A note on invariant Hilbert spaces of holomorphic functions on the unit ball in $\mathbb{C}^d$

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1 Introduction

Invariant Hilbert spaces of holomorphic functions on bounded symmetric domains have been extensively studied[Ara]. The study is motivated by the unitary representation of the automorphism group of the bounded symmetric domains.

Let $\Omega$ be a bounded symmetric domain, and $\operatorname{Aut}(\Omega)$ denote the automorphism group of $\Omega$. Let $G$ denote the connected component of the identity in $\operatorname{Aut}(\Omega)$. Then $G$ can be naturally represented on the Bergman space $L^2_a(\Omega)$, the representation map $\pi$ is defined by

$$\pi(\varphi)f = f \circ \varphi \cdot J\varphi, \quad f \in L^2_a(\Omega), \quad \varphi \in G,$$

where $J\varphi$ is the complex Jacobian of $\varphi$. Moreover, this representation is unitary, that is, for any $\varphi \in G$, the operator $\pi(\varphi)$ is unitary. For natural Hilbert space $H$ of holomorphic functions on $\Omega$, the similar action of $G$ on $H$ can also been defined. J. Arazy[Ara] shows that, with some mild assumptions, the only Hilbert space which makes $\pi$ be a unitary representation is the Bergman space. Of course, J. Arazy deals with a more complicated case. For detailed information, one can refer to [Ara].

In this note, we will mainly concern Hilbert spaces of holomorphic functions on the unit ball $\mathbb{B}_d$ in $\mathbb{C}^d$. In this case, the automorphism group $\operatorname{Aut}(\mathbb{B}_d)$

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can be written precisely. In fact, by [Ru, Theorem 2.2.5], Aut(\mathbb{B}_d) is generated by the unitary group U_d of \mathbb{C}^d and \{(\varphi_\lambda| \lambda \in \mathbb{B}_d\}$, where, for any \lambda \in \mathbb{B}_d, \varphi_\lambda is defined as follows. If \lambda = 0, \varphi_\lambda(z) = -z. If \lambda \neq 0,

$$
\varphi_\lambda = \frac{\lambda - P_\lambda z - \sqrt{1-|\lambda|^2}P_\lambda^\perp z}{1-(z,a)},
$$

(1.1)

where \text{P}_\lambda is the orthogonal projection from \mathbb{C}^d onto the complex line [\lambda] spanned in \mathbb{C}^d by \lambda, and \text{P}_\lambda^\perp = I - \text{P}_\lambda. Therefore, one can only consider the automorphism with the expression (1.1). We rewrite the above representation \pi(\varphi_\lambda) as \text{U}_\lambda in short, that is

$$
\text{U}_\lambda f = f \circ \varphi_\lambda \cdot J\varphi_\lambda.
$$

After some calculation, it is not difficult to see that the complex Jacobian \text{J}\varphi_\lambda = \frac{1}{(1-(z,\lambda))^{d+1}} is just the normalized Bergman kernel on \mathbb{B}_d multiplied by \text{(-1)}^d.

For many interesting unitary invariant reproducing Hilbert space \text{H} on \mathbb{B}_d, one can define the similar action by \text{V}_\lambda f = f \circ \varphi_\lambda \cdot \text{k}_\lambda, where \text{k}_\lambda is the normalized reproducing kernel of \text{H}. So, the question is, when \text{V}_\lambda is unitary? In other word, to ensure that \text{V}_\lambda is unitary, the complex Jacobian \text{J}\varphi_\lambda can be replaced to what kind of 'good' functions.

In this note, with some mild assumptions, we will prove that if \text{V}_\lambda is unitary, then there is a positive number \mu, such that \text{k}_\lambda = \text{((-1)}^d\text{J}\varphi_\lambda)^\mu.

We organize this note as follows. In section 2, we will introduce some notations of unitary invariant reproducing kernel. In section 3, we prove the main theorem.

2 Preliminaries

From a general theory of reproducing kernels [Aro], one sees that a reproducing function space is uniquely determined by its kernel. In this paper, we will mainly concern unitary invariant reproducing function space of holomorphic functions on \mathbb{B}_d. A reproducing function space is called unitary invariant, if for any unitary operator \text{U} on \mathbb{C}^d, \text{f} \circ \text{U} \in \text{H} whenever \text{f} \in \text{H}, and for all \text{f}, \text{g} \in \text{H},

$$
\langle \text{f} \circ \text{U}, \text{g} \circ \text{U} \rangle = \langle \text{f}, \text{g} \rangle.
$$
By [GHX], $H$ is unitary invariant if and only if for any unitary operator $U$ on $\mathbb{C}^d$

