

Solving Cubic Equations by ORIGAMI

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Extended Abstract

This article is an excerpt of the full version paper[1] with the other references omitted, and its purpose is to show an algebraic proof of the method for solving cubic equations by origami (paper folding).

In the early 1980's, H.Abe solved the construction problems using origami that are unsolvable in Euclidean geometry, such as angle trisection and doubling cubes. This paper expands H.Abe's methods to general cubic equations and shows the main results in three propositions. For an algebraic proof, we solve the radical membership problem in polynomial ideals computing a Gröbner basis together with constraints of parameters, which correspond to geometrically degenerate cases. Consequently, cubic equations are clearly solved as construction problems.

First, we expand H.Abe's angle trisection method(1980) to the case with obtuse angles, and prove it by a Gröbner basis. Assuming a semi-transparent sheet of origami, we can trisect an arbitrary obtuse angle as shown in Fig. 1.

Proposition 1 (Trisection of an obtuse angle by origami)

Let points $A(0, 1)$, $B(0, 0)$ and $C(c, 0)$ ($\exists c > 0$) be fixed, and $E(a, 1)$ be chosen with an arbitrary $a (< 0)$. Then the trisection of $\angle EBC$ is constructed as follows.

- (i) Mark two points $H(0, b)$, $F(0, 2b)$ with a proper length $b (> 0)$.
- (ii) Draw the horizontal line $y = b$ through H .
- (iii) Fold the paper to place simultaneously F onto BE and B onto $y = b$.
- (iv) If we let the point B be mapped to point B' , then we have $\angle B'BC = (\angle EBC)/3$.

Proof Let the points F and B be mapped to $F'(ay, y)$ and $B'(x, b)$ respectively. From the symmetry by folding with the crease PQ , we have an isosceles trapezoid $F'BB'F$.

- (1) Since $F'B = FB'$, we have $f_1 := (ay)^2 + y^2 - (x^2 + b^2)$.
- (2) Since $F'F \parallel BB'$ means $(2b - y)/(-ay) = b/x$, we have $f_2 := (2b - y)x + b(ay)$.
- (3) The slopes of BB' , BF' and $B'F$ are $k = b/x$, $k_1 = 1/a$ and $k_2 = b/(-x)$ respectively. Then we have

$$\tan \angle B'BF' = \frac{k_1 - k}{1 + k_1k} = \frac{x - ab}{ax + b}, \quad \tan \angle FB'B = \frac{k - k_2}{1 + k_2k} = \frac{2bx}{x^2 - b^2}.$$

Therefore, from $\angle B'BF' = \angle FB'B$, it is deduced that $f_3 := (x - ab)(x^2 - b^2) - 2bx(ax + b)$.

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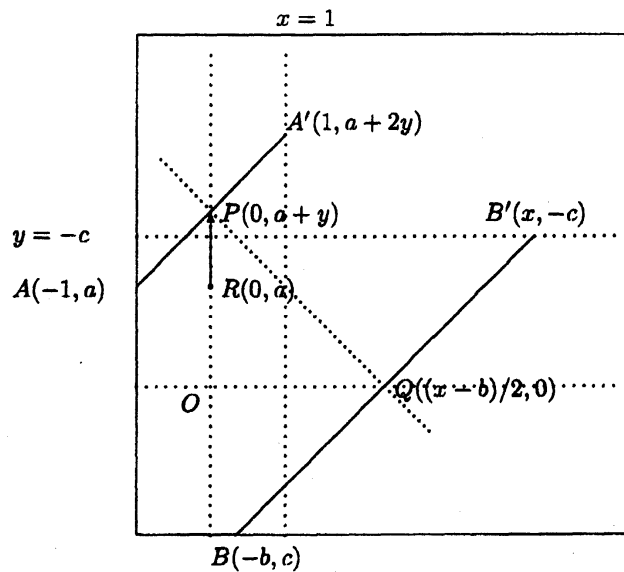


Fig. 2: A solution \overrightarrow{RP} for the cubic equation $t^3 + at^2 + bt + c = 0$

Proposition 2 (Construction for cubic equations - 1)

Given a cubic equation $t^3 + at^2 + bt + c = 0$, we arrange the points $A(-1, a)$, $B(-b, c)$ on a square origami as Fig. 2, where $c < 0$ is assumed. Then a real solution to the equation is constructed as follows.

- (i) Draw the lines $x = 1$ and $y = -c$.
- (ii) Fold the paper to place simultaneously A onto $x = 1$ and B onto $y = -c$.
- (iii) If we let P be the midpoint of the segment AA' , then \overrightarrow{RP} gives a real solution to the equation.

Proof Let the point $A(-1, a)$ be mapped to $A'(1, a + 2y)$, and $B(-b, c)$ to $B'(x, -c)$. Then, the midpoints of AA' and BB' are $P(0, a + y)$ and $Q((x - b)/2, 0)$ respectively.

- (1) From the symmetry by folding with the crease PQ , we deduce the following polynomials:

$$\begin{aligned} AB = A'B' &\Rightarrow f_1 := -x^2 + 2x - 4y^2 - 4(a + c)y + b^2 - 2b - 4ac, \\ AA' \parallel BB' &\Rightarrow f_2 := xy + by + 2c, \\ AA' \perp PQ &\Rightarrow f_3 := -x + 2y^2 + 2ay + b. \end{aligned}$$

- (2) The conclusion to be proved is that $\overrightarrow{RP} (= y)$ is a solution to the given equation. Hence, we let $p := y^3 + ay^2 + by + c$.
- (3) We regard $J = (f_1, f_2, f_3, 1 - zp)$ as an ideal in $\mathbf{Q}(a, b, c)[x, y, z]$. Computing its Gröbner basis with the total degree lexicographic order ($x > y > z$), we obtain $J = (1)$ and the conclusion p is correct.
- (4) Subsidiary condition for parameters is empty and it shows that this proposition holds for any real coefficients a, b, c .

