On Lie algebras of vector fields of manifolds with singularities

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§1. Introduction

In this talk we shall consider Pursell-Shanks type theorem for some manifolds with singularities.

Let $\mathcal{X}(M)$ be the Lie algebra of smooth vector fields on a connected smooth manifold $M$ with compact support. Then Pursell and Shanks proved the following.

Theorem 1.1 (Pursell-Shanks [PS])

Let $M$ and $N$ be connected smooth manifolds. If $\mathcal{X}(M)$ and $\mathcal{X}(N)$ are isomorphic as a Lie algebra, then $M$ and $N$ are diffeomorphic.

There are many analogous results on the Lie algebra of smooth vector fields which preserve a geometric structures (c.f. [AM], [BA], [FU], [GP], [GR], [OM], [KO]). We extended Theorem 1.1 to the case of smooth orbifold.

Theorem 1.2 (K. Abe [AB2])

Let $M$ and $N$ be connected smooth orbifold. If $\mathcal{X}(M)$ and $\mathcal{X}(N)$ are isomorphic as a Lie algebra, then $M$ and $N$ are diffeomorphic.

Note that a smooth orbifold is locally diffeomophic to the orbit space $V/\Gamma$ of a representation space $V$ of a finite group $\Gamma$. In this paper we consider when $\Gamma$ is a discrete subgroup of $SL(2, \mathbb{Z})$.

§2. Statement of the result

Let $\mathcal{H}$ denote the upper half complex plane. Let $SL(2, \mathbb{R})$ be the group of real matrix with determinant 1. Then $SL(2, \mathbb{R})$ acts on $\mathcal{H}$ by the Möbius as the following.

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $z \in \mathcal{H}$,

$$g \cdot z = \frac{az + b}{cz + d}.$$
Then $SL(2, \mathbb{R})$ acts transitively on $\mathcal{H}$ and the isotropy subgroup at $i = \sqrt{-1}$ is

$$SL(2, \mathbb{R})_i = SO(2).$$

The kernel of the action is $\mathbb{Z}_2 = \{\pm 1\}$ and $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{\pm 1\}$ acts effectively on $\mathcal{H}$ and

$$\mathcal{H} \cong SL(2, \mathbb{R})/SO(2).$$

The action can be extended to the Riemannian sphere $\bar{C} = \mathbb{C} \cup \{\infty\}$.

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \ z \in \bar{C}$,

$$g \cdot z = \begin{cases} \frac{az + b}{cz + d} & (z \neq -\frac{d}{c}, \infty) \\ \infty & (z = -\frac{d}{c}, \ z = d = 0) \\ \frac{a}{c} & (z = \infty) \end{cases}.$$  

Set

$$R_1 = \left\{ \pm \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a > 0 \right\}$$

and

$$R_2 = \left\{ \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ \pm \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \right\}.$$ 

Then each $g \in SL(2, \mathbb{R})$ is conjugate to one of the elements of $SO(2) \cup R_1 \cup R_2$, and $g \neq \pm 1$ is called elliptic, hyperbolic and parabolic if $g$ is conjugate to an element in $SO(2), R_1$ and $R_2$, respectively.

Let $\Gamma$ denote a discrete subgroup of $SL(2, \mathbb{R})$. $z \in \mathcal{H}$ is called elliptic point if there exists an elliptic element $g \in \Gamma$ such that $g \cdot z = z$. $x \in \mathbb{R} \cup \{\infty\}$ is called cusp point if there exists a parabolic element $g \in \Gamma$ such that $g \cdot z = z$.

**Proposition 2.1** (1) If $z$ is a elliptic point, then $\Gamma_z$ is a cyclic group which is conjugate to a cyclic subgroup of $SO(2)$.

(2) If $x$ is a cusp point, then $\Gamma_x$ is isomorphic to $\mathbb{Z}$ which is conjugate to a subgroup of the group

$$\Gamma_{\infty} = \left\{ \pm \begin{pmatrix} 1 & nk \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\} (\exists k \in \mathbb{Z}).$$

Let $E_{\Gamma}$ denote the set of all elliptic points in $\mathcal{H}$ and $C_{\Gamma}$ be the set of cusp points of $\Gamma$. Set $\mathcal{H}^* = \mathcal{H} \cup C_{\Gamma}$,
We shall give the following topology on $\mathcal{H}^*$.

(1) We give the canonical topology on $\mathcal{H}$.

(2) Let $x \in C_\Gamma$.

2.1 If $x \neq \infty$, then we take all the family of the form
$$\{x\} \cup \{\text{the interior of a circle in } \mathcal{H} \text{ tangent to the real axis at } x\}$$
as a fundamental system of open neighborhoods of $x$.

