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Polynomials having leading terms over $\mathbb{C}^2$ in the Fock space

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1. Introduction

Let $\mathcal{C}=\mathcal{C}(\mathbb{C}^2)$ be the set of polynomials and $\mathrm{Hol}(\mathbb{C}^2)$ be the space of entire functions on $\mathbb{C}^2$. We denote $L_a^2(\mathbb{C}^2)$ by the Hilbert space of functions $f\in \mathrm{Hol}(\mathbb{C}^2)$ satisfying

$$\|f\|^2=\int_{\mathbb{C}^2}|f(z,w)|^2e^{-\frac{|z|^2+|w|^2}{2}}dA/(2\pi)^2,$$

where dA denotes the Lebesgue measure on $\mathbb{C}^2$. It is easy to see that $\|z^nw^m\|^2=2^{n+m}n!m!$, $\{z^nw^m/\|z^nw^m\|\}_{n,m}$ is the orthonormal basis of $L_a^2(\mathbb{C}^2)$, and $\mathcal{C}$ is dense in $L_a^2(\mathbb{C}^2)$. The space $L_a^2(\mathbb{C}^2)$ is called the Fock space or the Segal-Bargmann space. The Fock space has arrested much attention because of the closed relationship between the operator theory on it and the Weyl quantization.

In [9], Guo and Zheng showed that if M is a non-zero closed subspace of $L_a^2(\mathbb{C}^2)$, then there are no non-constant multipliers of M, that is, if $\varphi M\subset M$ and $\varphi\in \mathrm{Hol}(\mathbb{C}^2)$ then φ is constant. So, in the Fock space we can not consider"invariant subspaces" for the multiplication operators T_z and T_w. As an appropriate substitution, Guo and Zheng defined "quasi-invariant subspaces". Let M be a closed subspace of $L_a^2(\mathbb{C}^2)$. M is called *quasi-invariant* if $pM\cap L_a^2(\mathbb{C}^2)\subset M$ for each polynomial p. They proved that for each finite codimensional ideal I of the polynomial ring $\mathcal{C}$, the closure of I is quasi-invariant. In [7], Guo proved that if $p\in\mathcal{C}$ is homogeneous, then $[p]=\overline{p\mathcal{C}}$ is quasi-invariant. As Douglas and Poulsen [4] and Guo [5, 6, 7, 8], it is natural to classify all quasi-invariant subspaces in a reasonable sense.

Let M_1 and M_2 be quasi-invariant subspaces of $L_a^2(\mathbb{C}^2)$. A bounded linear operator $T: M_1\to M_2$ is called a *quasi-module map* if $T(pf)=pT(f)$ whenever $pf\in M_1, p\in\mathcal{C}$, and $f\in M_1$. We say that M_1 and M_2 are *similar* if there exists an invertible quasi-module map $T: M_1\to$

M_2 such that $T^{-1} : M_2 \to M_1$ is a quasi-module map. Also we say that M_1 and M_2 are *quasi-similar* if there exist quasi-module maps $T_1 : M_1 \to M_2$ and $T_2 : M_2 \to M_1$ with dense range. In the case of the one dimensional Fock space, Chen, Guo, and Hou [2] showed that $[p]$ is quasi-invariant and determined the similarity orbit of $[p]$ for every $p \in \mathcal{C}$. In the multi-dimensional case, Guo [7] determined the similarity orbit of $[z^n]$. It is open to determine the similarity orbit of $[p]$. For the Fock space, see also [1, 3].

Let $p \in \mathcal{C}$ and $p(z,w) = \sum_{i=0}^{d(p)} p_i(z,w)$ the homogeneous expansion of p, where $d(p)$ denotes the homogeneous degree of p. We note $p_{d(p)} \neq 0$. When

$$p_{d(p)}(z,w) = a_{n,m} z^n w^m \quad \text{and} \quad p(z,w) = \sum_{i \le n, j \le m} a_{i,j} z^i w^j,$$

Guo and Hou [8] said that p has a leading term $z^n w^m$. And they showed that if p has a leading term $z^n w^m$, then $[p]$ is quasi-invariant, and a quasi-invariant subspace M is similar to $[p]$ if and only if $M = [q]$ for some $q \in \mathcal{C}$ having the same leading term as p. In the definiton due to Guo and Hou, the set of polynomials having leading terms $z^n w^m$ is a fairly restricted class. So in this paper, we generalize the concept of "leading terms" replacing $a_{n,m} z^n w^m$ by a general homogeneous polynomial.

