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Inner sequences and submodules in the Hardy space over the bidisk

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Abstract

We deal with infinite sequences of inner functions $\{q_j\}_{j\geq 0}$ with the property that q_j *is divisible by* q_{j+1}. It is shown that these sequences have close relations to the module structure of the Hardy space over the bidisk. This article is a résumé of recent papers. Some results of this research were obtained in joint work with R. Yang (SUNY).

1 Preliminaries

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, and let $H^2(z)$ denote the classical Hardy space over $\mathbb{D}$ with the variable z. The Hardy space over the bidisk H^2 is the tensor product Hilbert space $H^2(z) \otimes H^2(w)$ with variables z and w. A closed subspace $\mathcal{M}$ of H^2 is called a submodule if $\mathcal{M}$ is invariant under the action of multiplication operators of coordinate functions z and w. Let R_z (resp. R_w) denote the restriction of the Toeplitz operator T_z (resp. T_w) to a submodule $\mathcal{M}$. The quotient module $\mathcal{N} = H^2/\mathcal{M}$ is the orthogonal complement of a submodule $\mathcal{M}$ in H^2, and let S_z (resp. S_w) denote the compression of T_z (resp. T_w) to $\mathcal{N}$, that is, we set $S_z = P_{\mathcal{N}}T_z|_{\mathcal{N}}$ (resp. $S_w = P_{\mathcal{N}}T_w|_{\mathcal{N}}$) where $P_{\mathcal{N}}$ denotes the orthogonal projection from H^2 onto $\mathcal{N}$.

2 Rudin's submodule

Let $\mathcal{M}$ be the submodule consisting of all functions in H^2 which have a zero of order greater than or equal to n at $(\alpha_n, 0) = (1 - n^{-3}, 0)$ for any positive integer n. This module was given by Rudin in [1], and he proved that this is not finitely generated. Rudin's submodule can be decomposed as follows (cf. [3]):

$$\mathcal{M} = \sum_{j=0}^{\infty} \oplus q_j(z) H^2(z) w^j,$$

where we set $b_n(z) = (\alpha_n - z)/(1 - \alpha_n z)$, $q_0(z) = \prod_{n=1}^{\infty} b_n^n(z)$ and $q_j(z) = q_{j-1}(z)/\prod_{n=j}^{\infty} b_n(z)$ for any positive integer j.

Regarding this submodule, the following are known (cf. [4]):

$$\sigma_p(S_z) = \{\alpha_n : n \geq 1\}, \ \sigma_c(S_z) = \{1\}, \ \sigma_r(S_z) = \emptyset$$

and

$$\|[R_z^*, R_w]\|_2^2 = \sum_{j=1}^{\infty} \left(1 - \prod_{n=j}^{\infty} (1 - n^{-3})^2\right).$$

Moreover, we have obtained the following in [2]:

$$\sigma_p(S_w) = \{0\}, \ \sigma_c(S_w) = \overline{\mathbb{D}} \setminus \{0\}, \ \sigma_r(S_w) = \emptyset$$

and

$$\begin{aligned} \|[S_z^*, S_w]\|_2^2 &= \sum_{j=1}^{\infty} \left(1 - \prod_{n=j}^{\infty} (1 - n^{-3})^{2(n-j)}\right) \left(1 - \prod_{n=j}^{\infty} (1 - n^{-3})^2\right) \\ &= -1 + \sum_{j=1}^{\infty} \left(1 - \prod_{n=j}^{\infty} (1 - n^{-3})^2\right). \end{aligned}$$

3 Inner sequences

Definition 1 An infinite sequence of analytic functions $\{q_j(z)\}_{j \geq 0}$ is called *an inner sequence* if $\{q_j(z)\}_{j \geq 0}$ consists of inner functions and $(q_j/q_{j+1})(z)$ is inner for any j.

