数理解析研究所講究録
1512 巻 2006 年 95-101

SUMS OF FIVE CUBES OF PRIMES

Hiroshi Mikawa and Temenoujka Peneva*

Abstract

Let A, $\varepsilon > 0$ be arbitrary. We prove that the number of integers $n \in (x, x+H]$, satisfying some natural conditions, which cannot be represented as the sum of five cubes of primes is $\ll H(\log x)^{-A}$, provided that $x^{2/3+\varepsilon} \leq H \leq x$.

1. Introduction

It has been conjectured that every sufficiently large integer, satisfying some natural congruence conditions, can be written as the sum of four cubes of primes. While such a result appears to lie beyond the reach of present methods, Hua [3] has been able to show that every sufficiently large odd integer is the sum of nine cubes of primes. He also established that almost all integers $n \in \mathfrak{N} = \{n \in \mathbb{N} : n \equiv 1 \,(\mathrm{mod}\ 2),\, n \not\equiv 0, \pm 2 \,(\mathrm{mod}\ 9),\, n \not\equiv 0 \,(\mathrm{mod}\ 7)\}$, can be expressed as the sum of five cubes of primes. Here the term 'almost all' means that if $E(x)$ denotes the number of possible exceptions up to x, then $E(x) \ll x(\log x)^{-A}$ for a certain constant $A > 0$. In 1961, Schwarz [8] refined Hua's method to demonstrate the last estimate for any $A > 0$. In 2000, Ren [7] made a substantial improvement upon the latter result by showing that $E(x) \ll x^{152/153+\varepsilon}$ for any fixed $\varepsilon > 0$. Shortly afterward, the constant in the exponent was sharpened to 35/36 by Wooley [9], and to 79/84 by Kumchev [5].

In the present paper we gain further insight into the problem of representing integers as the sum of five cubes of primes by averaging over short intervals only. Let $\Lambda(n)$ and $\varphi(n)$ denote von Mangoldt's function and Euler's function, respectively, and write $e(\alpha) = e^{2\pi i\alpha}$ for real α. Following the notation introduced in [7], for a sufficiently large positive number x we define $U = (x/12)^{1/3}$,

$$R(n) = \sum_{\substack{k_1^3+\cdots+k_5^3=n \\ U<k_i\leq 2U}} \cdots \sum \Lambda(k_1)\ldots\Lambda(k_5),$$

$$\sigma(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \Big(\varphi(q)^{-1} \sum_{\substack{h=1 \\ (h,q)=1}}^{q} e(ah^3/q)\Big)^5 e(-an/q),$$

and

$$J(n) = 3^{-5} \int_{\mathcal{D}} (u_1 \ldots u_5)^{-2/3}\, du_1 \ldots du_4,$$

where

$$\mathcal{D} = \{(u_1, \ldots, u_4) : U^3 < u_1, \ldots, u_5 \leq 8U^3\}$$

with $u_5 = n - u_1 - \ldots - u_4$. Our first result states as follows.

*The second author was supported by the Japan Society for the Promotion of Science, P03763.

THEOREM 1. *Suppose that A, $\varepsilon > 0$ and $x^{2/3+\varepsilon} \le H \le x$. Then*

$$\sum_{\substack{x<n\le x+H\\ n\in\mathfrak{N}}} |R(n)-\sigma(n)\,J(n)|^2 \ll_{A,\varepsilon} Hx^{4/3}(\log x)^{-A}.$$

We recall that the singular series $\sigma(n)$ is absolutely convergent, and there exists a constant C such that $\sigma(n) \ge C > 0$ for every $n \in \mathfrak{N}$ (the reader may refer to Lemmas 8.10 and 8.12 of Hua's book [2]). We also note that the singular integral $J(n)$ trivially satisfies the inequality

$$U^2 \ll J(n) \ll U^2.$$

Employing a standard argument, we deduce from Theorem 1 the following

THEOREM 2. *Suppose that A, $\varepsilon > 0$ and $x^{2/3+\varepsilon} \le H \le x$. Then*

$$E(x+H) - E(x) \ll_{A,\varepsilon} H(\log x)^{-A}.$$

The proof of Theorem 1 is based on the Hardy–Littlewood circle method. The integral over the major arcs is evaluated by classical arguments, while the contribution of the minor arcs is bounded by adapting the technique of [6], applied to deal with sums of three squares of primes in short intervals. We also borrow an idea of Kawada [4, §6], which enables us to conveniently transform the short interval average over the minor arcs. It appears that the constant 2/3 is the best that our argument could yield.