2.2 If $x = \infty$, then
$$\{\infty\} \cup \cup_{c>0}\{z \in \mathcal{H} | \Im z > c\}$$
as a fundamental system of open neighborhood of the point $\infty$. Then $\Gamma$ acts on $\mathcal{H}^*$ as a topological transformation group. Set
$$\mathcal{R}_\Gamma = \mathcal{H}^*/\Gamma = \mathcal{H}/\Gamma \cup C_\Gamma/\Gamma$$
Then $\mathcal{R}_\Gamma$ is a Hausdorff space.

**Lemma 2.2** For each $x \in C_\Gamma$, there exists an open neighborhood $\tilde{U}_x$ of $x$ in $\mathcal{H}^*$ such that
$$\Gamma_x = \{\gamma \in \Gamma | \gamma \cdot \tilde{U}_x \cap \tilde{U}_x \neq \emptyset\}.$$

Take $x \in C_\Gamma$. Let $\iota_x : \tilde{U}_x/\Gamma_x \mapsto \mathcal{R}_\Gamma$ be a map defined by $\iota_x(\Gamma_x \cdot z) = \Gamma \cdot z$ for $z \in \tilde{U}_x$. Put $p = \Gamma \cdot x$. Then $U_p = \iota_x(\tilde{U}_x/\Gamma_x)$ is an open neighborhood of $p$ in $\mathcal{R}_\Gamma$.

For $x \in C_\Gamma$, there exist $g \in SL(2, \mathbb{R})$ and integer $k$ such that $g \cdot x = \infty$ and
$$g\Gamma_xg^{-1} = \left\{ \pm \begin{pmatrix} 1 & nk \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\}.$$

**Proposition 2.3** Let $\varphi_p : \tilde{U}_x/\Gamma_x \mapsto \mathbb{C}$ be a map given by
$$\varphi_p(\Gamma_xz) = \begin{cases} \exp\left(\frac{2\pi\sqrt{-1}}{k}(g \cdot z)\right) & (z \in \tilde{U}_x \setminus \{x\}), \\
0 & (z = x). \end{cases}$$

Then $\varphi_p$ is homeomorphic to an open subset $W_p$ of $\mathbb{C}$.

By Proposition 2.3, the map
$$\psi_p = \varphi_p \circ \iota_x^{-1} : U_p \mapsto \tilde{U}_x/\Gamma_x \mapsto W_p$$
is regarded as a local coordinate of $\mathcal{R}_\Gamma$ around $p$.

**Definition 2.4** $f : \mathcal{R}_\Gamma \mapsto \mathbb{R}$ is defined to be smooth if

1. $f \circ \pi_\Gamma$ is smooth, where $\pi_\Gamma : \mathcal{H} \mapsto \mathcal{H}/\Gamma$ is the natural projection,
2. for each $p \in \tilde{C}_\Gamma$, $f \circ \psi_p^{-1}$ is smooth.
Definition 2.4 (2) does not depend on the choice of $x$ with $\Gamma \cdot x = p$.

Let $C^\infty(\mathcal{R}_\Gamma)$ denote the set of all real valued smooth functions on $\mathcal{R}_\Gamma$.

**Definition 2.5** For discrete subgroups $\Gamma, \Gamma'$ of $SL(2, \mathbb{R})$, $h : \mathcal{R}_\Gamma \to \mathcal{R}_{\Gamma'}$ is said smooth if for each real valued smooth function $f : \mathcal{R}_{\Gamma'} \to \mathbb{R}$ $f \circ h$ is smooth. $h$ is said diffeomorphic if $h$ and $h^{-1}$ are smooth.

**Definition 2.6** A derivation $X$ of $C^\infty(\mathcal{R}_\Gamma)$ is called a smooth vector field on $\mathcal{R}_\Gamma$ if $X$ vanishes on $C_\Gamma$. Let $\mathcal{L}(\mathcal{R}_\Gamma)$ denote the set of all smooth vector field on $\mathcal{R}_\Gamma$ and let $\mathcal{X}(\mathcal{R}_\Gamma)$ be the subalgebra of $\mathcal{L}(\mathcal{R}_\Gamma)$ which consists of vector fields with compact support.

Then we have the following.

**Theorem 2.7** Let $\Gamma$ and $\Gamma'$ be discrete subgroups of $SL(2, \mathbb{R})$. Then $\mathcal{R}_\Gamma$ and $\mathcal{R}_{\Gamma'}$ are diffeomorphic if and only if $\mathcal{X}(\mathcal{R}_\Gamma)$ and $\mathcal{X}(\mathcal{R}_{\Gamma'})$ are isomorphic as a Lie algebra.