We note that the above condition is equivalent to that $q_j(z)H^2(z)$ is contained in $q_{j+1}(z)H^2(z)$. Therefore every inner sequence $\{q_j(z)\}_{j \geq 0}$ corresponds to a submodule $\mathcal{M}$ in H^2 as follows:

$$\mathcal{M} = \sum_{j=0}^{\infty} \oplus q_j(z) H^2(z) w^j.$$

In this submodule, we can calculate many subjects of operator theory, exactly.

Theorem 1 ([2, 3]) *Let $\mathcal{M}$ be the submodule arising from an inner sequence $\{q_j(z)\}_{j\geq 0}$. Then the following hold:*

(i) $\|[R_z^*, R_w]\|_2^2 = \sum_{j=0}^{\infty}(1 - |(q_j/q_{j+1})(0)|^2)$,

(ii) $\|[S_z^*, S_w]\|_2^2 = \sum_{j=0}^{\infty}(1 - |q_{j+1}(0)|^2)(1 - |(q_j/q_{j+1})(0)|^2)$.

Let $q_\infty(z)$ be the inner function defined as follows:

$$q_\infty(z) H^2(z) = \overline{\bigcup_{j=0}^{\infty} q_j(z) H^2(z)}.$$

Without loss of generality, we may assume that the first non-zero Taylor coefficient of $q_\infty(z)$ is positive.

Theorem 2 ([2]) *Let $\mathcal{N}$ be the quotient module arising from an inner sequence $\{q_j(z)\}_{j\geq 0}$. Then $\sigma(S_z) = \sigma(q_0(z))$, where $\sigma(q_0(z))$ is the spectrum of $q_0(z)$, that is, $\sigma(q_0(z))$ consists of all zero points of $q_0(z)$ in $\mathbb{D}$ and all points ζ on the unit circle $\partial\mathbb{D}$ such that $q_0(z)$ can not be continued analytically from $\mathbb{D}$ to ζ.*

Theorem 3 ([2]) *Let $\mathcal{N}$ be the quotient module arising from an inner sequence $\{q_j(z)\}_{j\geq 0}$.*

(i) *if $q_m(z) = 1$ for some finite m, then*

$$\sigma_p(S_w) = \{0\}, \ \sigma_c(S_w) = \emptyset \text{ and } \sigma_r(S_w) = \emptyset,$$

(ii) *if* $q_\infty(z) = 1$ *and* $q_j(z) \neq 1$ *for any* j, *then*

$$\sigma_p(S_w) = \{0\},\ \sigma_c(S_w) = \overline{\mathbb{D}} \setminus \{0\} \text{ and } \sigma_r(S_w) = \emptyset,$$

(iii) *if* $q_\infty(z) \neq 1$ *and* $q_j(z) \neq q_0(z)$ *for some* j, *then*

$$\sigma_p(S_w) = \{0\},\ \sigma_c(S_w) = \partial\mathbb{D} \text{ and } \sigma_r(S_w) = \mathbb{D} \setminus \{0\},$$

(iv) *if* $q_j(z) = q_0(z)$ *for any* j, *then*

$$\sigma_p(S_w) = \emptyset,\ \sigma_c(S_w) = \partial\mathbb{D} \text{ and } \sigma_r(S_w) = \mathbb{D}.$$

Let $\mathfrak{A}$ denote the weak closed subalgebra generated by S_z, S_w and the identity operator on $\mathcal{N}$, and let $\mathfrak{A}'$ denote the commutant of $\mathfrak{A}$.

Theorem 4 ([2]) *Let $\mathcal{N}$ be the quotient module arising from an inner sequence $\{q_j(z)\}_{j\geq 0}$. Then $\mathfrak{A} = \mathfrak{A}'$. Moreover, for any element A in $\mathfrak{A}'$, there exists a sequence of bounded analytic functions $\{\varphi_j(z)\}_{j\geq 0}$ in $H^\infty(z)$ such that $A = \sum_{j\geq 0} S_{\varphi_j(z)} S_w^j$ in the weak operator topology.*

References

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