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Discontinuity of the action of pure mapping class groups

Ege Fujikawa
藤川 英華
Department of Mathematics, Sophia University
上智大学 理工学部数学科

1 Teichmüller space

The *Teichmüller space* $T(R)$ of a Riemann surface R is the set of all equivalence classes $[f]$ of quasiconformal homeomorphisms f on R. Here we say that two quasiconformal homeomorphisms f_1 and f_2 on R are *equivalent* if there exists a conformal homeomorphism $h: f_1(R) \to f_2(R)$ such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity. All homotopies are consider to be relative to the ideal boundary at infinity. A distance between two points $[f_1]$ and $[f_2]$ in $T(R)$ is defined by $d([f_1], [f_2]) = (1/2) \log K(f)$, where f is an extremal quasiconformal homeomorphism in the sense that its maximal dilatation $K(f)$ is minimal in the homotopy class of $f_2 \circ f_1^{-1}$. Then d is a complete distance on $T(R)$ which is called the Teichmüller distance.

The *quasiconformal mapping class* is the homotopy equivalence class $[g]$ of quasiconformal automorphisms g of a Riemann surface, and the *quasiconformal mapping class group* $\mathrm{MCG}(R)$ of R is the set of all quasiconformal mapping classes on R. Every element $[g] \in \mathrm{MCG}(R)$ induces a biholomorphic automorphism $[g]_*$ of $T(R)$ by $[f] \mapsto [f \circ g^{-1}]$, which is also isometric with respect to the Teichmüller distance. Let $\mathrm{Aut}(T(R))$ be the group of all biholomorphic automorphisms of $T(R)$. Then we have a homomorphism

$$\iota : \mathrm{MCG}(R) \to \mathrm{Aut}(T(R))$$

given by $[g] \mapsto [g]_*$. It is proved in [2] that the homomorphism ι is injective (faithful) for all Riemann surfaces R of non-exceptional type. See also [6] and [19] for other proofs. Here we say that a Riemann surface R is of *exceptional type* if R has finite hyperbolic area and satisfies $2g + n \leq 4$, where g is the genus of R and n is the number of punctures of R. The homomorphism ι is also surjective for all Riemann surfaces R of non-exceptional type. The proof is a combination of the results of [1] and [18]. See [10] for a survey of the proof.

Definition 1.1 We say that a subgroup $G \subset \mathrm{MCG}(R)$ acts at a point $p \in T(R)$ *discontinuously* if the following equivalent conditions are satisfied:

(a) there exists a neighborhood U of p such that the number of elements $[g] \in G$ satisfying $[g]_*(U) \cap U \neq \emptyset$ is finite.

(b) there exist no distinct elements $[g_n] \in G$ such that $d([g_n]_*(p), p) \to 0$ as $n \to \infty$,

(c) the orbit $G(p)$ is a discrete set and the stabilizer subgroup $\mathrm{Stab}_G(p)$ is finite.

Set
$$\Omega(G) = \{p \in T(R) \mid G \text{ acts at } p \text{ discontinuously}\}.$$
We call $\Omega(G)$ the region of discontinuity of G. By definition, $\Omega(G)$ is an open subset on $T(R)$. For a Riemann surface R of analytically finite type, the quasiconformal mapping class group $\mathrm{MCG}(R)$ acts on $T(R)$ discontinuously, namely $\Omega(\mathrm{MCG}(R)) = T(R)$ (see Section 8 in [14]). However, for a Riemann surface of analytically infinite type, the action of $\mathrm{MCG}(R)$ is not discontinuous, in general.

On the basis of this fact, Gardiner and Lakic [16] considered the special case as follows. For the standard middle-thirds Cantor set C in the unit interval as a subset of the complex sphere $\hat{\mathbb{C}}$, the pure mapping class group $P(\hat{\mathbb{C}} - \mathrm{C})$ of the complement $\hat{\mathbb{C}} - \mathrm{C}$ of the Cantor set C is the set of all elements $[g] \in \mathrm{MCG}(\hat{\mathbb{C}} - \mathrm{C})$ such that g fixes all points of C. Then they proved the following.

Proposition 1.2 **([16])** *For the complement $\hat{\mathbb{C}} - \mathrm{C}$ of the middle-thirds Cantor set C, the pure mapping class group $P(\hat{\mathbb{C}} - \mathrm{C})$ acts on the Teichmüller space $T(\hat{\mathbb{C}} - \mathrm{C})$ discontinuously.*

We extend Proposition 1.2 for general Riemann surfaces. First we define the pure mapping class group for all Riemann surfaces.

Definition 1.3 The *pure mapping class group* $P(R)$ of a Riemann surface R is the set of all quasiconformal mapping classes $[g] \in \mathrm{MCG}(R)$ such that g fixes all topological (Stoïlow) ends of R.