§3. Maximal ideals of $\mathcal{X}(\mathcal{R}_\Gamma)$

In order to prove Theorem 2.7 we investigate the maximal ideals of $\mathcal{X}(\mathcal{R}_\Gamma)$. Let $\Gamma$ be a discrete subgroup of $SL(2, \mathbb{R})$. Let $E_\Gamma = E_\Gamma / \Gamma$ and $C_\Gamma$ denote the set of elliptic singularities and cusp singularities in $\mathcal{R}_\Gamma$, respectively. Set $S_\Gamma = E_\Gamma \cup C_\Gamma$ which is the set of singularities in $\mathcal{R}_\Gamma$. We abbreviate $\mathcal{R}_\Gamma, S_\Gamma$ and $E_\Gamma$ to $\mathcal{R}, S$ and $E$, respectively. Let $\mathcal{R}_1 = \mathcal{R} \setminus S$ be the set of regular points of $\mathcal{R}$. For each $p \in \mathcal{R}_1$, set

$$\mathcal{X}_p(\mathcal{R}) = \{X \in \mathcal{X}(\mathcal{R})| X(p) = 0\}.$$  

**Proposition 3.1** For each $p \in \mathcal{R}_1$, there exists a unique maximal ideal $\mathcal{M}_p$ of $\mathcal{X}(\mathcal{R})$ which is contained in $\mathcal{X}_p(\mathcal{R})$. Moreover $\mathcal{M}_p$ is an infinite codimensional subalgebra in $\mathcal{X}(\mathcal{R})$.

Next we shall find the maximal ideals of $\mathcal{X}(\mathcal{R})$ which correspond to the singularities in $\mathcal{R}$. Here we recall the results by Bierstone and Schwarz. Let $G$ be a finite group and $V$ be a representation space of $G$. Let $\pi : V \to V/G$ be the natural projection. $\mathcal{X}_G(V)$ denotes the Lie algebra of $G$-invariant smooth vector fields on $V$ with compact support.

**Theorem 3.2** (Bierstone [BI] and Schwarz [SC])

The induced map $\pi_* : \mathcal{X}_G(V) \to \mathcal{X}(V/G)$ is a Lie algebra isomorphism.
(I) For each \( p \in \overline{E} \), take \( x_p \in E \) with \( \Gamma \cdot x_p = p \). Let \( V_{x_p} \) be the linear slice at \( x_p \). Then \( V_{x_p} \) is a \( \Gamma_{x_p} \)-module. Let

\[
(\pi_{x_p})_* : \mathcal{X}_{\Gamma_{x_p}}(V_{x_p}) \to \mathcal{X}(V_{x_p}/\Gamma_{x_p}) \hookrightarrow \mathcal{X}(\mathcal{R})
\]

be the natural Lie algebra homomorphism. By Theorem 3.2, for each \( X \in \mathcal{X}(\mathcal{R}) \) there exists \( Y_{x_p} \in \mathcal{X}_{\Gamma_{x_p}}(V_{x_p}) \) such that \( (\pi_{x_p})_*(Y_{x_p}) = X \) on a neighborhood of \( p \) in \( \mathcal{R} \). Let \( \mathfrak{gl}_{\Gamma_{x_p}}(V_{x_p}) \) be the set of \( \Gamma_{x_p} \)-invariant linear endmorphisms. Let

\[
J_p : \mathcal{X}(\mathcal{R}) \to \mathfrak{gl}_\Gamma(V_{x_p})
\]

be the homomorphism defined by \( J_p(X) = j^1_{x_p}(Y_{x_p}) \), where \( j^1_{x_p}(Y_{x_p}) \) is the 1-jet of \( Y_{x_p} \) at \( x_p \).

(II) For \( p \in \overline{C} \) there is a chart \( \psi_p : U_p \to W_p \subset \mathbb{C} = \mathbb{R}^2 \) around the open neighborhood \( U_p \) of \( p \) in \( \mathcal{R} \). Let

\[
J_p : \mathcal{X}(\mathcal{R}) \to \mathfrak{gl}(2, \mathbb{R})
\]

be the Lie algebra homomorphism defined by \( J_p(X) = j^1_{p}(X|_{U_p}) \).

