Inner sequences and submodules in the Hardy space over the bidisk

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Abstract

We deal with infinite sequences of inner functions \( \{q_j\}_{j \geq 0} \) with the property that \( q_j \) is divisible by \( q_{j+1} \). It is shown that these sequences have close relations to the module structure of the Hardy space over the bidisk. This article is a résumé of recent papers. Some results of this research were obtained in joint work with R. Yang (SUNY).

1 Preliminaries

Let \( \mathbb{D} \) be the open unit disk in the complex plane \( \mathbb{C} \), and let \( H^2(z) \) denote the classical Hardy space over \( \mathbb{D} \) with the variable \( z \). The Hardy space over the bidisk \( H^2 \) is the tensor product Hilbert space \( H^2(z) \otimes H^2(w) \) with variables \( z \) and \( w \). A closed subspace \( \mathcal{M} \) of \( H^2 \) is called a submodule if \( \mathcal{M} \) is invariant under the action of multiplication operators of coordinate functions \( z \) and \( w \). Let \( R_z \) (resp. \( R_w \)) denote the restriction of the Toeplitz operator \( T_z \) (resp. \( T_w \)) to a submodule \( \mathcal{M} \). The quotient module \( \mathcal{N} = H^2/\mathcal{M} \) is the orthogonal complement of a submodule \( \mathcal{M} \) in \( H^2 \), and let \( S_z \) (resp. \( S_w \)) denote the compression of \( T_z \) (resp. \( T_w \)) to \( \mathcal{N} \), that is, we set \( S_z = P_{\mathcal{N}}T_z|_{\mathcal{N}} \) (resp. \( S_w = P_{\mathcal{N}}T_w|_{\mathcal{N}} \)) where \( P_{\mathcal{N}} \) denotes the orthogonal projection from \( H^2 \) onto \( \mathcal{N} \).

2 Rudin's submodule

Let \( \mathcal{M} \) be the submodule consisting of all functions in \( H^2 \) which have a zero of order greater than or equal to \( n \) at \( (\alpha_n, 0) = (1 - n^{-3}, 0) \) for any positive
integer $n$. This module was given by Rudin in [1], and he proved that this is not finitely generated. Rudin’s submodule can be decomposed as follows (cf. [3]):

$$\mathcal{M} = \sum_{j=0}^{\infty} \oplus q_j(z)H^2(z)w^j,$$

where we set $b_n(z) = (\alpha_n - z)/(1 - \alpha_n z)$, $q_0(z) = \prod_{n=1}^{\infty} b_n^n(z)$ and $q_j(z) = q_{j-1}(z)/\prod_{n=j}^{\infty} b_n(z)$ for any positive integer $j$.

Regarding this submodule, the following are known (cf. [4]):

$$\sigma_p(S_z) = \{\alpha_n : n \geq 1\}, \quad \sigma_c(S_z) = \{1\}, \quad \sigma_r(S_z) = \emptyset$$

and

$$\|[R_z^*, R_w]\|_2^2 = \sum_{j=1}^{\infty} \left( 1 - \prod_{n=j}^{\infty} (1 - n^{-3})^2 \right).$$

Moreover, we have obtained the following in [2]:

$$\sigma_p(S_w) = \{0\}, \quad \sigma_c(S_w) = \overline{\mathbb{D}} \setminus \{0\}, \quad \sigma_r(S_w) = \emptyset$$

and

$$\|[S_z^*, S_w]\|_2^2 = \sum_{j=1}^{\infty} \left( 1 - \prod_{n=j}^{\infty} (1 - n^{-3})^{2(n-j)} \right) \left( 1 - \prod_{n=j}^{\infty} (1 - n^{-3})^2 \right)$$

$$= -1 + \sum_{j=1}^{\infty} \left( 1 - \prod_{n=j}^{\infty} (1 - n^{-3})^2 \right).$$

3 Inner sequences

**Definition 1** An infinite sequence of analytic functions $\{q_j(z)\}_{j \geq 0}$ is called an *inner sequence* if $\{q_j(z)\}_{j \geq 0}$ consists of inner functions and $(q_j/q_{j+1})(z)$ is inner for any $j$.