We also define a condition on Riemann surfaces in terms of hyperbolic geometry.

Definition 1.4 We say that a Riemann surface R has the *bounded geometry* if R satisfies the following three conditions:

(i) *the lower bound condition*: the injectivity radius at any point of R except cusp neighborhoods are uniformly bounded away from zero.

(ii) *the upper bound condition*: there exists a subdomain R^* of R such that the injectivity radius at any point of R^* is uniformly bounded from above and that the simple closed curves in R^* carry the fundamental group of R.

(iii) R has no ideal boundary at infinity, namely the Fuchsian model of R is of the first kind.

The bounded geometry condition is quasiconformally invariant, and every non-universal normal cover of a Riemann surface of analytically finite type has the bounded geometry. The complement of the Cantor set also satisfies the bounded geometry.

Now we state our theorem.

Theorem 1.5 *Let R be a Riemann surface that has the bounded geometry and has more than two topological ends. Then the pure mapping class group $P(R)$ acts on the Teichmüller space $T(R)$ discontinuously.*

2 Proof of theorem

A proof of Theorem 1.5 is given in [11]. In this section, we explain our approach to the proof. First we define a stationary subgroup of the quasiconformal mapping class group, which is a generalization of the mapping class group of a topologically finite Riemann surface.

Definition 2.1 A subgroup G of $\mathrm{MCG}(R)$ is said to be *stationary* if there exists a compact subsurface W of R such that $g(W) \cap W \neq \emptyset$ for every representative g of every element of G. Furthermore, an element $[g] \in \mathrm{MCG}(R)$ is said to be stationary if the cyclic group generated by $[g]$ is stationary.

Remark 2.2 There exists a subgroup $G \subset \mathrm{MCG}(R)$ such that each element of G is stationary but G is not stationary. Indeed, there exists an abstract countable infinite group Γ such that every element of Γ is of finite order, and for any countable group Γ, there exists a Riemann surface R such that the group $\mathrm{Conf}(R)$ of all conformal automorphisms of R contains a subgroup G isomorphic to Γ. Then we may regard G as a subgroup of $\mathrm{MCG}(R)$. Every element $[g] \in G$ is stationary since it is of finite order. On the other hand, G is not stationary since $\mathrm{Conf}(R)$ acts on R properly discontinuously.

It is known that a sequence of normalized quasiconformal homeomorphisms whose maximal dilatations are uniformly bounded is sequentially compact in compact open topology. The stationary property of mapping classes corresponds to the normalization in this context and hence such a sequence of mapping classes also has the compactness property if they are uniformly bounded. By using this observation, we have the following.

Proposition 2.3 *Let R be a Riemann surface of non-exceptional type that has the bounded geometry. Then* (i) $\Omega(\mathrm{MCG}(R)) \neq \emptyset$; (ii) $\Omega(G) = T(R)$ *for every stationary subgroup G of* $\mathrm{MCG}(R)$.

See [7] and [8] for a proof of Proposition 2.3.

Remark 2.4 There exist Riemann surfaces R such that $\emptyset \neq \Omega(\mathrm{MCG}(R)) \subsetneq T(R)$. A typical example is a non-universal normal covering surface of an analytically finite Riemann surface.

Remark 2.5 There exist a Riemann surface R and a subgroup G of $\mathrm{MCG}(R)$ such that G is non-stationary but $\Omega(G) = T(R)$. See Proposition 3.1 in [12]. In the paper [12], we further constructed a Riemann surface R satisfying the bounded geometry such that $\mathrm{MCG}(R)$ is non-stationary but $\Omega(\mathrm{MCG}(R)) = T(R)$.

By Proposition 2.3 (ii), the following proposition completes a proof of Theorem 1.5.

Proposition 2.6 *If R has more than two topological ends, then the pure mapping class group $P(R)$ is stationary.*

Proof. By considering a canonical exhaustion of R by a sequence of compact subsurfaces, we have a compact subsurface W whose complement consists of more than two connected components. Since a mapping class $[g] \in P(R)$ preserves each topological end, any representative g of $[g]$ satisfies $g(U) \cap U \neq \emptyset$ for every connected component U of $R - W$. This implies that $g(W) \cap W \neq \emptyset$ and hence $P(R)$ is stationary. ∎

Remark 2.7 We have an example of another stationary subgroup. For a simple closed geodesic c on R, let $G_c(R)$ be the set of all elements $[g] \in \mathrm{MCG}(R)$ such that $g(c)$ is freely homotopic to c. Then $G_c(R)$ is stationary. See [13].

In the last of this section, we define a subgroup of the pure mapping class group.

