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EQUIVARIANT COHOMOLOGY DETERMINES (QUASI)TORIC MANIFOLDS

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1. Results

We denote a compact torus of dimension n by T. Let M be a toric manifold (i.e., a compact non-singular toric variety) of complex dimension n with restricted T-action or a quasitoric manifold of real dimension $2n$. The notion of quasitoric manifold was introduced by Davis-Januszkiewicz [2] as a topological counterpart to toric manifold, see [1] for details. The equivariant cohomology $H_T^*(M)$ of M is defined as

$$H_T^*(M) := H^*(ET \times_T M)$$

where ET is the total space of the universal principal T-bundle and $ET \times_T M$ is the oribit space of $ET \times M$ by the T-action defined by $t(x,p) = (xt^{-1}, tp)$ for $(x,p) \in ET \times M$ and $t \in T$. $H_T^*(M)$ is not only a ring but also an algebra over $H^*(BT)$ through the first projection from $ET \times_T M$ onto $ET/T = BT$.

Theorem 1.1. *Two toric (or quasitoric) manifolds are equivariantly diffeomorphic if and only if their equivariant cohomology algebras are isomorphic.*

Remark. The theorem above is proved for some special toric or quasitoric manifolds such as Bott towers in [6] and [7].

Corollary 1.2. *For two toric (or quasitoric) manifolds M and M', the following are equivalent:*

(1) $H_T^*(M)$ *is isomorphic to* $H_T^*(M')$ *as algebra over* $H^*(BT)$,
(2) M *is* T*-homotopic to* M',
(3) M *is* T*-diffeomorphic to* M'.

2. Outline of proof

For $\xi \in H^2_T(M)$, we denote its restriction to $p \in M^T$ by $\xi|p$ and define

$$Z(\xi) := \{p \in M^T \mid \xi|p = 0\}.$$

Let M_i $(i = 1, \dots, m)$ be characteristic submanifolds of M. We give an omniorientation for M and denote the Thom class of M_i by τ_i. Then ξ can be expressed as $\sum_{i=1}^m a_i\tau_i$ with integers a_i.

Lemma 2.1. *If $a_i \neq 0$ for some i, then $Z(\xi) \subset Z(\tau_i)$. Moreover, if $a_i \neq 0$ and $a_j \neq 0$ for some different i and j, then $Z(\xi) \subsetneq Z(\tau_i)$.*

Proof. Let $p \in Z(\xi)$. Then $0 = \xi|p = \sum_{i=1}^m a_i\tau_i|p$. Here non-zero $\tau_k|p$'s form a basis of $H^*_T(p) = H^*(BT)$, $\tau_i|p = 0$ if $a_i \neq 0$. This proves the former statement in the lemma.

If both a_i and a_j are non-zero, then $Z(\xi) \subset Z(\tau_i) \cap Z(\tau_j)$ by the former statement. Therefore, it suffices to prove that $Z(\tau_i) \cap Z(\tau_j) \subsetneq Z(\tau_i)$. Suppose that $Z(\tau_i) \cap Z(\tau_j) = Z(\tau_i)$. Then $Z(\tau_j) \supset Z(\tau_i)$, i.e., $M_j^T \subset M_i^T$. Since M is a (quasi)toric manifold, this implies that $M_j \subset M_i$ and hence $M_j = M_i$, a contradiction. □

Let $S = H^*(BT)\backslash\{0\}$[1]. Since $H^{odd}(M) = 0$, the natural map

$$H^*_T(M) \to S^{-1}H^*_T(M) = \bigoplus_{p \in M^T} S^{-1}H^*_T(p)$$

is injective. The annihilator $\mathrm{Ann}(\xi) := \{\eta \in S^{-1}H^*_T(M) \mid \eta\xi = 0\}$ of ξ in $S^{-1}H^*_T(M)$ is nothing but sum of $S^{-1}H^*_T(p)$ over p with $\xi|p = 0$. Therefore it is a free $S^{-1}H^*(BT)$ module of rank $|Z(\xi)|$. Since $\mathrm{Ann}(\xi)$ is defined using the algebra structure of $H^*_T(M)$, $|Z(\xi)|$ is an invariant of ξ depending only on the algebra structure of $H^*_T(M)$. We note that $|Z(\xi)|$ is preserved under any algebra isomorphism. We call $|Z(\xi)|$ the zero-length of ξ.

Lemma 2.2. *Let M and M' be (quasi)toric manifolds. If $f\colon H^*_T(M) \to H^*_T(M')$ is an algebra isomorphism, then f maps Thom classes of M to Thom classes of M' up to sign.*

Proof. We classify the Thom classes τ_i's of M according to zero-length. Let T_1 be the subset of Thom classes of M with largest zero-length, and let T_2 be the subset of Thom classes of M with second largest zero-length, and so on. Similarly we define T_1', T_2' and so on for Thom classes of M'.

