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Mock Theta Functions in Ramanujan's Lost Notebook

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1. INTRODUCTION

In his last letter to G. H. Hardy [8], S. Ramanujan proclaimed, "I discovered very interesting functions recently which I call 'Mock' *vartheta*-functions." But, the introduction to this letter has evidently been lost; a portion of it can be found in Ramanujan's *Collected Papers* [18]. However, the 'mathematical' portion of the letter has been completely preserved. Extracts from it can be found in the *Collected Papers* [18]. The complete mathematical portion is given in G. N. Watson's paper [21], with the publication of Ramanujan's lost notebook [19] (a photocopy of the original letter), in G. E. Andrews' survey paper [4], and in B. C. Berndt and R. A. Rankin's book [8].

To understand mock theta functions, we need to read a part of the letter where Ramanujan explained mock theta functions.

"If we consider a ϑ-function in the transformed Eulerian form, e.g.

$$\text{(A)}1+\frac{q}{(1-q)^2}+\frac{q^4}{(1-q)^2(1-q^2)^2}+\frac{q^9}{(1-q)^2(1-q^2)^2(1-q^3)^2}+\cdots,$$

$$\text{(B) } 1+\frac{q}{1-q}+\frac{q^4}{(1-q)(1-q^2)}+\frac{q^9}{(1-q)(1-q^2)(1-q^3)}+\cdots,$$

and determine the nature of the singularities at the points $q=1$, $q^2=1$, $q^3=1$, $q^4=1$, $q^5=1,\ldots$, we know how beautifully the asymptotic form of the function can be expressed in a very neat and closed exponential form. For instance, when $q=e^{-t}$ and $t\to 0$,

$$\text{(A)}=\sqrt{\frac{t}{2\pi}}\exp\left(\frac{\pi^2}{6t}-\frac{t}{24}*\right)+o(1)\dagger,$$

$$\text{(B)}=\frac{\exp\left(\frac{\pi^2}{15t}-\frac{t}{60}*\right)}{\sqrt{\frac{5-\sqrt{5}}{2}}}+o(1)\dagger,$$

and similar results an other singularities.$*$ It is not necessary that there should be only one term like this. There may be many terms, but the number of terms must be finite.$\dagger$ Also $o(1)$ may turn out to be $O(1)$. That is all. For instance when $q\to 1$ the function

$$\frac{1}{\{(1-q)(1-q^2)(1-q^3)\cdots\}^{120}}$$

is equivalent to the sum of five terms like $(*)$ together with $O(1)$ instead of $o(1)$.

If we take a number of functions like (A) and (B), it is only in a limited number of cases the terms close as above: but in the majority of cases they never close as above. For instance, when $q = e^{-t}$ and $t \to 0$,

$$(\mathrm{C}) 1 + \frac{q}{(1-q)^2} + \frac{q^3}{(1-q)^2(1-q^2)^2} + \frac{q^6}{(1-q)^2(1-q^2)^2(1-q^3)^2} + \cdots$$
$$= \sqrt{\frac{t}{2\sqrt{5}}} \exp\left(\frac{\pi^2}{5t}^{**} + a_1^{\dagger\dagger} t + a_2 t^2 + \cdots + O(a_k t^k)\right),$$

where $a_1 = \frac{1}{8\sqrt{5}}$, and so on. The function (C) is a simple example of a function behaving in an unclosed form at the singularities.

**The coefficient (of) $1/t$ in the index of e happens to be $\frac{\pi^2}{5}$ in this particular case. In may be some other transcendental numbers in other cases.

††The coefficients of t, t^2, $\cdots$ happen to be $\frac{1}{8\sqrt{5}}, \cdots$ in this case. In other cases they may turn out to be some other algebraic numbers.

Now a very interesting question arises. Is the converse of the statements concerning the forms (A) and (B) true? That is to say: Suppose there is a function in the Eulerian form and suppose that all or an infinity of points $q = e^{2i\pi m/n}$ are exponential singularities, and also suppose that at these points the asymptotic form of the function closes as neatly as in the cases of (A) and (B). The question is: Is the function taken the sum of two functions one of which is an ordinary ϑ-function and the other a (trivial) function which is $O(1)$ at all the points $e^{2i\pi m/n}$? The answer is it is not necessarily so. When it is not so, I call the function a Mock ϑ-function. I have not proved rigorously that it is not necessarily so. But I have constructed a number of examples in which it is inconceivable to construct a ϑ-function to cut out the singularities of the original function. Also I have shown that if it is necessarily so then it leads to the following assertion-viz. It is possible to construct two power series in x, namely $\sum_0^\infty a_n x^n$ and $\sum_0^\infty b_n x^n$, both of which have essential singularities on the unit circle, are convergent when $|x| < 1$, and tend to finite limits at every point $x = e^{2i\pi r/s}$, and that at the same time the limit of $\sum_0^\infty a_n x^n$ at the point $x = e^{-2i\pi r/s}$ is equal to the limit of $\sum_0^\infty b_n x^n$ at the point $x = e^{-2i\pi r/s}$.

