \tau_s^{(m,n)} \mu_t^{(m,n)}\right) x^l,$$

where $\left|\overline{\sum_{s+t=l} \tau_s^{(m,n)} \mu_t^{(m,n)}}\right| \le (\theta^{6mL})^{lr^n}$ and $\operatorname{den}\left(\sum_{s+t=l} \tau_s^{(m,n)} \mu_t^{(m,n)}\right) \le (\theta^{5mL})^{lr^n}$. We put

$$f_{m,n}(x) = a_H x^H + a_{H+1} x^{H+1} + \dots, \; a_H \neq 0,$$

where $1 \le H \le I$. Then

$$f_{m,n}(\alpha^{r^n}) = a_H \alpha^{Hr^n}\left(1 + \frac{a_{H+1}}{a_H}\alpha^{r^n} + \frac{a_{H+2}}{a_H}\alpha^{2r^n} + \cdots\right).$$

Since $||a_H|| \le (\theta^{6mL})^{Hr^n}$, we obtain

$$\left|\frac{a_{H+l}}{a_H}\alpha^{lr^n}\right| \le (\theta^{6mL})^{(1+2[K:\mathbb{Q}])Ir^n} |\theta^{6mL}\alpha|^{lr^n}.$$

Choosing $\theta > 1$ with $\eta_m = (\theta^{6mL})^{(1+2[K:\mathbb{Q}])I}|\theta^{6mL}\alpha| < 1$, we have

$$0 < |f_{m,n}(\alpha^{r^n})| < 2|\theta^{6mL}\alpha|^{Hr^n} < \eta_m^{Hr^n}$$

for sufficiently large $n$; which implies the lemma.

**Lemma 8.** *$\Phi_0(\alpha)$ is algebraic if and only if $\Phi_0(x)^m$ is a rational function with coefficients in $K$ for some positive integer $m$.*

**Proof.** We prove that if $\Phi_0(\alpha)$ is algebraic then there exists a positive integer $m$ such that $\Delta_{m,n}(x) = 0$ for every large $n$, which implies $\Phi_0(x)^m$ is a rational function by Lemma 6. For every integer $m$, suppose that there exist infinitely many $n$ satisfying $\triangle_{m,n}(x) \neq 0$. Denote by $\{l(m,n)\}_{n\geq 0}$ the sequence satisfying

$$\triangle_{m,l(m,n)}(x) \neq 0, \quad \triangle_{m,k}(x) = 0$$

for every $n \geq 0$ and every $k$ with $l(m,n) < k < l(m,n+1)$. Then two cases occur:
(i) For every $m$, $l(m,n+1) - l(m,n) \leq C_m$ for some positive constant $C_m$. Then it is clear that the determinant

$$\begin{vmatrix} Q^*_{m,l(m,n)}(x) & P^*_{m,l(m,n)}(x) \\ Q^*_{m,l(m,n+1)}(x) & P^*_{m,l(m,n+1)}(x) \end{vmatrix} = \begin{vmatrix} Q_{m,n}(x) & P_{m,n}(x) \\ Q_{m,n+1}(x) & P_{m,n+1}(x) \end{vmatrix} \neq 0,$$

namely, the condition (5) in Lemma 4 is satisfied. Hence we can apply Lemma 1 and find that $\Phi_0(\alpha)$ is transcendental.
(ii) For some $m$, $\overline{\lim_{n\to\infty}}(l(m,n+1) - l(m,n)) = +\infty$. In this case, we have by using Lemma 6

$$\left(\frac{E_k(x)}{F_k(x)}\right)^m = \frac{B_{m,k}(x)A_{m,k+1}(x^r)}{A_{m,k}(x)B_{m,k+1}(x^r)}$$

for every $k$ satisfying $l(m,n) < k < l(m,n+1)$, so that

$$\prod_{k=l(m,n)+1}^{l(m,n+1)-1} \left(\frac{E_k(x^{r^k})}{F_k(x^{r^k})}\right)^m = \frac{B_{m,l(m,n)+1}(x^{r^{l(m,n)+1}})A_{m,l(m,n+1)}(x^{r^{l(m,n+1)}})}{A_{m,l(m,n)+1}(x^{r^{l(m,n)+1}})B_{m,l(m,n+1)}(x^{r^{l(m,n+1)}})}. \tag{9}$$

Let

$$f_{m,l(m,n+1)}(x) = \frac{A_{m,l(m,n+1)}(x)}{B_{m,l(m,n+1)}(x)}\Phi_{l(m,n+1)}(x)^m - 1,$$

where we may assume $A_{m,l(m,n+1)}(0) = B_{m,l(m,n+1)}(0) = 1$. Since $\triangle_{m,l(m,n+1)}(x) \neq 0$, we have $D_{m,l(m,n+1)}(x) \neq 0$ by Lemma 6. Therefore by Lemma 5

$$\begin{aligned} \operatorname{ord} f_{m,l(m,n+1)}(x) &\leq \operatorname{ord}\left(A_{m,l(m,n+1)}(x)\Phi_{l(m,n+1)}(x)^m - B_{m,l(m,n+1)}(x)\right) \\ &\leq r(2mL+1). \end{aligned}$$