Let P be a homogeneous polynomial. We can write P as

$$P(z,w) = a w^{l_0} \prod_{j=1}^{k} (z - \alpha_j w)^{l_j}, \tag{1.1}$$

where $a, \alpha_j \in \mathbb{C}, \alpha_i \neq \alpha_j$ for $i \neq j$. We note $d(P) = \sum_{j=0}^{k} l_j$. Associate with P, let

$$A = \{\alpha_j; 1 \le j \le k\}$$

and we define domains by

$$\Omega_{A,r} = \bigcup_{j=1}^{k} \{(z,w) \in \mathbb{C}^2; |z - \alpha_j w| < r\},$$

for every $r > 0$. Let q be another polynomial with $d(q) \le d(P)$. If q has the following form

$$q(z,w) = \sum_{l'_j \le l_j} a_{(l'_0,\cdots,l'_k)} w^{l'_0} \prod_{j=1}^{k} (z - \alpha_j w)^{l'_j},$$

q is said to be *dominated* by P, and we write as $q \ll P$. If p is a polynomial and $p \ll p_{d(p)}$, we say that p has a *leading term* $p_{d(p)}$.

Generally some polynomials may not have leading terms. But the set of polynomials having leading terms is a fairly big class in $\mathcal{C}$. In this paper, we study polynomials having leading terms and prove the same type of assertions given by Guo and Hou in [8].

In Section 2, under the condition $l_0 = 0$ in (1.1), we characterize $q \in \mathcal{C}$ satisfying $q \ll P$.

Even if $p_{d(p)} = P$ and $l_0 = 0$, p may vanish in $\mathbb{C}^2 \setminus \Omega_{A,r}$ for every $r > 0$. In Section 3, we characterize polynomials p satisfying $|p| > 0$ on $\mathbb{C}^2 \setminus \Omega_{A,r}$ for some $r > 0$.

Let $\mathcal{C}_A$ be the set of homogeneous polynomials q such that

$$q(z,w) = aw^{i_0} \prod_{j=1}^{k} (z - \alpha_j w)^{i_j}, \quad a \neq 0.$$

In Section 4, we prove that p has a leading term $p_{d(p)}$ in $\mathcal{C}_A$ if and only if $|p| > 0$ on $\mathbb{C}^2 \setminus \Omega_{A,r}$ for some $r > 0$.

In Section 5, we study functions $f, g \in \mathrm{Hol}(\mathbb{C}^2)$ satisfying $|f| \leq K|g|$ on $\mathbb{C}^2 \setminus \Omega_{A,r}$ for some $K, r > 0$.

In Section 6, we study the case $l_0 \neq 0$ in (1.1) using unitary transformations.

So far we studied function theoretic properties of polynomials having leading terms. Applying them, we study quasi-invariant subspaces in the Fock space. In Section 7, we show that if $p \in \mathcal{C}$ has a leading term $p_{d(p)}$, then $[p]$ is quasi-invariant, and in Section 8, we prove that a quasi-invariant subspace M is similar to $[p]$ if and only if $M = [q]$ for some $q \in \mathcal{C}$ having the same leading term as p.

This is a summary of the paper [10].

2. Dominated polynomials

Let $\mathcal{C}_h = \mathcal{C}_h(\mathbb{C}^2)$ be the sets of homogeneous polynomials on $\mathbb{C}^2$. Let $p \in \mathcal{C}_h$. If we set $\zeta = z/w, w \neq 0$, then p has the form as

$$p(z,w) = w^{d(p)} p(\zeta, 1) = aw^{d(p)} \prod_{j=1}^{k} (\zeta - \alpha_j)^{l_j} = aw^{l_0} \prod_{j=1}^{k} (z - \alpha_j w)^{l_j},$$

where $a, \alpha_j \in \mathbb{C}, \alpha_i \neq \alpha_j$ for $i \neq j$, $l_j \in \mathbb{Z}_+$, and $d(p) = \sum_{j=0}^{k} l_j$. By this fact, for $s, t \in \mathbb{Z}_+$ with $s+t = d(p)$, there exist $p_1, p_2 \in \mathcal{C}_h$ such that $p = p_1 p_2 \in \mathcal{C}_h$, $d(p_1) = s$, and $d(p_2) = t$. Let $q \in \mathcal{C}$ with $d(q) \leq d(p)$. If q has the following form

$$q(z,w) = \sum_{l'_j \leq l_j} a_{(l'_0, \cdots, l'_k)} w^{l'_0} \prod_{j=1}^{k} (z - \alpha_j w)^{l'_j},$$

q is said to be dominated by p, and write as $q \ll p$. In this section, we characterize q satisfying $q \ll p$ under the condition $l_0 = 0$. The following is the main theorem in this section.