This assertion seems to be untrue. Anyhow, we shall go to the examples and see how far our assertions are true.

I have proved that if

$$f(q) = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \cdots,$$

then

$$f(q) + (1-q)(1-q^3)(1-q^5)\cdots(1 - 2q + 2q^4 - 2q^9 + \cdots) = O(1)$$

at all the points $q = -1, q^3 = -1, q^5 = -1, q^7 = -1, \ldots$, and at the same time

$$f(q) - (1-q)(1-q^3)(1-q^5)\cdots(1 - 2q + 2q^4 - \cdots) = O(1)$$

at all the points $q^2 = -1, q^4 = -1, q^6 = -1, \ldots$. Also obviously $f(q) = O(1)$ at all the points $q = 1, q^3 = 1, q^5 = 1, \ldots$. And so $f(q)$ is a Mock ϑ-function."

With this explanation, Ramanujan introduced 'third order', 'fifth order', and 'seventh order' mock theta functions in his letter. He also provided identities satisfied by third order and fifth order mock theta functions, but he didn't provide any identity for seventh order mock theta functions. In long papers [21, 22], Watson proved all of the results on third and fifth order mock theta functions, and introduced three new third order mock theta functions, and A. Selberg [20] provided a full account of the behavior of the seventh order mock theta function near the unit circle.

In 1976, Andrews rediscovered Ramanujan's lost notebook [19] in the library of Trinity College, Cambridge. This lost notebook contains many further theorems on mock theta functions. In this survey, we will see the results on mock theta functions in his lost notebook and methods which have been used to prove the results.

In Ramanujan's last letter to Hardy, he used the term "order", but he provided no formal definition of "order". We use the terms "sixth order" in section 5 and 'tenth order" in section 6. These are based on combinatorial interpretations of Hecke type series for sixth and tenth order mock theta functions.

2. Andrews' conjecture

We can find seven third order mock theta functions in Watson's paper [21]. To state these functions, we need to define some notations.

Notation. *For a complex number q with $|q|<1$, $|bc|<1$, and any integer n,*

$$(a;q)_\infty := \prod_{m=0}^{\infty}(1-aq^m), \quad (a;q)_n := \frac{(a;q)_\infty}{(aq^n;q)_\infty},$$

and

$$f(b,c) := \sum_{j=-\infty}^{\infty} b^{j(j+1)/2}c^{j(j-1)/2} = (-b;bc)_\infty(-c;bc)_\infty(bc;bc)_\infty.$$

Mock theta functions of third order:

$$f(q) = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n^2}, \quad \phi(q) = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q^2;q^2)_n},$$

$$\psi(q) = \sum_{n=1}^{\infty}\frac{q^{n^2}}{(q;q^2)_n}, \quad \chi(q) = \sum_{n=0}^{\infty}\frac{q^{n^2}}{\prod_{i=1}^{n}(1-q^i+q^{2i})},$$

$$\omega(q) = \sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2}, \quad \nu(q) = \sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(-q;q^2)_{n+1}},$$

$$\rho(q) = \sum_{n=1}^{\infty}q^{2n(n+1)}\prod_{i=1}^{n+1}(1+q^{2i-1}+q^{4i-2}).$$

Ramanujan provided the first four functions above in his letter, and Watson added the other three functions. Also, Watson proved all third order mock theta function identities in Ramanujan's letter.

In his last letter to Hardy, Ramanujan asserted that the coefficient of q^n in $f(q)$ is

$$(-1)^{n-1}\frac{\exp\left(\pi\sqrt{\frac{n}{6}-\frac{1}{144}}\right)}{2\sqrt{n-\frac{1}{24}}} + O\left(\frac{\exp\left(\frac{\pi}{2}\sqrt{\frac{n}{6}-\frac{1}{144}}\right)}{\sqrt{n-\frac{1}{24}}}\right)$$

without any proof. Then, L. Dragonette proved Ramanujan's claim in her Ph.D thesis [14]. Later, Andrews [1] improved Dragonette's result and conjectured the exact formula for the coefficients of $f(q)$.