Let m_k (resp. m_k') be the zero-length of elements in T_k (resp. T_k'). Since f and f^{-1} preserve zero-length and isomorphisms, $m_1 = m_1'$ and f maps T_1 to T_1' bijectively up to sign by Lemma 2.1. Then, if τ_{i_2} is

[1] The localization theorem holds for a much smaller multiplicative set S. In fact one can take S to be a multiplicative set consisting of equivariant Euler classes of T-representations with no trivial factor.

an element of T_2, then $f(\tau_{i_2})$ is not a linear combination of elements in T_1' (because T_1 and T_1' are preserved under f and f^{-1})). This together with Lemma 2.1 means that $m_2 \leq m_2'$. The same argument for f^{-1} instead of f shows that $m_2' \leq m_2$, so that $m_2 = m_2'$. Again, this together with Lemma 2.1 implies that f maps T_2 to T_2' bijectively up to sign. The lemma follows by repeating this argument. □

Now suppose that there is an algebra isomorphism $f\colon H_T^*(M) \to H_T^*(M')$. By Lemma 2.2, the number of Thom classes of M is same as that of M' and there is a permutation $\bar{f}$ on $[m] := \{1, 2, \dots, m\}$ such that $f(\tau_i) = \epsilon_i \tau'_{\bar{f}(i)}$ with $\epsilon_i = \pm 1$. Let Σ_M (resp. $\Sigma_{M'}$) be the (abstract) simplicial complex associated with M (resp. M'), which is formed by subsets I of $[m]$ such that $\tau_I := \prod_{i \in I} \tau_i$ is non-zero. If I is an element of Σ_M, then τ_I is non-zero and so is $f(\tau_I) = \prod_{i \in I} \epsilon_i \tau'_{\bar{f}(i)}$. Therefore the subset $\bar{f}(I) := \{\bar{f}(i) \mid i \in I\}$ is a simplex in $\Sigma_{M'}$. This shows that $\bar{f}$ induces an isomorphism from Σ_M to $\Sigma_{M'}$.

There are elements $v_i \in H_2(BT)$ which satisfy

$$u = \sum_{i=1}^{m} \langle u, v_i \rangle \tau_i \quad \text{for any } u \in H^2(BT) \tag{2.1}$$

where $u \in H^2(BT)$ at the left hand side is regarded as an element of $H_T^2(M)$ through the projection from $ET \times_T M$ onto BT. In fact, the elements v_i's are characterized by the identity above. Similarly we have $v_i' \in H_2(BT)$ which satisfy

$$u = \sum_{i=1}^{m} \langle u, v_i' \rangle \tau_i' \quad \text{for any } u \in H^2(BT). \tag{2.2}$$

We recall

Lemma 2.3. *If there is a simplicial isomorphism $\bar{f}\colon \Sigma_M \to \Sigma_{M'}$ such that $v_i = \pm v'_{\bar{f}(i)}$, then M is T-diffeomorphic to M'.*

We send the identity (2.1) by f. Since f is an algebra map, $f(u) = u$; so we have

$$u = \sum_{i=1}^{m} \langle u, v_i \rangle f(\tau_i) = \sum_{i=1}^{m} \langle u, v_i \rangle \epsilon_i \tau'_{\bar{f}(i)}.$$

Comparing this with (2.2) and noting that $\bar{f}$ is a permutation on $[m]$, we have that $\epsilon_i v_i = v'_{\bar{f}(i)}$ for each i. Thus, the theorem follows from Lemma 2.3.

3. Comments

The family of toric manifolds is not contained in the family of quasitoric manifolds and vice versa although they have projective toric manifolds in their intersection. So it is natural to expect that Theorem 1.1 would hold for a more general family of T-manifolds. A torus manifold, which was introduced in [4], is a closed smooth manifold with an effective action of T. Sometimes an orientation date called an omniorientation is incorporated in the definition but we do not need it here. Clearly toric or quasitoric manifolds are torus manifolds. The T-orbit space of a quasitoric manifold is a simple polytope by definition, and that of a toric manifold is not necessarily a simple convex polytope but always a manifold with corners whose faces are all contractible. This is not true for torus manifolds, but Theorem 1.1 might hold for a family of torus manifolds whose T-orbit spaces are manifolds with corners such that all faces, even the orbit space itself, are contractible[2].

It is intriguing to ask whether the non-equivariant version of Theorem 1.1 holds and I pose it as a problem.

Problem. Are two toric or quasitoric manifolds diffeomorphic if and only if their cohomology rings are isomorphic?

References

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[2]It is proved in [5] that the T-orbit space of a torus manifold is a manifold M with corners such that all faces, even the orbit space itself, are *acyclic* if $H^{odd}(M)$ vanishes.