2. Auxiliary lemmas

Much of our analysis is concerned with the exponential sum

$$S(\alpha) = \sum_{k\sim U} \Lambda(k) e(\alpha k^3),$$

where $k \sim U$ denotes $U < k \le 2U$. Our first lemma states the famous Vinogradov's estimate in a form due to Fujii [1, Lemma 2].

LEMMA 1. *Suppose that $|\alpha - a/q| \le q^{-2}$ with $(a,q) = 1$. Then*

$$S(\alpha) \ll U(q^{-1} + qU^{-3} + U^{-1/2})^{1/32}(\log qU)^{C_1},$$

where $C_1 > 0$ and the implied constant are absolute.

In the next lemma we recall the well-known Hua's estimate [2, Theorem 4].

LEMMA 2. *We have*

$$\int_0^1 |S(\alpha)|^8\, d\alpha \ll U^5(\log U)^{C_2},$$

where $C_2 > 0$ and the implied constant are absolute.

We introduce the Fejér kernel

$$K(\alpha) = K(\alpha, H) = \sum_{|m| \le 2H} M(m) e(\alpha m),$$

where

$$M(m) = M(m, H) = \max\left(1 - \frac{|m|}{2H}, 0\right).$$

Then $K(\alpha) \ge 0$ for all real α, see for example [4, §6]. We define

$$\begin{aligned} \Phi(\alpha) &= \int_{-1/2}^{1/2} |S(\alpha+\beta)|^2 K(\beta)\, d\beta \\ &= \sum_{k,\, l \sim U}\sum \Lambda(k)\Lambda(l)\, M(k^3 - l^3) e(\alpha(k^3 - l^3)), \end{aligned}$$

and

$$\Psi(\alpha) = \sum_{k,\, l \sim U}\sum M(k^3 - l^3) e(\alpha(k^3 - l^3)).$$

In the next statement we collect some properties of the above quantities. Let $\tau_3(k)$ denote, as usual, the divisor function.

LEMMA 3. *For every real α:*

(i) $0 \le \Phi(\alpha) \le \Phi(0) \ll U(1 + HU^{-2})(\log U)^2$;

(ii) $0 \le \Psi(\alpha) \le \Psi(0) \ll U(1 + HU^{-2})$;

(iii) *There exists a function $\Xi(\alpha)$, such that $\Psi(\alpha)^2 \ll \Xi(\alpha)$ and*

$$\Xi(\alpha) = \mathcal{O}(U^2 + H^2U^{-3}) + HU^{-2} \sum_{0<|h| \le 2H} c(h) e(\alpha h),$$

with $c(h) \ll \tau_3(|h|)$.

PROOF. First we consider (iii). Supposing that $0 < k^3 - l^3$, we put $k = l + d$ and change the summation variable. Subsequently, l, $l+d \sim U$ and $k^3 - l^3 = (l+d)^3 - l^3 = 3l^2d + 3ld^2 + d^3$. Since $M(k^3 - l^3) = 0$ unless $k^3 - l^3 < 2H$, we see that $2H > k^3 - l^3 = (k-l)(k^2 + kl + l^2) > (k-l)3U^2$, or $d < HU^{-2}$. On writing

$$M(k^3 - l^3) = M'(l, d),$$

we find that

$$\Psi(\alpha) \ll U + \sum_{d < HU^{-2}} \left| \sum_{l}{}' M'(l, d)\, e(\alpha(3l^2 d + 3ld^2)) \right|, \tag{1}$$