Conjecture

Let $f(q) = \sum_{n=0}^{\infty} \alpha(n) q^n$. Then,

$$\alpha(n) = \pi(24n-1)^{-\frac{1}{4}} \sum_{k=1}^{\infty} \frac{(-1)^{\lfloor \frac{k+1}{2} \rfloor} A_{2k}(n - \frac{k(1+(-1)^k)}{4}) I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12k}\right)}{k},$$

where $I_s(x)$ is the usual I–Bessel function of order s and

$$A_k(n) := \frac{1}{2}\sqrt{\frac{k}{12}} \sum_{x \pmod{24k},\, x^2 \equiv -24n+1 \pmod{24k}} \left(\frac{12}{x}\right) e^{\frac{x}{12k}}.$$

K. Bringmann and K. Ono in [9] proved this conjecture with using the work of S. P. Zwegers [23] and theories of modular forms.

3. Mock Theta Conjectures

We are able to find several further results on fifth order mock theta functions in Ramanujan's lost notebook. We provide two groups of fifth order mock theta functions which are given by Ramanujan in his letter.

Mock theta functions of fifth order:

$$f_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n}, \quad \phi_0(q) = \sum_{n=0}^{\infty} q^{n^2}(-q;q^2)_n,$$
$$\psi_0(q) = \sum_{n=1}^{\infty} q^{n(n+1)/2}(-q;q)_{n-1}, \quad F_0(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n},$$
$$\chi_0(q) = \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1};q)_n} = 1 + \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q^{n+1};q)_{n+1}}.$$

$$f_1(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q)_n}, \quad \phi_1(q) = \sum_{n=1}^{\infty} q^{n^2}(-q;q^2)_{n-1},$$
$$\psi_1(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}(-q;q)_n, \quad F_1(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}},$$
$$\chi_1(q) = \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1};q)_{n+1}}.$$

In Ramanujan's lost notebook, we are able to find the following identities satisfied by fifth order mock theta functions:

$$M_1(q) \equiv \chi_0(q) - 2 - 3\Phi(q) + A(q) = 0,$$
$$M_2(q) \equiv F_0(q) - 1 - \Phi(q) + q\psi(q^5)H(q^2) = 0,$$
$$M_3(q) \equiv \phi_0(-q) + \Phi(q) - \frac{(q^5;q^5)_\infty G(q^2)H(q)}{H(q^2)} = 0,$$
$$M_4(q) \equiv \psi_0(q) - \Phi(q^2) + qH(q)\sum_{n=-\infty}^{\infty}(-1)^n q^{5n^2+4n} = 0,$$
$$M_5(q) \equiv f_0(q) + 2\Phi(q^2) - \vartheta_4(q^5)G(q) = 0,$$

$$M_6(q) \equiv q\chi_1(q) - 3\Psi(q) - qD(q) = 0,$$
$$M_7(q) \equiv qF_1(q) - \Psi(q) - q\psi(q^5)G(q^2) = 0,$$
$$M_8(q) \equiv \phi_1(-q) + \Psi(q) - \frac{q(q^5;q^5)_\infty G(q)H(q^2)}{G(q^2)} = 0,$$
$$M_9(q) \equiv \psi_1(q) - \frac{1}{q}\Psi(q^2) - G(q)\sum_{n=-\infty}^{\infty}(-1)^n q^{5n^2+2n} = 0,$$
$$M_{10}(q) \equiv f_1(q) + \frac{2}{q}\Psi(q^2) - \vartheta_4(q^5)H(q) = 0,$$

where

$$\vartheta_4(q) = \frac{(q;q)_\infty}{(-q;q)_\infty}, \quad \psi(q) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$
$$\Phi(q) = -1 + \sum_{n=0}^{\infty}\frac{q^{5n^2}}{(q;q^5)_{n+1}(q^4;q^5)_n},$$
$$\Psi(q) = -1 + \sum_{n=0}^{\infty}\frac{q^{5n^2}}{(q^2;q^5)_{n+1}(q^3;q^5)_n},$$
$$G(q) = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}, \quad H(q) = \sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q;q)_n},$$
$$A(q) = \frac{G(q)^2(q^5;q^5)_\infty}{H(q)}, \quad D(q) = \frac{H(q)^2(q^5;q^5)_\infty}{G(q)}.$$

Note that $G(q)$ and $H(q)$ which occur in the Rogers-Ramanujan identities [7].