Theorem 2.1. *Let $\{\alpha_j\}_{j=1}^k \subset \mathbb{C}$ be such that $\alpha_i \neq \alpha_j$ for $i \neq j$. Let $p \in C_h$ be such that*

$$p(z,w) = \prod_{j=1}^{k} (z - \alpha_j w)^{l_j}$$

and $l_j \geq 1$ for ever $1 \leq j \leq k$. Let $q \in C$ and $q = \sum_{i=0}^{d(q)} q_i$ be the homogeneous expansion of q with $d(q_i) = i$ if $q_i \neq 0$. Then $q \ll p$ if and only if $d(q) \leq d(p)$ and

$$q_{d(p)-i} = q' \prod_{j=1}^{k} (z - \alpha_j w)^{(l_j - i)_+}$$

for every i with $0 \leq i < \max\limits_{1 \leq j \leq k} l_j$ and $q' \in C_h$.

To prove our theorem, we need some lemmas. It is not difficult to prove the following.

Lemma 2.1. *Let $p_1, p_2 \in C_h$ and $q_1, q_2 \in C$.*

(i) *If $q_1 \ll p_1$ and $q_2 \ll p_1$, then $q_1 + q_2 \ll p_1$.*
(ii) *If $q_1 \ll p_1$ and $q_2 \ll p_2$, then $q_1 q_2 \ll p_1 p_2$.*
(iii) *If $q_1 \ll p_1$ and $p_1 \ll p_2$, then $q_1 \ll p_2$ and $q_1 + p_2 \ll p_2$.*

Lemma 2.2. *Let $\{\alpha_j\}_{j=1}^k \subset \mathbb{C}$ be such that $\alpha_i \neq \alpha_j$ for $i \neq j$. Let $p \in C_h$ be such that*

$$p(z,w) = \prod_{j=1}^{k} (z - \alpha_j w)^{l_j}$$

and $l_j \geq 1$ for every $1 \leq j \leq k$. If $q \in C$ and $d(q) \leq d(p) - \max\limits_{1 \leq j \leq k} l_j$, then $q \ll p$.

For each integer m, let $m_+ = \max\{m, 0\}$.

Lemma 2.3. *Let $\{\alpha_j\}_{j=1}^k \subset \mathbb{C}$ be such that $\alpha_i \neq \alpha_j$ for $i \neq j$. Let $p \in \mathcal{C}_h$ be such that*

$$p(z,w) = \prod_{j=1}^{k} (z - \alpha_j w)^{l_j}$$

and $l_j \geq 1$ for every $1 \leq j \leq k$. Let i_0 be an integer with $0 \leq i_0 < \max_{1 \leq j \leq k} l_j$. Let $q \in \mathcal{C}_h$ with $d(q) = d(p) - i_0$. Then $q \ll p$ if and only if

$$q = q' \prod_{j=1}^{k} (z - \alpha_j w)^{(l_j - i_0)_+}$$

for some $q' \in \mathcal{C}_h$.

Combining Lemmas 2.2 and 2.3, we can get Theorem 2.1.

3. Zeros of polynomials in two variables

Let $A = \{\alpha_j\}_{j=1}^k \subset \mathbb{C}$ be such that $\alpha_i \neq \alpha_j$ for $i \neq j$. For $r > 0$, let

$$\Omega_{\alpha_j,r} = \Omega_{(\alpha_j,r)} = \{(z,w) \in \mathbb{C}^2; |z - \alpha_j w| < r\}$$

and

$$\Omega_{A,r} = \Omega_{(A,r)} = \bigcup_{j=1}^{k} \Omega_{\alpha_j,r}.$$

In this section, we prove the following.