where $'$ in $\sum'$ indicates the condition l, $l + d \sim U$. An appeal to Cauchy's inequality reveals that

$$\begin{aligned} \Psi(\alpha)^2 &\ll U^2 + HU^{-2} \sum_{d < HU^{-2}} \left| \sum_{l}{}' M'(l, d)\, e(\alpha(3l^2 d + 3ld^2)) \right|^2 \\ &= \Xi(\alpha), \end{aligned}$$

say. The sum above is

$$\begin{aligned}
&= \mathcal{O}\Bigg(\sum_{d<HU^{-2}} {\sum_{l}}' 1\Bigg) \\
&\quad + \sum_{d<HU^{-2}} {\sum\sum_{l\neq m}}' M'(l,d)M'(m,d)\, e(\alpha(3d(l^2-m^2)+3d^2(l-m))) \\
&= \mathcal{O}(HU^{-2}U) + \sum_{0<|h|\le 2H} c(h)e(\alpha h)\,,
\end{aligned}$$

where

$$c(h) = \sum_{d<HU^{-2}} \quad {\sum\sum_{\substack{l,\,m \\ 3d(l-m)(l+m+d)=h}}}' \quad M'(l,d)\,M'(m,d) \ll \tau_3(|h|)\,,$$

which completes the proof of (iii).

We now turn to (ii). By (1), we trivially have

$$\Psi(0) \ll U + HU^{-2}U,$$

which delivers the last inequality in (ii), and the other two are obvious. The proof of (i) is analogous.

3. Proof of Theorem 1

Hereafter we assume that $\varepsilon > 0$ is sufficiently small, and $U^{2+\varepsilon} \ll H \ll U^3$ so that $1 + HU^{-2} \ll HU^{-2}$ in Lemma 3. We have

$$R(n) = \int_0^1 S(\alpha)^5 e(-\alpha n)\, d\alpha\,.$$

Put

$$L = \log x, \qquad P = L^B, \qquad Q = xP^{-2},$$

where the constant $B > 0$ will be specified later. Define the set of major arcs $\mathfrak{M}$ as the union of all intervals $\{\alpha \in \mathbb{R} : |q\alpha - a| \le Q^{-1}\}$ with $1 \le a \le q \le P$ and $(a,q) = 1$. Denote the corresponding set of minor arcs by $\mathfrak{m} = [1/Q, 1+1/Q]\backslash\mathfrak{M}$. Then,

$$R(n) = \left(\int_{\mathfrak{M}} + \int_{\mathfrak{m}}\right) S(\alpha)^5 e(-\alpha n) d\alpha = R_{\mathfrak{M}}(n) + R_{\mathfrak{m}}(n)\,,$$

say. By classical arguments based on the Siegel-Walfisz theorem (see [2], for example), we derive that for all $n \in \mathfrak{N} \cap (x, x+H]$, in the notation introduced above,

$$|R_{\mathfrak{M}}(n) - \sigma(n)\, J(n)| \ll U^2 L^{-A/2}\,,$$

provided that $B \ge A + 1$. Our choice of the constant B at the end of Section 4.2 satisfies this inequality, thus yielding the desired bound for the contribution of the major arcs. It remains to prove that

$$\sum_{x<n\le x+H} |R_{\mathfrak{m}}(n)|^2 \ll HU^4 L^{-A}\,, \tag{2}$$

which is the objective of the next section.