Remark 2

If the given cubic equation has three real zeros, there exist three ways of folding.

For the case $c > 0$, Fig. 2 should be turned upside down. If $c = 0$, this construction essentially gives a real solution to quadratic equations. Therefore, given cubic equations $t^3 + at^2 + bt + c = 0$ with arbitrary real coefficients, its real zero(s) can be constructed by origami.

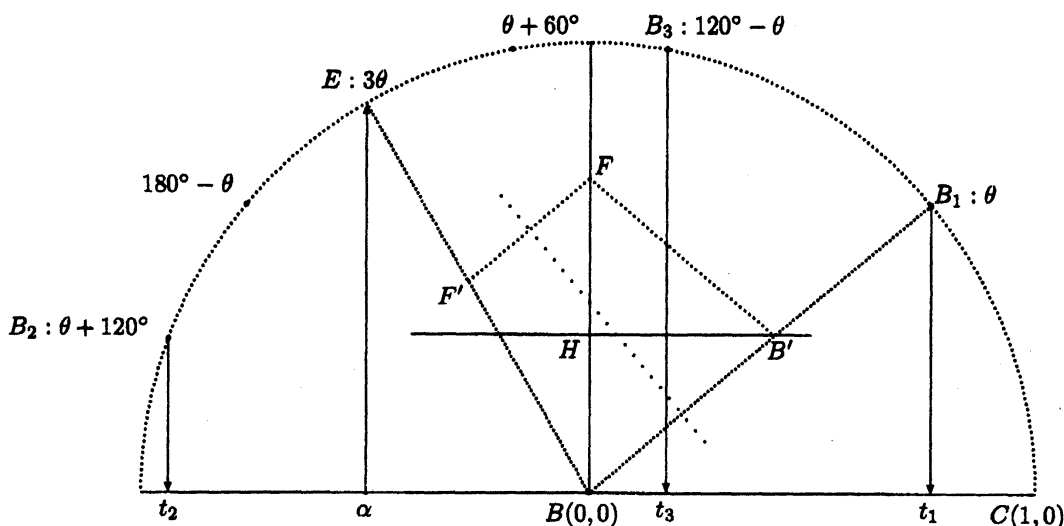


Fig. 3: Solutions for the cubic equation $4t^3 - 3t - \alpha = 0$ ($|\alpha| < 1$)

Finally, we show another origami solution to cubic equations $t^3 + at^2 + bt + c = 0$ ($a, b, c \in \mathbf{R}$) with three real zeros. The following proposition is based on a classical formula, but it cannot be constructed within Euclidean geometry because angle trisection is necessary.

Proposition 3 (Construction for cubic equations - 2)

A cubic equation $x^3 + 3px + 2q = 0$ has three real zeros, iff $p^3 + q^2 < 0$ ($p < 0$) from its discriminant. If we let $x = 2\sqrt{-p} \cdot t$, then the above equation is transformed into

$$4t^3 - 3t - \alpha = 0 \quad \alpha = \frac{q}{\sqrt{-p^3}} \quad (-1 < \alpha < 1). \quad (\text{I})$$

Compared with the triple angle formula $\cos 3\theta = 4\cos^3 \theta - 3\cos \theta$, let $\alpha = \cos 3\theta$ as $0^\circ < 3\theta < 180^\circ$. Then, we obtain the three real zeros for (I):

$$t = \cos \theta, \quad \cos(\theta + 120^\circ), \quad \cos(\theta + 240^\circ) \quad (\text{II})$$

The third one is rewritten as $\cos(\theta + 240^\circ) = \cos(\theta - 120^\circ) = \cos(120^\circ - \theta) = \cos((180^\circ - \theta) - 60^\circ)$. Since $0^\circ < \theta < 60^\circ$, all the zeros (II) are found in the upper semicircle.

In Fig. 3, we show an example of construction for a given equation $4t^3 - 3t + 1/2 = 0$ ($\alpha = -1/2$).

- (i) Mark the point E on the upper semicircle whose orthogonal projection is $x = \alpha$. In this example, $3\theta = 120^\circ$.

- (ii) Trisect $\angle EBC$ by Prop. 1 using origami, and we obtain the trisector BB' . Then, mark the point B_1 on the semicircle.
- (iii) The orthogonal projection of $\overrightarrow{BB_1}$ gives the first zero : $t_1 = \cos \theta = \cos 40^\circ$.
- (iv) Using a compass, construct $\theta + 120^\circ (= \theta + 2 \times 60^\circ)$, and we obtain B_2 . Then, the projection of $\overrightarrow{BB_2}$ gives the second zero : $t_2 = \cos 160^\circ$.
- (v) Using a compass, construct $(180^\circ - \theta) - 60^\circ$, and we obtain B_3 . Then, the projection of $\overrightarrow{BB_3}$ gives the third zero : $t_3 = \cos 80^\circ$.

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In summary, constructing real zeros for cubic equations by origami is discussed in this paper, and two methods are proposed together with algebraic proofs using Gröbner bases. Those are respectively related to classical origami construction problems as follows.

- Prop. 2 : Expansion of the method for doubling cubes.
- Prop. 3 : Application of the method for angle trisection.

In future work, following problems should be studied to expand the present results.

- How to represent complex solutions using origami.
- How to solve quartic equations by origami. In principle, it is reduced to solving auxiliary equations with degrees two and three, but its clear representation is not known yet.

References

- [1] Moritsugu, S.: Solving Cubic Equations by ORIGAMI, *Trans. Japan Soc. Indust. Appl. Math.*, **16**(1), 2006, 79–92. (in Japanese).