$$K_{U\lambda}(Uz) = K_{\lambda}(z);$$

and this holds if and only if there is a holomorphic function on the unit disk $f(z) = \sum_{n=1}^{\infty} a_n z^n$ with $a_n \geq 0$, such that

$$K_{\lambda}(z) = f(\langle z, \lambda \rangle).$$

Without loss of generality, we will consider the case that all the $a_n > 0$, and $a_0 = 1$. Hence, by [GHX, Proposition 4.1], $H$ has a canonical orthonormal basis $\{[a_{|\alpha|}]\frac{|\alpha!}{\alpha!}^{1/2}z^\alpha\}$, and $\|z^\alpha\| = [\frac{\alpha^1}{a_{|a|}\alpha|!}\mathrm{i}]^{1/2}$. Particularly, $\|1\| = 1$.

**Example.** Let $H^2_{\mu}(\mathbb{B}_d)$ be the reproducing function space defined by the reproducing kernel $K^{(\mu)}_{\lambda} = \frac{1}{(1-\langle z, \lambda \rangle)^\mu}$ ($\mu > 0$). It is easy to verify that $H^2_{\mu}(\mathbb{B}_d)$ is unitary invariant. When $\mu = 1$, $H^2_{\mu}(\mathbb{B}_d)$ is the symmetric Fock space $H^2_d$, which is deeply studied by W. Arveson[Arv]. When $\mu = d$, $H^2_{\mu}(\mathbb{B}_d)$ is the Hardy space $H^2(\mathbb{B}_d)$. When $\mu > d$, $H^2_{\mu}(\mathbb{B}_d)$ is the weighted Bergman space $L^2_d[(1-|z|^2)\mu^{-d-1}dV]$, and in particular $H^2_{d+1}(\mathbb{B}_d)$ is the usual Bergman space.

By [Guo, Section 4], for a given $\mu > 0$, the operator

$$V_{\lambda}f = f \circ \varphi_{\lambda} \cdot \frac{(1-|\lambda|^2)^{\frac{\mu}{2}}}{(1-\langle \cdot, \lambda \rangle)^\mu}$$

is a unitary operator on $H^2_{\mu}(\mathbb{B}_d)$ (For the case $\mu = 1$, this is also proved by D. Greene[Gr, Theorem 3.3]). Notice that $\frac{(1-|\lambda|^2)^{\frac{\mu}{2}}}{(1-\langle \cdot, \lambda \rangle)^\mu}$ is the normalized reproducing kernel of $H^2_{\mu}(\mathbb{B}_d)$.

**3 The proof of the main theorem**

In this section, we will prove the main theorem. As in Section 2, let $H$ be a unitary invariant reproducing functions space with the reproducing kernel $K_{\lambda}$. For any $\lambda \in \mathbb{B}_d$, define an operator $V_{\lambda}$ on $H$ by $V_{\lambda}f = f \circ \varphi_{\lambda} \cdot k_{\lambda}$, where $k_{\lambda}$ is the normalized reproducing kernel. We have the following theorem.
Theorem 3.1. With the above notations, if $V_\lambda$ is a unitary operator on $H$, then there is a positive number $\mu$ such that,

$$k_\lambda = \frac{(1-|\lambda|^2)^{\mu}}{(1-\langle\cdot, \lambda\rangle)^\mu}.$$ 

Proof. Below, we will prove that if $V_\lambda$ is unitary, then the reproducing kernel $K_\lambda = \sum_{n=0}^\infty a_n (z, \lambda)^n$ is uniquely determined by $a_1$, that is,

Claim. For $n > 1$, each $a_n$ can be uniquely expressed by $a_1$.

We will prove the claim by induction.

At first, we will calculate $a_2$. Taking $\lambda = (r, 0, \ldots, 0)$, we simply write $\phi_\lambda = \phi_r$ and $k_\lambda = k_r$. Since $z_1 = z_1 \circ \phi_r \circ \phi_r$, we have

$$\|z_1 k_r\| = \|z_1 \circ \phi_r\|^2$$

(3.1)

We first calculate the left side of (1). By [GHX, Proposition 4.1], $\|z_1^n\|^2 = \frac{1}{a_n}$, and $\langle z_1^n, z_1^m \rangle = 0$ whenever $n \neq m$.

$$\|z_1 k_r\|^2 = \sum_{n=0}^\infty a_n r^{2n} \|z_1^n\|^2 = \sum_{n=0}^\infty a_n r^{2n} \sum_{n=0}^\infty a_n r^{2n} = \sum_{n=0}^\infty \frac{a_n^2 r^{2n}}{a_{n+1}}.$$ 