Theorem 3.1. *Let $A = \{\alpha_j\}_{j=1}^k \subset \mathbb{C}$ be such that $\alpha_i \neq \alpha_j$ for $i \neq j$. Let $p \in \mathcal{C}$ be such that $d(p) \geq 1$ and*

$$p_{d(p)} = \prod_{j=1}^{k} (z - \alpha_j w)^{l_j}.$$

Then $|p| > 0$ on $\mathbb{C}^2 \setminus \Omega_{A,r}$ for some $r > 0$ if and only if there exists $r > 0$ such that

$$\limsup_{|w| \to \infty} \frac{\sup_{z \in D^1_{(\alpha_j,r)}(w)} |p(z,w)|}{|w|^{d(p) - l_j}} < \infty$$

for every $1 \leq j \leq k$.

To prove our theorem, we need some lemmas. For each fixed w, $|w| > 1$, let

$$D^1_{\alpha_j,r}(w) = D^1_{(\alpha_j,r)}(w) = \{z \in \mathbb{C}; |z - \alpha_j w| < r\},$$

$$D^2_{\alpha_j,r}(w) = D^2_{(\alpha_j,r)}(w) = \{\zeta \in \mathbb{C}; |\zeta - \alpha_j| < r/|w|\},$$
$$D^3_{\alpha_j,r}(w) = D^3_{(\alpha_j,r)}(w) = \{z \in \mathbb{C}; |z - \alpha_j w| < r|w|\},$$

and

$$D^4_{\alpha_j,r} = D^4_{(\alpha_j,r)} = \{\zeta \in \mathbb{C}; |\zeta - \alpha_j| < r\}.$$

Then $D^1_{\alpha_j,r}(w) \subset D^3_{\alpha_j,r}(w)$ and the mappings

$$D^1_{\alpha_j,r}(w) \ni z \to \zeta = z/w \in D^2_{\alpha_j,r}(w), \tag{3.3}$$

$$D^3_{\alpha_j,r}(w) \ni z \to \zeta = z/w \in D^4_{\alpha_j,r} \tag{3.4}$$

are one to one and onto. It is not difficult to show the following.

Lemma 3.1. *Let $A = \{\alpha_j\}_{j=1}^k \subset \mathbb{C}$ be such that $\alpha_i \neq \alpha_j$ for $i \neq j$ and $r > 0$. Then we have the following.*

(i) *For a large w, $D^1_{\alpha_i,r}(w) \cap D^1_{\alpha_j,r}(w) = \emptyset$ for $i \neq j$.*

(ii) *If $|\alpha_i - \alpha_j| > r_0 > 0$, then for a large w, $D^1_{\alpha_j,r}(w) \subset D^3_{\alpha_j,r_0}(w)$ and $D^1_{\alpha_i,r}(w) \cap D^3_{\alpha_j,r_0}(w) = \emptyset$.*

(iii) *If $\alpha \in \mathbb{C} \setminus A$, then for a large w, $(\alpha w, w) \in \mathbb{C}^2 \setminus \Omega_{A,r}$.*

(iv) *$(z, w) \in \Omega_{A,r}$ if and only if $z \in \bigcup_{j=1}^k D^1_{\alpha_j,r}(w)$.*

Lemma 3.2. *Let $A = \{\alpha_j\}_{j=1}^k \subset \mathbb{C}$ be such that $\alpha_i \neq \alpha_j$ for $i \neq j$. Let $p \in C$ be such that*

$$p_{d(p)} = \prod_{j=1}^k (z - \alpha_j w)^{l_j}.$$

Let $r_0 > 0$ be such that $2r_0 < \min_{i \neq j} |\alpha_i - \alpha_j|$. Then for each j with $1 \leq j \leq k$, we have the following.

(i) *For a large w, the function $p(\zeta w, w)/w^{d(p)}$ in ζ has l_j-zeros in $D^4_{\alpha_j,r_0}$ counting multiplicites.*

(ii) *For a large w, the function $p(z, w)$ in z has l_j-zeros in $D^3_{\alpha_j,r_0}(w)$ counting multiplicites.*

Proposition 3.1. *Let $A = \{\alpha_j\}_{j=1}^k \subset \mathbb{C}$ be such that $\alpha_i \neq \alpha_j$ for $i \neq j$. Let $p \in C$ be such that*

$$p_{d(p)} = \prod_{j=1}^k (z - \alpha_j w)^{l_j}.$$

Then for each j with $1 \leq j \leq k$, we have the following.