Combining (I) and (II) we set

\[
J(\mathcal{R}) = \bigoplus_{p \in \overline{E}} \mathfrak{gl}_{\Gamma_{x_p}}(V_{x_p}) \bigoplus \bigoplus_{p \in \overline{C}} \mathfrak{gl}(2, \mathbb{R}).
\]

Let \( J : \mathcal{X}(\mathcal{R}) \to J(\mathcal{R}) \) be a Lie algebra homomorphism defined by

\[
J(X) = \bigoplus_{p \in \overline{E}} J_p(X) \bigoplus \bigoplus_{p \in \overline{C}} J_p(X).
\]

**Lemma 3.3** \( J \) is an onto Lie algebra homomorphism.

**Proposition 3.4** If \( \mathcal{M} \) is a maximal ideal of \( \mathcal{X}(\mathcal{R}) \), then we have the following.

1. If \( \mathcal{M} \) is contained in \( \mathcal{X}_p(\mathcal{R}) \) for some \( p \in \mathcal{R}_1 \), then \( \mathcal{M} = \mathcal{M}_p \), and \( \mathcal{M} \) is an infinite codimensional subalgebra of \( \mathcal{X}(\mathcal{R}) \).

2. If \( \mathcal{M} \not\subset \mathcal{X}_p(\mathcal{R}) \) for any \( p \in \mathcal{R}_1 \), then there exists a maximal ideal \( \mathcal{L} \) of \( J(\mathcal{R}) \) such that \( \mathcal{M} = J^{-1}(\mathcal{L}) \), and \( \mathcal{M} \) is a finite codimensional subalgebra of \( \mathcal{X}(\mathcal{R}) \).
§4. Stone topology of the maximal ideals

Let $\mathcal{R}^*$ be the set of all maximal ideals of $\mathcal{X}(\mathcal{R})$.

Definition 4.1 The Stone topology on $\mathcal{R}^*$ is defined by the closure operator $\mathcal{C}\ell$ as following.

1. $\mathcal{C}\ell(\emptyset) = \emptyset$,
2. For a subset $B$ of $\mathcal{R}^*$ with $B \neq \emptyset$,
   \[
   \mathcal{C}\ell(B) = \left\{ \mathfrak{M} \in \mathcal{R}^* \mid \mathfrak{M} \supset \bigcap_{\mathfrak{M}' \in B} \mathfrak{M}' \right\}.
   \]

Let $\mathcal{O}(S)$ denote the family of all subsets of $S$. We define a map

\[ \tau_{\mathcal{R}} : \mathcal{R}_1 \rightarrow \mathcal{R}_1 \cup \mathcal{O}(S) \]

by the following way.

1. For $p \in \mathcal{R}_1$, $\tau_{\mathcal{R}}(\mathcal{M}_p) = p$.
2. If $\mathfrak{M} \in \mathcal{R}^*$ such that $\mathfrak{M} \not\subset \mathcal{X}_p(\mathcal{R})$ for any $p \in \mathcal{R}_1$, then
   \[ \tau_{\mathcal{R}}(\mathfrak{M}) = \{ p \in S \mid J(\mathfrak{M}) \not\supset J_p(\mathcal{X}(\mathcal{R})) \}. \]

Set $\mathcal{R}_1^* = \{ \mathcal{M}_p \in \mathcal{R}^* \mid p \in \mathcal{R}_1 \}$.

Proposition 4.2

The map $\tau_{\mathcal{R}} : \mathcal{R}_1^* \rightarrow \mathcal{R}_1$ is homeomorphic.

Definition 4.3 (End)

Let $\mathcal{K}(\mathcal{R}_1) = \{ K_i \mid i \in I \}$ denote the family of compact subset in $\mathcal{R}_1$. For each $K \in \mathcal{K}(\mathcal{R}_1)$, let $\mathcal{C}_K :$ be the set of connected component of $\mathcal{R}_1 \setminus K$.

\[ \prod_{K_i \in \mathcal{K}(\mathcal{R}_1)} C_{K_i} \in \prod_{K_i \in \mathcal{K}(\mathcal{R}_1)} \mathcal{C}_{K_i} \]

is said to be an end of $\mathcal{R}_1$ if $C_{K_i} \subset C_{K_j}$ for any pair $i, j \in I$ with $K_j \subset K_i$.