And now we calculate the right side of (3.1),

$$\|z_1 \circ \phi_r\|^2 = \|(r - z_1) \sum_{n=0}^\infty (rz_1)^n\|^2$$

$$= \sum_{n=0}^\infty (r^{n+1}z_1^n - r^n z_1^{n+1})\|^2$$

$$= \|r + \sum_{n=1}^\infty (r^{n+1} - r^{n-1})z_1^n\|^2$$

$$= r^2 + \sum_{n=1}^\infty \frac{r^{2n-2}(r^4 - 2r^2 + 1)}{a_n}.$$
Hence
\[
\sum_{n=0}^{\infty} \frac{a_{n}^{2}}{a_{n+1}} r^{2n} = (\sum_{m=0}^{\infty} a_{m} r^{2m}) (r^{2} + \sum_{n=1}^{\infty} \frac{r^{2n-2}(r^{4} - 2r^{2} + 1)}{a_{n}}).
\]  
(3.2)

Comparing the coefficients of \(r^{2}\) in both sides of (3.2) first, we have
\[
\frac{a_{1}^{2}}{a_{2}} = 1 - \frac{2}{a_{1}} + \frac{1}{a_{2}} + \frac{a_{1}}{a_{1}}.
\]

Therefore, when \(a_{1} \neq 1\),
\[
a_{2} = \frac{a_{1}(a_{1}+1)}{2}.
\]  
(3.3)

When \(a_{1} = 1\), to determine \(a_{2}\), we compare the coefficient of \(r^{4}\) in both sides of (3.2). After some simple computation, we have
\[
\frac{a_{2}^{2}}{a_{3}} = \frac{1}{a_{3}} - \frac{1}{a_{2}} + a_{2}.
\]  
(3.4)

We also need the following equation.
\[
\|z_{1}^{2} \circ \varphi_{r} \cdot K_{r}\|^{2} = z_{1}^{2} = \frac{1}{a_{2}}.
\]

Thus,
\[
\|z_{1}^{2} \circ \varphi_{r} \cdot K_{r}\|^{2} = \frac{1}{a_{2}} \sum_{n=0}^{\infty} a_{n} r^{2n}.
\]  
(3.5)

Now, let us calculate the left side of (3.5). A careful verification shows that
\[
\|z_{1}^{2} \circ \varphi_{r} \cdot K_{r}\|^{2} = \| (r - z_{1})^{2} K_{r} \|^{2}
\]
\[
= \| (r - z_{1})^{2} \left[ \sum_{n=0}^{\infty} (n+1)(r z_{1})^{n} \right] \sum_{m=0}^{\infty} a_{m} (r z_{1})^{m} \|^{2}
\]
\[
= \| r^{2} + (r^{2}(2r + a_{1}r) - 2r) z_{1} + \sum_{n=2}^{\infty} a_{n+1} r^{2n-2} (r^{4} \sum_{j=1}^{n+1} a_{n+1-j} - 2r^{2} \sum_{j=1}^{n} a_{n-j} + \sum_{j=1}^{n-1} a_{n-1-j} ) z_{1}^{n} \|^{2}
\]
Now, set $b_n = \sum_{j=1}^{n-1} ja_{n-1-j}$, and the above equation can be simplified as follows.

\[
\|z_1^2 \circ \varphi_r \cdot K_r\|^2
\]

\[
= \|r^2 + (r^2(2r + a_1r) - 2r)z_1 + \sum_{n=2}^{\infty} r^{n-2}(r^4b_{n+2} - 2r^2b_{n+1} + b_n)z_1^n\|^2
\]

\[
= r^4 + [r^2(2r + a_1r) - 2r]^2 \frac{1}{a_1} + \sum_{n=2}^{\infty} [r^{n-2}(r^4b_{n+2} - 2r^2b_{n+1} + b_n)]^2 \frac{1}{a_n}
\]

\[
= r^4 + [r^3(2 + a_1) - 2r]^2 \frac{1}{a_1} + \sum_{n=2}^{\infty} r^{2n-4} [r^8b_{n+2}^2 - 4r^6b_{n+2}b_n + r^4(4b_{n+1}^2 + 2b_{n+2}b_n) - 4r^2b_{n+1}b_n + b_n^2] \frac{1}{a_n}
\]

\[
= \frac{b_2^2}{a_2} + r^2 \left( \frac{4}{a_1} - \frac{4b_3b_2}{a_2} + \frac{b_3^2}{a_3} \right)
\]

\[+ \sum_{n=2}^{\infty} r^{2n} \left( \frac{b_{n+2}^2}{a_{n+2}} + C(a_1, \cdots, a_{n+1}, b_2, \cdots, b_{n+2}) \right),
\]

where $C(a_1, \cdots, a_{n+1}, b_1, \cdots, b_{n+2})$ can be uniquely expressed by $\{a_i\}_{i=1}^{n+1}$ and $\{b_i\}_{i=2}^{n+2}$. Now comparing the coefficients of $r^2$ in both sides of (3.5), we have

\[
\frac{4}{a_1} - \frac{2\cdot 2(2+a_1)}{a_2} + (\frac{2+a_1}{a_3})^2 = \frac{1}{a_2}.
\]