Applying Lemma 7, we see that there exists a positive number $\eta_m < 1$ such that

$$0 < |f_{m,l(m,n+1)}(\alpha^{r^{l(m,n+1)}})| < \eta_m{}^{r^{l(m,n+1)}} \tag{10}$$

for every large $n$. Since

$$\Phi_0(x)^m = \prod_{k=0}^{l(m,n)} \left(\frac{E_k(x^{r^k})}{F_k(x^{r^k})}\right)^m \prod_{k=l(m,n)+1}^{l(m,n+1)-1} \left(\frac{E_k(x^{r^k})}{F_k(x^{r^k})}\right)^m \Phi_{l(m,n+1)}(x^{r^{l(m,n+1)}})^m,$$

we get by (9)

$$
\begin{aligned}
&f_{m,l(m,n+1)}(\alpha^{r^{l(m,n+1)}})\frac{B_{m,l(m,n)+1}(\alpha^{r^{l(m,n)+1}})}{A_{m,l(m,n)+1}(\alpha^{r^{l(m,n)+1}})}\prod_{k=0}^{l(m,n)}\left(\frac{E_k(\alpha^{r^k})}{F_k(\alpha^{r^k})}\right)^m \\
= \; & \Phi_0(\alpha)^m - \prod_{k=0}^{l(m,n)}\left(\frac{E_k(\alpha^{r^k})}{F_k(\alpha^{r^k})}\right)^m \frac{B_{m,l(m,n)+1}(\alpha^{r^{l(m,n)+1}})}{A_{m,l(m,n)+1}(\alpha^{r^{l(m,n)+1}})}.
\end{aligned}
$$

If $\Phi_0(\alpha)^m$ is algebraic, then $f_{m,l(m,n+1)}(\alpha^{r^{l(m,n+1)}})$ is also algebraic and so there exists a constant $C_m > 1$ such that

$$||f_{m,l(m,n+1)}(\alpha^{r^{l(m,n+1)}})|| \le C_m^{r^{l(m,n)}}. \tag{11}$$

These inequalities (10) and (11) contradict the fundamental inequality if $n$ is large. Hence $\Phi_0(\alpha)$ is transcendental, also in this case. The converse is trivial.

The next lemma together with Lemma 8 implies Theorem 1.

**Lemma 9.** *$\Phi_0(x)$ is a rational function with coefficients in $K$ if and only if $\Phi_0(x)^m$ is so for some positive integer $m$.*

**Proof.** Suppose that $\Phi_0(x)^m \in K(x)$ for some integer $m \ge 1$, then $\Phi_0(\alpha)$ is algebraic. By Lemma 8 there exists a positive integer $m'$ such that $\triangle_{m',n}(x) = 0$ for every large $n$, that is,

$$\left(\frac{E_n(x)}{F_n(x)}\right)^{m'} = \frac{B_{m',n}(x)A_{m',n+1}(x^r)}{A_{m',n}(x)B_{m',n+1}(x^r)}, \quad n \ge N.$$

Hence we have

$$\Phi_0(x)^{mm'} = \left(\prod_{k=0}^{n-1}\frac{E_k(x^{r^k})}{F_k(x^{r^k})}\right)^{mm'}\left(\frac{B_{m',n}(x^{r^n})}{A_{m',n}(x^{r^n})}\right)^m = \left(\frac{P(x)}{Q(x)}\right)^{m'}, \quad n \ge N$$

for some $P(x)$, $Q(x) \in K[x]$. We can put

$$\frac{B_{m',n}(x)}{A_{m',n}(x)} = C_n(x)p_n(x)^{m'}, \quad \frac{P(x)}{Q(x)} = R(x)q_n(x)^m,$$

where $p_n(x), q_n(x) \in K(x)^\times$, $p_n(0) = 1$ and $C_n(x)$, $R(x) \in K[x]$ with orders less than $m'$ and $m$ at each zero, respectively. Since $B_{m',n}(x)/A_{m',n}(x) = 1 + O(x)$, we may assume $C_n(0) = 1$. If $\deg C_n(x) \ge 1$, there exists an $\alpha \ne 0$ with $C_n(\alpha) = 0$. Since $C_n(x^{r^n})^m \in R(x)^{m'}(K(x)^\times)^{mm'}$ and the order of $C_n(x^{r^n})$ at $\alpha^{\frac{1}{r^n}}$ is less than $m'$, we see that $\alpha^{\frac{1}{r^n}}$ is a root of $R(x)$. This implies $mr^n \le m' \deg R(x)$. Hence $C_n(x) = 1$ for every large $n$. Therefore we obtain

$$\frac{B_{m',n}(x)}{A_{m',n}(x)} = \left(\frac{B_n(x)}{A_n(x)}\right)^{m'}, \quad n \ge M$$

for some $A_n(x), B_n(x) \in K[x]$ satisfying $A_n(0) = B_n(0) = 1$, $(A_n(x), B_n(x)) = 1$, and $\deg A_n(x), \deg B_n(x) \leq L$. Then we have

$$\frac{E_n(x)}{F_n(x)} = \frac{B_n(x)A_{n+1}(x^r)}{A_n(x)B_{n+1}(x^r)}, \quad n \geq M, \tag{12}$$

that is, $\Phi_0(x)$ is a rational function with coefficients in $K$. The converse is trivial. Hence the proof is completed.

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