$\mathcal{E}(\mathcal{R}_1) :$ the set of all ends of $\mathcal{R}_1$

For each $p \in S$ there exists a unique end $\mathcal{E}_p = \prod_{K_i \in \mathcal{K}(\mathcal{R}_1)} C_{K_i}$ in $\mathcal{R}_1$ such that

\[ \bigcap_{K_i \in \mathcal{K}(\mathcal{R}_1)} cl(C_{K_i}) = \{ p \}, \text{ where } cl(C_{K_i}) \text{ is the closure of } C_{K_i} \text{ in } \mathcal{R}. \]

Set

\[ \mathcal{E}_0(\mathcal{R}_1) = \{ \mathcal{E}_p \mid p \in S \}, \quad \bar{\mathcal{R}}_1 = \mathcal{R}_1 \cup \mathcal{E}(\mathcal{R}_1). \]
Then $\overline{\mathcal{R}}_1$ has the natural topology such that

$$\{C_{K_j} \cup \prod_{K_i \in \mathcal{R}(\mathcal{R}_1)} C_{K_i} | K_j \in \mathcal{R}(\mathcal{R}_1)\}$$

is the fundamental system of neighborhood of a point $\prod_{K_i \in \mathcal{R}(\mathcal{R}_1)} C_{K_i} \in \mathcal{E}(\mathcal{R}_1)$.

Put $\overline{\mathcal{R}}_0 = \mathcal{R}_1 \cup \mathcal{E}_0(\mathcal{R}_1)$. Let $\kappa_\mathcal{R} : \mathcal{R} \to \overline{\mathcal{R}}_0$ be the natural map defined by

$$\kappa_\mathcal{R}(p) = \begin{cases} p & \text{for } p \in \mathcal{R}_1 \\ \mathcal{E}_p & \text{for } p \in \mathcal{S}. \end{cases}$$

**Lemma 4.4** The map $\kappa_\mathcal{R} : \mathcal{R} \to \overline{\mathcal{R}}_0$ is a homeomorphism.

§5. Outline of the proof of Theorem 2.7

Let $\Gamma, \Gamma'$ be discrete subgroups. Assume that there exists a Lie algebra isomorphism $\Phi : \mathcal{X}(\mathcal{R}_\Gamma) \to \mathcal{X}(\mathcal{R}_{\Gamma'})$. We abbreviate $\mathcal{R}_{\Gamma'}, \mathcal{S}_{\Gamma'}, \overline{E}_{\Gamma'}, \ldots$ to $\mathcal{R}', \mathcal{S}', \overline{E}', \ldots$, respectively. By Proposition 4.2 we have.

**Proposition 5.1**

(1) $\Phi_* : \mathcal{R}^* \to \mathcal{R}'^*$ is homeomorphic.

(2) The composition $\sigma_1 = \tau_{\mathcal{R}'} \circ \Phi_* \circ \tau_{\mathcal{R}}^{-1} : \mathcal{R}_1 \to \mathcal{R}'_1$ is homeomorphic.

By Proposition 5.1 we have.

**Corollary 5.2** There exists a homeomorphism $\bar{\sigma} : \overline{\mathcal{R}} \to \overline{\mathcal{R}}'$ which is an extension of $\sigma_1$ such that the following diagram is commutative:

$$\begin{array}{ccc}
\mathcal{R}^* & \xrightarrow{\Phi_*} & \mathcal{R}'^* \\
\tau_\mathcal{R} \downarrow & & \tau_{\mathcal{R}'} \downarrow \\
\overline{\mathcal{R}} & \xrightarrow{\bar{\sigma}} & \overline{\mathcal{R}}'
\end{array}$$

**Lemma 5.3** For $p \in \mathcal{S}$ let $U$ be a neighborhood of $p$ in $\mathcal{R}$ such that $\text{cl}(U) \cap \mathcal{S} = \{p\}$. Then we have

$$\text{Cl}(\tau_{\mathcal{R}}^{-1}(U)) = \tau_{\mathcal{R}}^{-1}(\text{cl}(U))$$

From Corollary 5.2, Lemma 5.3 and Lemma 4.4, we have the following.
Proposition 5.4  We can extend the homeomorphism $\sigma_1 : \mathcal{R}_1 \to \mathcal{R}'_1$ to the homeomorphism $\sigma : \mathcal{R} \to \mathcal{R}'$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{R} & \xrightarrow{\sigma} & \mathcal{R}' \\
\kappa_{\mathcal{R}} \downarrow & & \kappa_{\mathcal{R}'} \\
\mathcal{R}_0 & \xrightarrow{\bar{\sigma}} & \mathcal{R}'_0 \\
\end{array}
\]

Lemma 5.5  Let $p \in \mathcal{R}_1$ and $X \in \mathcal{X}(\mathcal{R})$. Then $X_p \neq 0$ if and only if

$[X, \mathcal{X}(\mathcal{R})] + \mathcal{M}_p = \mathcal{X}(\mathcal{R})$.

Corollary 5.6  $\sigma_1 : \mathcal{R}_1 \to \mathcal{R}'_1$ is diffeomorphic.

By the method Koriyama [KO] and Abe [AB1] we can prove that $\sigma : \mathcal{R} \to \mathcal{R}'$ is diffeomorphic.

References