In [5], Andrews and F. Garvan showed that five identities for functions in Ramanujan's first group of fifth order mock theta functions are equivalent to each other. Namely. the five are either true of false together. They call these the "First Mock Theta Conjecture". Similarly, they showed that five identities for the second group are equivalent, and called these "Second Mock Theta Conjecture". Furthermore, they provided the combinatorial interpretations, we need to define the rank of partitions and $N(b, 5, n)$. The rank of a partition is the largest part minus the number of parts, and $N(b, 5, n)$ is the number of partition of n with rank$\equiv b($ mod 5).

First Mock Theta Conjecture: $N(1,5,5n) - N(0,5,5n)$ equals the number of partitions of n with unique smallest part and no parts exceeding the double of the smallest part.

Second Mock Theta Conjecture: $2N(2,5,5n+3) - N(1,5,5n+3) - N(0,5,5n+3) - 1$ equals the number of partitions of n with unique smallest part and all other parts at most one larger than the double of the smallest part.

However, they were unable to provide the proofs of two mock theta conjectures. These two conjecture were proved by D. Hickerson [15] with using the constant term method and Bailey's lemma.

4. Bailey's Lemma and constant term method

Andrews [2] introduced the method of Bailey's lemma. To understand the method of Bailey's lemma, we need to see Bailey's lemma.

Lemma 1 (Bailey's Lemma). *If for $r \geq 0$ the sequences $\{\alpha_r\}$ and $\{\beta_r\}$ are related by*

$$\beta_r = \sum_{n=0}^{r} \frac{\alpha_n}{(q;q)_{r-n}(aq;q)_{r+n}},$$

then for $r \geq 0$,

$$\beta_r' = \sum_{n=0}^{r} \frac{\alpha_n'}{(q;q)_{r-n}(aq;q)_{r+n}},$$

where for any given numbers ρ_1 and ρ_2,

$$\alpha_n' = \frac{(\rho_1;q)_n(\rho_2;q)_n(aq/\rho_1\rho_2)^n\alpha_n}{(aq/\rho_1;q)_n(aq/\rho_2;q)_n},$$

and

$$\beta_r' = \frac{1}{(aq/\rho_1;q)_r(aq/\rho_2;q)_r}\sum_{j=0}^{r}\frac{(\rho_1;q)_j(\rho_2;q)_j(aq/\rho_1\rho_2;q)_{r-j}(aq/\rho_1\rho_2)^j\beta_j}{(q;q)_{r-j}}.$$

for any given number ρ_1 and ρ_2.

Note that a pair of sequences α_n, β_n is called a Bailey pair.

Anderws [2] proved that a pair of

$$\frac{1}{(-q;q)_n} \quad \text{and} \quad q^{n(n+1)/2}\sum_{j=-n}^{n}(-1)^j q^{-j^2} - q^{n(n-1)/2}\sum_{j=-n+1}^{n-1}(-1)^j q^{-j^2}$$

is a Bailey pair. With the above Bailey pair and Bailey's lemma, he obtained that Hecke type series for fifth order mock theta function $f_0(q)$ equals

$$\frac{1}{(q;q)_\infty}\sum_{n=0,|j|\geq n}^{\infty}(-1)^j q^{n(5n+1)/2-j^2}(1-q^{4n+2}).$$

Similarly, by the method of Bailey's lemma, Andrews provided Hecke type series for all fifth and seventh order mock theta functions in Ramanujan's last letter to Hardy. Also, Andrews and Hickerson [6] derived Hecke type series for sixth order mock theta functions, and Y.-S. Choi [10, 11] derived Hecke type series for tenth order mock thteat functions with the method of Bailey's lemma. These Hecke type series for mock theta functions are playing a key role to prove the results on mock theta functions in Ramanujan's lost notebook.

And, Andrews [3] showed that the constant term method can be applied to the fifth order mock theta functions. For instance, he showed that fifth order mock theta function $f_0(q)$ equals the coefficient of z^0 in

$$\frac{(q^5;q^5)_\infty^3 f(zq^4, z^{-1}q^{-1}) f(-z, -z^{-1}q^5) f(-q, -q^4)}{f(zq^3, z^{-1}q^2) f(zq^2, z^{-1}q^3) f(q^2, q^3)}.$$