4. The minor arcs

Employing an argument of Kawada [4, §6], we find that

$$\begin{aligned}\sum_{x<n\le x+H} |R_{\mathfrak{m}}(n)|^2 &\le 2\sum_{|m|\le 2H} M(m)\left|\int_{\mathfrak{m}} S(\alpha)^5 e(-\alpha(x+m))\,d\alpha\right|^2\\ &\ll \int_{\mathfrak{m}}\int_{\mathfrak{m}} |S(\alpha)|^5|S(\beta)|^5|K(\beta-\alpha)|\,d\alpha\,d\beta\\ &\ll W_5, \end{aligned}\tag{3}$$

where

$$W_l = W_l(H) = \int_{\mathfrak{m}}\int_{\mathfrak{m}} |S(\alpha)|^l|S(\beta)|^l K(\beta-\alpha,H)\,d\alpha\,d\beta.$$

Hence our principal task is to bound W_5. However, our argument in Section 4.2 reduces the estimate of W_5 to that of W_8 and therefore it is convenient to start with the latter quantity.

4.1. *The estimate of W_8*

First we observe that for any $\xi\in\mathfrak{m}$ there exists a rational number a/q such that $|\xi - a/q|\le q^{-2}$, $(a,q)=1$ and $P\le q\le Q$, by Dirichlet's approximation theorem. Since

$$|S(\alpha)|^8|S(\beta)|^8 \ll |S(\alpha)|^{14}|S(\beta)|^2 + |S(\alpha)|^2|S(\beta)|^{14},$$

we have by symmetry,

$$\begin{aligned}W_8 &\ll \int_{\mathfrak{m}}\int_{\mathfrak{m}} |S(\alpha)|^{14}|S(\beta)|^2 K(\beta-\alpha)\,d\alpha\,d\beta\\ &\ll \int_{\mathfrak{m}} |S(\alpha)|^{14}\left(\int_{-1/2}^{1/2} |S(\alpha+\beta)|^2 K(\beta)\,d\beta\right)d\alpha\\ &= \int_{\mathfrak{m}} |S(\alpha)|^{14}\Phi(\alpha)\,d\alpha\\ &\ll \Phi(0)\left(\sup_{\alpha'\in\mathfrak{m}} |S(\alpha')|^6\right)\int_0^1 |S(\alpha)|^8\,d\alpha, \end{aligned}$$

by Lemma 3. Combining Lemmas 1, 2 and 3, we obtain

$$W_8 = W_8(H) \ll HU^{10}P^{-3/16}L^{6C_1+C_2+2}.\tag{4}$$

4.2. *The estimate of W_5*

Following the argument from the previous section, we find that

$$\begin{aligned}W_5 &\ll \int_{\mathfrak{m}} |S(\alpha)|^8\Phi(\alpha)\,d\alpha\\ &\ll L^2\sum\sum_{k,l\sim U} M(k^3-l^3)\left|\int_{\mathfrak{m}} |S(\alpha)|^8 e(\alpha(k^3-l^3))\,d\alpha\right|. \end{aligned}$$

An application of Cauchy's inequality yields

$$\begin{aligned}(W_5)^2 &\ll L^4\Psi(0)\sum_{k,\,l\sim U}\sum M(k^3-l^3)\left|\int_{\mathfrak{m}}|S(\alpha)|^8 e(\alpha(k^3-l^3))\,d\alpha\right|^2 \\ &\ll L^4\Psi(0)\int_{\mathfrak{m}}\int_{\mathfrak{m}}|S(\alpha)|^8|S(\beta)|^8\Psi(\beta-\alpha)\,d\alpha\,d\beta.\end{aligned}$$

Another application of Cauchy's inequality, Lemmas 2 and 3 show that

$$\begin{aligned}(W_5)^4 &\ll L^8\Psi(0)^2\left(\int_{\mathfrak{m}}|S(\alpha')|^8\,d\alpha'\right)^2 \\ &\qquad\times\int_{\mathfrak{m}}\int_{\mathfrak{m}}|S(\alpha)|^8|S(\beta)|^8\Psi(\beta-\alpha)^2\,d\alpha\,d\beta \\ &\ll L^8(HU^{-1})^2\left(\int_0^1|S(\alpha')|^8\,d\alpha'\right)^2 \\ &\qquad\times\left((U^2+H^2U^{-3})\left(\int_0^1|S(\alpha)|^8\,d\alpha\right)^2+HU^{-2}J\right) \\ &\ll H^4U^{16}(H^{-2}U^4+U^{-1})L^{4C_2+8}+H^3U^6L^{2C_2+8}J, \qquad (5)\end{aligned}$$

where

$$J=\sum_{h\le 2H}\tau_3(h)\left|\int_{\mathfrak{m}}|S(\alpha)|^8 e(\alpha h)\,d\alpha\right|^2.$$