Definition 2.8 A quasiconformal mapping class $[g] \in \mathrm{MCG}(R)$ is said to be *eventually trivial* if there exists a compact subsurface V_g of R with geodesic boundary such that, for each connected component W of $R - V_g$, the restriction $g|_W : W \to R$ is homotopic to the inclusion map $id|_W : W \hookrightarrow R$. The *eventually trivial mapping class group* $E(R)$ is the set of all eventually trivial mapping classes.

Since $E(R)$ is a subgroup of $P(R)$, Theorem 1.5 yields that $E(R)$ acts on $T(R)$ discontinuously if a Riemann surface R has the bounded geometry and has more than two topological ends. However we see that the assumption on the number of ends can be removed as follows.

Theorem 2.9 *Let R be an analytically infinite Riemann surface having the bounded geometry. Then the eventually trivial mapping class group $E(R)$ acts on the Teichmüller space $T(R)$ discontinuously.*

We prove Theorem 2.9 in [11].

3 Asymptotic Teichmüller space

In this section, we consider the asymptotic Teichmüller space of a Riemann surface R, which is a quotient space of the Teichmüller space. It was introduced in [17] when R is the upper half-plane and in [2], [3] and [15] when R is an arbitrary hyperbolic Riemann surface. We say that a quasiconformal homeomorphism f on R is *asymptotically conformal* if for every $\epsilon > 0$, there exists a compact subset V of R such that the maximal dilatation $K(f|_{R-V})$ of the restriction of f to $R - V$ is less than $1 + \epsilon$. We say that two quasiconformal homeomorphisms f_1

and f_2 on R are *asymptotically equivalent* if there exists an asymptotically conformal homeomorphism $h : f_1(R) \to f_2(R)$ such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity by a homotopy that keeps every point of the ideal boundary at infinity fixed throughout. The *asymptotic Teichmüller space* $AT(R)$ with the base Riemann surface R is the set of all asymptotic equivalence classes $[[f]]$ of quasiconformal homeomorphisms f on R. The asymptotic Teichmüller space $AT(R)$ is of interest only when R is analytically infinite. Otherwise $AT(R)$ is trivial, that is, it consists of just one point. Conversely, if R is analytically infinite, then $AT(R)$ is not trivial. In fact, it is infinite dimensional. Since a conformal homeomorphism is asymptotically conformal, there is a natural projection $\pi : T(R) \to AT(R)$ that maps each Teichmüller equivalence class $[f] \in T(R)$ to the asymptotic Teichmüller equivalence class $[[f]] \in AT(R)$. The asymptotic Teichmüller space $AT(R)$ has a complex manifold structure such that π is holomorphic. See also [4] and [5].

For a quasiconformal homeomorphism f of R, the *boundary dilatation* of f is defined by $H^*(f) = \inf K(f|_{R-E})$, where infimum is taken over all compact subsets E of R. Furthermore, for a Teichmüller equivalence class $[f] \in T(R)$, the *boundary dilatation* of $[f]$ is defined by $H([f]) = \inf H^*(g)$, where infimum is taken over all elements $g \in [f]$. A distance between two points $[[f_1]]$ and $[[f_2]]$ in $AT(R)$ is defined by $d_A([[f_1]], [[f_2]]) = (1/2) \log H([f_2 \circ f_1^{-1}])$, where $[f_2 \circ f_1^{-1}]$ is a Teichmüller equivalence class of $f_2 \circ f_1^{-1}$ in $T(f_1(R))$. Then d_{AT} is a complete distance on $AT(R)$, which is called the asymptotic Teichmüller distance. For every point $[[f]] \in AT(R)$, there exists an asymptotically extremal element $f_0 \in [[f]]$ in the sense that $H([f]) = H^*(f_0)$.

Every element $[g] \in \mathrm{MCG}(R)$ induces a biholomorphic automorphism $[g]_*$ of $AT(R)$ by $[[f]] \mapsto [[f \circ g^{-1}]]$, which is also isometric with respect to d_A. Let $\mathrm{Aut}(AT(R))$ be the group of all biholomorphic automorphisms of $AT(R)$. Then we have a homomorphism

$$\iota_A : \mathrm{MCG}(R) \to \mathrm{Aut}(AT(R))$$

given by $[g] \mapsto [g]_*$. It is different from the case of $\iota : \mathrm{MCG}(R) \to \mathrm{Aut}(T(R))$ that the homomorphism ι_A is not injective, namely $\mathrm{Ker}\,\iota_A \neq \{[id]\}$, unless R is either the unit disc or a once-punctured disc. Moreover there exists a Riemann surface R of analytically infinite type such that $\mathrm{MCG}(R) = \mathrm{Ker}\,\iota_A$, namely the action of $\mathrm{MCG}(R)$ on $AT(R)$ is trivial. Such a Riemann surface was constructed in [20]. On the basis of this fact, first we give a sufficient condition for non-trivial action.