(3.6)

When $a_1 = 1$, combining (3.4) with (3.6), we have

\[
a_2 = 1 = \frac{a_1(a_1+1)}{2}
\]

Hence, by (3.3) and (3.7), the equality $a_2 = \frac{a_1(a_1+1)}{2}$ is always true.

And now we assume that $a_j$ is uniquely expressed by $a_1$ for $1 < j \leq m$. To prove $a_{m+1}$ is uniquely expressed by $a_1$, we compare the coefficient of $r^{2(m-1)}$ in both sides of (3.5).

\[
\frac{a_{m-1}}{a_2} = \frac{b_{m+1}^2}{a_{m+1}} + C(a_1, \cdots, a_m, b_2, \cdots, b_{m+1}).
\]
By the definition of $b_i$, we know that $b_i$ is uniquely expressed by \{a_j\}_{j=1}^{i-2}$. By the inductive assumption, both $a_{m-1}$ and $C(a_1, \ldots, a_m, b_2, \ldots, b_{m+1})$ are uniquely expressed by $a_1$, and so is $a_{m+1}$. Thus the claim is proved.

Set $\mu = a_1$. By section 2, if

$$K_\lambda(z) = \frac{1}{(1 - \langle z, \lambda \rangle)^\mu} = 1 + \mu \langle z, \lambda \rangle + \sum_{n=2}^{\infty} \frac{\mu(\mu + 1) \cdots (\mu + n - 1)}{n!} \langle z, \lambda \rangle^n,$$

then $V_\lambda$ is unitary. The above reasoning thus shows that

$$a_n = \frac{\mu(\mu + 1) \cdots (\mu + n - 1)}{n!}.$$

This means $K_\lambda(z) = \frac{1}{(1 - \langle z, \lambda \rangle)^\mu}$, which implies that $k_\lambda = \frac{(1-|\lambda|^2)^{\frac{\mu}{2}}}{(1-\langle \cdot, \lambda \rangle)^\mu}$. \hfill \Box

**Proposition 3.2.** Let $H$ and $H'$ be two unitary invariant reproducing function spaces on $\mathbb{B}_d$ with the reproducing kernels $K_\lambda$ and $K'_\lambda$ relatively. If

$$\|f \circ \varphi \cdot k'_\lambda\| = \|f\| \text{ for } \forall f \in H,$$

then $H = H'$, and hence by Theorem 3.1 $H = H^{\mu}_2(\mathbb{B}_d)$ for some $\mu > 0$.

**Proof.** Write $K_\lambda(z) = \sum_{n=0}^{\infty} a_n \langle z, \lambda \rangle^n$ and $K'_\lambda(z) = \sum_{n=0}^{\infty} b_n \langle z, \lambda \rangle^n$. Denote the inner product of $H$ by $\| \cdot \|$ and the inner product of $H'$ by $\| \cdot \|'$ . Since $\|1\| = 1$, we have

$$\|1 \circ \varphi \cdot k'_\lambda\|^2 = \left\| \frac{K_\lambda}{\|K'_\lambda\|'} \right\|^2 = 1.$$

On the one hand, since $\langle z^\alpha, z^\beta \rangle = 0$ whenever $\alpha \neq \beta$,

$$\|K'_\lambda\|^2 = \sum_{n=0}^{\infty} b_n \|\langle z, \lambda \rangle^n\|^2.$$

On the other hand

$$\|K_\lambda\|^2 = \sum_{n=0}^{\infty} b_n |\lambda|^{2n}.$$
Hence
\[\sum_{n=0}^{\infty} b_n \|\langle z, \lambda \rangle^n\|^2 = \sum_{n=0}^{\infty} b_n |\lambda|^{2n}.\]

Taking \(\lambda = (r, 0 \cdots, 0)\), we know \(\|z_1^n\|^2 = \frac{1}{b_n}\). By [GHX, Proposition 4.1], \(\frac{1}{a_n} = \|z_1^n\|^2 = \frac{1}{b_n}\), and hence \(K_\lambda = K'_\lambda\), which implies \(H = H'\).

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References


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