The estimate of J is reduced to that of W_8. Indeed, by Cauchy's inequality and Lemma 2, we find that

$$\begin{aligned}J^2 &\ll \sum_{h'\le 2H}\tau_3(h')^2\left(\int_{\mathfrak{m}}|S(\alpha')|^8\,d\alpha'\right)^2\sum_{h\le 2H}\left|\int_{\mathfrak{m}}|S(\alpha)|^8 e(\alpha h)\,d\alpha\right|^2 \\ &\ll HL^8(U^5L^{C_2})^2\sum_{|h|\le 4H}M(h,2H)\left|\int_{\mathfrak{m}}|S(\alpha)|^8 e(\alpha h)\,d\alpha\right|^2 \\ &= HU^{10}L^{2C_2+8}W_8(2H), \qquad (6)\end{aligned}$$

since $M(h,2H)=\max\left(1-\frac{|h|}{4H},0\right)\ge\frac{1}{2}$ for $0<h\le 2H$.

Substituting (6) into (5), and recalling (4), we conclude that

$$\begin{aligned}(W_5)^4 &\ll H^4U^{16-\varepsilon}+H^3U^6L^{2C_2+8}(HU^{10}L^{2C_2+8}W_8(2H))^{1/2} \\ &\ll H^4U^{16-\varepsilon}+H^4U^{16}P^{-3/32}L^{3C_1+4C_2+13}. \qquad (7)\end{aligned}$$

On choosing $B=44(A+C_1+C_2+4)$, the inequality (2) follows from (3) and (7). The proof of Theorem 1 is complete.

References

[1] Fujii, A., *Some additive problems of numbers*, Elementary and analytic theory of numbers (Warsaw, 1982), 121–141, Banach Center Publ. 17, PWN, Warsaw, 1985.

[2] Hua, L. K., *Additive theory of prime numbers*, Amer. Math. Soc., Providence RI, 1965.

[3] Hua, L. K., *On the representations of a number as the sum of two cubes*, Math. Z. 44 (1938), 335–346.

[4] Kawada, K., *On the sum of four cubes*, Mathematika 43 (1996), 323–348.

[5] Kumchev, A. V., *On the Waring-Goldbach problem: exceptional sets for sums of cubes and higher powers*, Canad. J. Math. 57 (2005), 298–327.

[6] Mikawa, H., *On the sum of three squares of primes*, Analytic Number Theory (Kyoto, 1996), 253–264, London Math. Soc. Lecture Note Ser. 247, Cambridge Univ. Press, Cambridge, 1997.

[7] Ren, H. M., *The Waring-Goldbach problem for cubes*, Acta Arith. 94 (2000), 287–301.

[8] Schwarz, W., *Zur Darstellung von Zahlen durch Summen von Primzahlpotenzen, II*, J. Reine Angew. Math. 206 (1961), 78–112.

[9] Wooley, T. D., *Slim exceptional sets for sums of cubes*, Canad. J. Math. 54 (2002), 417–448.

INSTITUTE OF MATHEMATICS
UNIVERSITY OF TSUKUBA
TSUKUBA 305–8571
JAPAN
E-mail: mikawa@math.tsukuba.ac.jp

INSTITUTE OF MATHEMATICS
UNIVERSITY OF TSUKUBA
TSUKUBA 305–8571
JAPAN

FACULTY OF MATHEMATICS
AND INFORMATICS
UNIVERSITY OF PLOVDIV
236, BULGARIA BLVD.
PLOVDIV 4003
BULGARIA
E-mail: tpeneva@pu.acad.bg