5. FIFTH, SIXTH, AND SEVENTH ORDER MOCK THETA FUNCTIONS

Hickerson [15] proved two mock theta conjectures with Bailey's lemma and a modification of constant term method. To prove mock theta conjectures, he rewrote Heck type series for fifth order mock theta functions $f_0(q)$ and $f_1(q)$. Namely, he showed that $f_0(q)$ equals

$$\frac{1}{(q;q)_\infty} \sum_{sg(r)=sg(s), r\equiv s (\mod 2)} sg(r)(-1)^{(r-s)/2} q^{rs+\frac{3}{8}(r+s)^2+\frac{1}{4}(r+s)},$$

and $f_1(q)$ equals

$$\frac{1}{(q;q)_\infty} \sum_{sg(r)=sg(s), r\equiv s (\mod 2)} sg(r)(-1)^{(r-s)/2} q^{rs+\frac{3}{8}(r+s)^2+\frac{3}{4}(r+s)},$$

where $sg(t)$ equals 1 if $t \geq 0$, or -1 if $t < 0$. With the new representations for $f_0(q)$ and $f_1(q)$, he showed that $qf_0(q)$ equals the coefficient of z in the Laurent series expansion of $B(q,z)$ which is $z^2(q^2;q^2)_\infty f(z, z^{-1}q) f(-z, -z^{-1}q^3)/f(-z, -z^{-1}q^2)$, and $f_1(q)$ is the coefficient of z^2. Then, he derived two different representations for $B(q,z)$. The first is given in terms of theta functions, generalized Lambert series, and the $f_0(q)$ and $f_1(q)$. (In a generalized Lambert series, the powers of q in the numerator are quadratic forms in the summation variable.) The second is represented solely in terms of theta functions. Upon equating the coefficients in the two representations for $B(q,z)$, he then proved the mock theta function conjectures.

Eleven identities for sixth order mock theta functions are found in the lost notebook; these were established by Andrews and Hickerson [6] with Bailey's lemma and constant term method.

Mock theta functions of sixth order:

$$\phi(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} (q;q^2)_n}{(-q;q)_{2n}}, \quad \psi(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2} (q;q^2)_n}{(-q;q)_{2n+1}},$$

$$\rho(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2} (-q;q)_n}{(q;q^2)_{n+1}}, \quad \sigma(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)/2} (-q;q)_n}{(q;q^2)_{n+1}},$$

$$\lambda(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^n (q;q^2)_n}{(-q;q)_n}, \quad \mu(q) = \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n}{(-q;q)_n},$$

$$\nu(q) = \sum_{n=0}^{\infty} \frac{q^{n^2} (q;q)_n}{(q^3;q^3)_n}.$$

$$
\begin{aligned}
\phi(q^9) - \psi(q) - q^{-3}\psi(q^9) &= \frac{(q^6;q^6)_\infty^2 f(q^3,q^9)}{f(q,q^3)f(q^9,q^{27})},\\
\frac{\psi(\omega q) - \psi(\omega^2 q)}{(\omega - \omega^2)q} &= \frac{(q^3;q^6)_\infty^2 f(q,q^3)f(q^9,q^{27})}{f(q^3,q^9)},\\
(q;q)_\infty \phi(q) &= 1 - 2\sum_{n=-\infty}^{\infty} \frac{q^{(2n+1)(3n+1)}}{1+q^{3n+1}} + 2\sum_{n\geq 1} \frac{q^{n(6n+1)}}{1-q^n+q^{2n}},\\
f(q,q^2)\phi(q) &= 2\sum_{n=-\infty}^{\infty} \frac{q^{n(3n+1)/2}}{1+q^{3n}},\\
f(q^3,q^9)\phi(q) &= \sum_{n=-\infty}^{\infty} \frac{q^{3n(n+1)/2}}{1+q^{3n+1}},\\
f(q,q^2)\psi(q) &= \sum_{n=-\infty}^{\infty} \frac{q^{(n+1)(3n+2)/2}}{1+q^{3n+1}},\\
q^{-1}\psi(q^2) + \rho(q) &= (-q;q^2)_\infty^2 f(q,q^5),\\
\phi(q^2) + 2\rho(q) &= (-q;q^2)_\infty^2 f(q^3,q^3),\\
2\phi(q^2) - 2\mu(-q) &= (-q;q^2)_\infty^2 f(q^3,q^3),\\
2q^{-1}\psi(q^2) + \lambda(-q) &= (-q;q^2)_\infty^2 f(q,q^5),\\
2\nu(q) &= 3\phi(q) - \frac{f(-q,-q)^2}{f(q,q^2)},
\end{aligned}
$$

where ω is a primitive cube root of unity.