Theorem 3.1 *Let R be a Riemann surface of topologically infinite type. Suppose that R satisfies the upper bound condition. Then* $\mathrm{Ker}\,\iota_A \subsetneq \mathrm{MCG}(R)$.

A proof of Theorem 3.1 is given in [9]. In the proof, we show that there exists a quasiconformal automorphism of R that is not homotopic to any asymptotically conformal automorphism of R if R satisfies the upper bound condition. Then the base point of $AT(R)$ is not a common fixed point of $\mathrm{MCG}(R)$ and we have the assertion. Since the upper bound condition is quasiconfomally invariant, we can apply the same argument for all points $[[f]] \in AT(R)$ to prove that there exists a quasiconformal automorphism of $f(R)$ that is not homotopic to any asymptotically conformal automorphism of $f(R)$. Thus we have the following.

Theorem 3.2 *Let R be a Riemann surface of topologically infinite type. Suppose that R satisfies the upper bound condition. Then* $\mathrm{MCG}(R)$ *has no common fixed points on $AT(R)$.*

Next we characterize the subgroup $\mathrm{Ker}\, \iota_A$. The following theorem gives a condition for a quasiconformal homeomorphism which does not belong to $\mathrm{Ker}\, \iota_A$.

Theorem 3.3 ([9]) *Let g be a quasiconformal automorphism of a Riemann surface R. Suppose there exists a constant $\delta > 1$ such that, for every compact subset E of R, there is a simple closed geodesic c on R outside of E satisfying either*

$$\frac{\ell(g(c))}{\ell(c)} \leq \frac{1}{\delta} \quad \text{or} \quad \frac{\ell(g(c))}{\ell(c)} \geq \delta.$$

Then g is not homotopic to any asymptotically conformal automorphism of R. In particular, $[g] \notin \mathrm{Ker}\, \iota_A$.

On the other hand, we have a property of elements of $\mathrm{Ker}\, \iota_A$. We say that an end is *essential* if it does not correspond to a puncture, and we define the essential pure mapping class group as follows.

Definition 3.4 The *essential pure mapping class group* $P_e(R)$ of R is the set of all quasiconformal mapping classes $[g] \in \mathrm{MCG}(R)$ such that g fixes all essential ends of R.

Clearly $P(R) \subset P_e(R)$. Now we state our theorem.

Theorem 3.5 *We have* $E(R) \subset \mathrm{Ker}\, \iota_A \subset P_e(R)$.

We prove Theorem 3.5 in [11]. We also remark that each inclusion relation is proper, in general. However it is conjectured that $E(R) = \mathrm{Ker}\, \iota_A$ under the assumption that R satisfies the bounded geometry.

In the last of this section, we consider the dynamics of the geometric automorphisms on $AT(R)$. Similar to the definition of the region of discontinuity on Teichmüller space, we define the region of discontinuity of $G \subset \mathrm{MCG}(R)$ on the asymptotic Teichmüller space as

$$\Omega_A(G) = \{p \in AT(R) \mid G \text{ acts at } p \text{ discontinuously}\}.$$

As we have seen in the previous section, every stationary subgroup of the mapping class group acts on the Teichmüller space discontinuously under the bounded geometry condition. However, on the asymptotic Teichmüller space, a situation is different.

Theorem 3.6 *There exists a Riemann surface R having the bounded geometry and more than two topological ends such that $\Omega_A(P(R)) \subsetneqq AT(R)$ for the pure mapping class group $P(R)$.*

Proof. Let R_0 be a normal cover of a compact Riemann surface of genus 2 whose covering transformation group is a cyclic group $\langle\phi\rangle$ generated by a conformal automorphism ϕ of R_0 of infinite order. Set $R = R_0 - \{x\}$ for a point $x \in R_0$. Then R has the bounded geometry and three topological ends. We see that there exists a quasiconformal automorphism ψ of R of infinite order such that it fixes the three ends and it is coincident with ϕ outside a topologically finite subsurface whose boundary consists of x and two dividing simple closed geodesics. By a similar argument to the proofs of Proposition 4.3 and Lemmas 4.4 and 4.5 in [9], we can construct a point $p \in AT(R)$ satisfying the following two properties:

(i) $d_A([\psi^{3^k}]_*(p), p) \to 0$ $(k \to \infty)$;

(ii) $[\psi^{3^k}]_* \neq [\psi^{3^m}]_*$ in $\mathrm{Aut}(AT(R))$ for every $k \neq m$.

Then $p \notin \Omega_A(P(R))$ and we have the assertion. ∎

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