We note that the above condition is equivalent to that $q_j(z)H^2(z)$ is contained in $q_{j+1}(z)H^2(z)$. Therefore every inner sequence $\{q_j(z)\}_{j \geq 0}$ corre-
sponds to a submodule $\mathcal{M}$ in $H^2$ as follows:

$$\mathcal{M} = \sum_{j=0}^{\infty} \oplus q_j(z)H^2(z)w^j.$$ 

In this submodule, we can calculate many subjects of operator theory, exactly.

**Theorem 1 ([2, 3])** Let $\mathcal{M}$ be the submodule arising from an inner sequence $\{q_j(z)\}_{j\geq 0}$. Then the following hold:

(i) $\|[R_z^*, R_w]\|_2^2 = \sum_{j=0}^{\infty} (1 - |(q_j/q_{j+1})(0)|^2)$,

(ii) $\|[S_z^*, S_w]\|_2^2 = \sum_{j=0}^{\infty} (1 - |q_{j+1}(0)|^2)(1 - |(q_j/q_{j+1})(0)|^2)$.

Let $q_{\infty}(z)$ be the inner function defined as follows:

$$q_{\infty}(z)H^2(z) = \bigcup_{j=0}^{\infty} q_j(z)H^2(z).$$

Without loss of generality, we may assume that the first non-zero Taylor coefficient of $q_{\infty}(z)$ is positive.

**Theorem 2 ([2])** Let $\mathcal{N}$ be the quotient module arising from an inner sequence $\{q_j(z)\}_{j\geq 0}$. Then $\sigma(S_z) = \sigma(q_0(z))$, where $\sigma(q_0(z))$ is the spectrum of $q_0(z)$, that is, $\sigma(q_0(z))$ consists of all zero points of $q_0(z)$ in $\mathbb{D}$ and all points $\zeta$ on the unit circle $\partial\mathbb{D}$ such that $q_0(z)$ cannot be continued analytically from $\mathbb{D}$ to $\zeta$.

**Theorem 3 ([2])** Let $\mathcal{N}$ be the quotient module arising from an inner sequence $\{q_j(z)\}_{j\geq 0}$.

(i) if $q_m(z) = 1$ for some finite $m$, then

$$\sigma_p(S_w) = \{0\}, \ \sigma_c(S_w) = \emptyset \text{ and } \sigma_r(S_w) = \emptyset,$$
(ii) if $q_{\infty}(z) = 1$ and $q_{j}(z) \neq 1$ for any $j$, then
\[ \sigma_{p}(S_{w}) = \{0\}, \quad \sigma_{c}(S_{w}) = \overline{D} \setminus \{0\} \quad \text{and} \quad \sigma_{r}(S_{w}) = \emptyset, \]

(iii) if $q_{\infty}(z) \neq 1$ and $q_{j}(z) \neq q_{0}(z)$ for some $j$, then
\[ \sigma_{p}(S_{w}) = \{0\}, \quad \sigma_{c}(S_{w}) = \partial D \quad \text{and} \quad \sigma_{r}(S_{w}) = D \setminus \{0\}, \]

(iv) if $q_{j}(z) = q_{0}(z)$ for any $j$, then
\[ \sigma_{p}(S_{w}) = \emptyset, \quad \sigma_{c}(S_{w}) = \partial D \quad \text{and} \quad \sigma_{r}(S_{w}) = D. \]

Let $\mathfrak{A}$ denote the weak closed subalgebra generated by $S_{z}, S_{w}$ and the identity operator on $\mathcal{N}$, and let $\mathfrak{A}'$ denote the commutant of $\mathfrak{A}$.

**Theorem 4 ([2])** Let $\mathcal{N}$ be the quotient module arising from an inner sequence $\{q_{j}(z)\}_{j \geq 0}$. Then $\mathfrak{A} = \mathfrak{A}'$. Moreover, for any element $A$ in $\mathfrak{A}'$, there exists a sequence of bounded analytic functions $\{\varphi_{j}(z)\}_{j \geq 0}$ in $H^{\infty}(z)$ such that $A = \sum_{j \geq 0} S_{\varphi_{j}(z)} S_{w}^{j}$ in the weak operator topology.

**References**