With Bailey's lemma and the constant term method, Hickerson [16] derived analogs of $M_5(q)$ for each of the seventh order mock theta functions.

6. TENTH ORDER MOCK THETA FUNCTIONS

Lastly, the lost notebook contains eight identities for tenth order mock theta functions which were proved by Choi [10, 11, 12, 13].

Mock theta functions of tenth order:

$$
\begin{aligned}
\phi(q) &:= \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q^2)_{n+1}}, & \psi(q) &:= \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)/2}}{(q;q^2)_{n+1}},\\
X(q) &:= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-q;q)_{2n}}, & \chi(q) &:= \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}}{(-q;q)_{2n+1}},
\end{aligned}
$$

$$q^{2/3}\phi(q^3) - \frac{\psi(\omega q^{1/3}) - \psi(\omega^2 q^{1/3})}{\omega - \omega^2} = -q^{1/3}\frac{f(-q^{1/3}, -q^{1/3})f(-q, -q^4)}{f(-q, -q)(q; q^2)_\infty},$$

$$q^{-2/3}\psi(q^3) + \frac{\omega\phi(\omega q^{1/3}) - \omega^2\phi(\omega^2 q^{1/3})}{\omega - \omega^2} = \frac{f(-q^{1/3}, -q^{1/3})f(-q^2, -q^3)}{f(-q, -q)(q; q^2)_\infty},$$

$$X(q^3) - \frac{\omega\chi(\omega q^{1/3}) - \omega^2\chi(\omega^2 q^{1/3})}{\omega - \omega^2} = \frac{f(q^{1/3}, q)f(-q^4, -q^6)}{f(-q, -q^3)(-q^3; q^3)_\infty},$$

$$\chi(q^3) + q^{2/3}\frac{X(\omega q^{1/3}) - X(\omega^2 q^{1/3})}{\omega - \omega^2} = -q\frac{f(q^{1/3}, q)f(-q^2, -q^8)}{f(-q, -q^3)(-q^3; q^3)_\infty},$$

$$\phi(q^{1/2}) - q^{-1/2}\psi(-q^2) + q^{-1}\chi(q^4) = f(q^{1/2}, q^{1/2})\frac{f(q, -q^4)}{f(-q, -q^3)},$$

$$\psi(q^{1/2}) + q^{1/2}\phi(-q^2) + X(q^4) = f(q^{1/2}, q^{1/2})\frac{f(-q^2, q^3)}{f(-q, -q^3)},$$

$$\int_0^\infty \frac{e^{-\pi n x^2}}{\cosh\frac{2\pi x}{\sqrt{5}} + \frac{1+\sqrt{5}}{4}}dx + \frac{1}{\sqrt{n}}e^{\frac{\pi}{5n}}\psi(-e^{-\frac{\pi}{n}})$$

$$= \sqrt{\frac{5+\sqrt{5}}{2}}e^{-\frac{\pi n}{5}}\phi(-e^{-\pi n}) - \frac{\sqrt{5}+1}{2\sqrt{n}}e^{-\frac{\pi}{5n}}\phi(-e^{-\frac{\pi}{n}}),$$

$$\int_0^\infty \frac{e^{-\pi n x^2}}{\cosh\frac{2\pi x}{\sqrt{5}} + \frac{1-\sqrt{5}}{4}}dx + \frac{1}{\sqrt{n}}e^{\frac{\pi}{5n}}\psi(-e^{-\frac{\pi}{n}})$$

$$= -\sqrt{\frac{5-\sqrt{5}}{2}}e^{\frac{\pi n}{5}}\psi(-e^{-\pi n}) + \frac{\sqrt{5}-1}{2\sqrt{n}}e^{-\frac{\pi}{5n}}\phi(-e^{\frac{\pi}{n}}).$$

Bailey's lemma and the constant term method were also playing a key role in proofs of eight identities, but additional techniques and ideas were needed. For instance, several complicated theta function identities were needed to be established. Some of the theta function identities can be proved by classical means, but for others the theory of modular forms must be invoked.

Among eight identities above, seventh and eighth identities are only identities which involve definite integrals. To prove these, the transformation formula for the Mordell type integral given by L. J. Mordell [17] was needed.

7. Conclusion

I would like to conclude this survey with Watson's comment in his paper [21]. "Ramanujan's discovery of the mock theta functions makes it obvious that his skill and ingenuity did not desert him at the oncoming of his untimely end. As much as any of his earlier work, the mock theta functions are an achievement sufficient to cause his name to be held in lasting remembrance."

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