A remark on quadratic differentials vanishing at infinity

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For a Fuchsian group $\Gamma$ acting on the upper half-plane $H$, let $B(\Gamma)$ denote the Banach space of all holomorphic functions on $H$ satisfying $\gamma^*\phi = \phi$ for every $\gamma \in \Gamma$ and $\|\phi\| < \infty$, where $(\gamma^*\phi)(z) := \phi(\gamma(z))\gamma'(z)^2$ and $\|\phi\| := \sup_{z \in H} 4(\text{Im } z)^2|\phi(z)|$. An element in $B(\Gamma)$ is called a bounded holomorphic quadratic differential for $\Gamma$. Let $S(\Gamma)$ be a subset of $B(\Gamma)$ consisting of those $\phi = S_f$, where $S_f$ is the Schwarzian derivative for a $\Gamma$-compatible univalent function $f$ on $H$. The Nehari theorem says that if $\phi \in S(\Gamma)$ then $\|\phi\| \leq 6$. Also $S(\Gamma)$ is closed in $B(\Gamma)$.

The boundary semi-norm for $\phi \in B(\Gamma)$ is defined by

$$
\|\phi\|_0 = \inf_V \|\phi|_{H-\Gamma(V)}
$$

where the infimum is taken over all compact subsets $V \subset H$. It is said that $\phi \in B(\Gamma)$ vanishes at infinity if $\|\phi\|_0 = 0$. Let $B_0(\Gamma)$ be a Banach subspace of $B(\Gamma)$ consisting of all $\phi$ vanishing at infinity. An element $[\phi]$ in the quotient Banach space $B(\Gamma)/B_0(\Gamma)$ is identified with the coset $\phi + B_0(\Gamma)$ in $B(\Gamma)$. For each $\phi \in B(\Gamma)$, we set

$$
\|\phi\|_{B(\Gamma)} = \inf \{\|\phi + \psi\| : \psi \in B_0(\Gamma) \},
$$

which induces the quotient norm for $[\phi]$ in $B(\Gamma)/B_0(\Gamma)$.

The purpose of this note is to remark the following theorem. An idea of the proof is contained in [2]. This article as well as [1] studies the case where $\phi(z) = \frac{1}{2}z^{-2}$ and $\beta = 2 + \varepsilon$ in the statement below.

**Theorem.** Let $\tilde{\phi} \in B(\Gamma)$ satisfy $\|\tilde{\phi}\|_{B(\Gamma)} < \beta$ for a positive constant $\beta > 0$ and $\phi \in B(1)$ (for the trivial group 1) satisfy $r\phi \notin S(1)$ for all $r > 1$. Assume that there exists a sequence $\{h_n\}$ of conformal automorphisms of $H$ such that the orbit $\{h_n(z)\}$ eventually exits from $\Gamma(V)$ for any compact subset $V \subset H$ and such that $h_n^*\phi$ converge to $\phi$ locally uniformly. Then there exists $\phi \in B(\Gamma)$ with $\|\phi\|_{B(\Gamma)} < \beta$ satisfying

$$
(\phi + B_0(\Gamma)) \cap S(\Gamma) = \emptyset.
$$
Proof. Suppose to the contrary that every $\varphi \in B(\Gamma)$ with $\|\varphi\|_{\mathfrak{h}(\Gamma)} < \beta$ satisfies $(\varphi + B_0(\Gamma)) \cap S(\Gamma) \neq \emptyset$. We take $(1 + \delta)\tilde{\phi}$ as this $\varphi$, where $\delta > 0$ is chosen so that $(1 + \delta)\|\tilde{\phi}\|_{\mathfrak{h}(\Gamma)} < \beta$. Then there exists some $\psi \in B_0(\Gamma)$ such that $(1 + \delta)\tilde{\phi} + \psi \in S(\Gamma)$. Set $\tilde{\phi}_n = h_{n}^{*}\tilde{\phi}$ and $\psi_n = h_{n}^{*}\psi$. By assumption, $\tilde{\phi}_n$ converge to $\phi$ locally uniformly. Since $\psi$ vanishes at infinity, $\psi_n$ converge to 0 locally uniformly. Hence

$$(1 + \delta)\tilde{\phi}_n + \psi_n \rightarrow (1 + \delta)\phi.$$  

On the other hand, since $(1 + \delta)\tilde{\phi}_n + \psi_n$ belong to $S(1)$ for all $n$, there exist univalent functions $f_n$ on $\Delta$ such that $(1 + \delta)\tilde{\phi}_n + \psi_n = S_{f_n}$. We may give a certain normalization to $f_n$ so that a subsequence converges to a univalent function $f$ on $\Delta$ locally uniformly. Then $S_{f_n} \rightarrow S_f$ and hence $(1 + \delta)\phi = S_f \in S(1)$. However, this contradicts the assumption that $r\phi \notin S(1)$ for all $r > 1$. \hfill $\square$

Corollary. Suppose that a Fuchsian group $\Gamma$ is contained in another Fuchsian group $\tilde{\Gamma}$ as a normal subgroup of infinite index. Let $\phi \in B_0(\Gamma)$ satisfy $r\phi \notin S(\Gamma)$ for all $r > 1$. Then, for every $\varepsilon > 0$, there exists $\varphi \in B(\Gamma)$ with $\|\varphi\|_{\mathfrak{h}(\Gamma)} < \|\phi\| + \varepsilon$ satisfying

$$(\varphi + B_0(\Gamma)) \cap S(\Gamma) = \emptyset.$$  

Proof. Take a system of representatives $\{h_1, h_2, \ldots\} \subset \tilde{\Gamma}$ for the coset decomposition of $\tilde{\Gamma}$ modulo $\Gamma$. Then the sequence $\{h_n\}$ of conformal automorphisms of $H$ holds a property that the orbit $\{h_n(z)\}$ eventually exits from $\Gamma(V)$ for any compact subset $V \subset H$. Moreover, for a given $\varepsilon > 0$, we can choose a subsequence $h_{n_k}$ so that

$$\tilde{\phi} = \sum_{k=1}^{\infty} (h_{n_k}^{-1})^{*}\phi \in B(\Gamma)$$  

satisfies $(\|\tilde{\phi}\|_{\mathfrak{h}(\Gamma)} \leq) \|\tilde{\phi}\| < \|\phi\| + \varepsilon$ and so that $h_{n_k}^{*}\tilde{\phi}$ converge to $\phi$ locally uniformly. Then we can apply the above theorem. \hfill $\square$

References


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