(i) *If* $|p| > 0$ *on* $\mathbb{C}^2 \setminus \Omega_{A,r}$ *for some* $r > 0$*, then for a large* w *the function* $p(z, w)$ *in* z *has* l_j*-zeros in* $D^1_{\alpha_j,r}(w)$ *counting multiplicites for every* $1 \leq j \leq k$.

Conversely, for $r > 0$ *and a large* w*, the function* $p(z, w)$ *in* z *has* l_j*-zeros in* $D^1_{\alpha_j,r}(w)$ *counting multiplicites for every* $1 \leq j \leq k$*, then there exists* $r_1 > 0$ *such that* $|p| > 0$ *on* $\mathbb{C}^2 \setminus \Omega_{A,r_1}$.

(ii) *If* $|p| > 0$ *on* $\mathbb{C}^2 \setminus \Omega_{A,r}$ *for some* $r > 0$*, then for a large* w *the function* $p(\zeta w, w)$ *in* ζ *has* l_j*-zeros in* $D^2_{\alpha_j,r}(w)$ *counting multiplicites for every* $1 \leq j \leq k$.

Conversely, for $r > 0$ *and a large* w*, the function* $p(\zeta w, w)$ *in* ζ *has* l_j*-zeros in* $D^2_{\alpha_j,r}(w)$ *counting multiplicites for every* $1 \leq j \leq k$*, then there exists* $r_1 > 0$ *such that* $|p| > 0$ *on* $\mathbb{C}^2 \setminus \Omega_{A,r_1}$.

For $R > 0$, let

$$B_R = \{(z, w) \in \mathbb{C}^2; |z|^2 + |w|^2 < R^2\}.$$

Lemma 3.3. *Let* $A = \{\alpha_j\}_{j=1}^k \subset \mathbb{C}$ *be such that* $\alpha_i \neq \alpha_j$ *for* $i \neq j$*. Let* $p \in C$ *be such that*

$$p_{d(p)}(z, w) = \prod_{j=1}^{k-1} (z - \alpha_j w)^{l_j}$$

and $l_j \geq 1$ *for every* $1 \leq j \leq k - 1$*. Then we have the following.*

(i)

$$\limsup_{|w| \to \infty} \frac{\sup_{z \in D^1_{(\alpha_k, r)}(w)} |p(z, w)|}{|w|^{d(p)}} < \infty.$$

(ii) $|p| > 0$ *on* $\Omega_{\alpha_k,r} \setminus B_R$ *for some* $r, R > 0$.

Remark 3.1. By the proof, if

$$p_{d(p)} = \prod_{j=1}^{k} (z - \alpha_j w)^{l_j}$$

and $|p| > 0$ on $\mathbb{C}^2 \setminus \Omega_{A,r}$, then

$$\limsup_{|w|\to\infty} \frac{\sup_{z\in D^1_{(\alpha_j,r)}(w)} |p(z,w)|}{|w|^{d(p)-l_j}} < \infty.$$

If $r_1 > r$, then $\mathbb{C}^2 \setminus \Omega_{A,r_1} \subset \mathbb{C}^2 \setminus \Omega_{A,r}$, so that $|p| > 0$ on $\mathbb{C}^2 \setminus \Omega_{A,r_1}$. Hence

$$\limsup_{|w|\to\infty} \frac{\sup_{z\in D^1_{(\alpha_j,r_1)}(w)} |p(z,w)|}{|w|^{d(p)-l_j}} < \infty.$$

If $r_2 < r$, then $D^1_{\alpha_j,r_2}(w) \subset D^1_{\alpha_j,r}(w)$, so that

$$\limsup_{|w|\to\infty} \frac{\sup_{z\in D^1_{(\alpha_j,r_2)}(w)} |p(z,w)|}{|w|^{d(p)-l_j}} < \infty.$$

Hence if $|p| > 0$ on $\mathbb{C}^2 \setminus \Omega_{A,r}$ for some $r > 0$, then

$$\limsup_{|w|\to\infty} \frac{\sup_{z\in D^1_{(\alpha_j,r)}(w)} |p(z,w)|}{|w|^{d(p)-l_j}} < \infty.$$

for every $r > 0$.

4. Leading terms

The following is the main theorem in this section.

Theorem 4.1. *Let $A = \{\alpha_j\}_{j=1}^k \subset \mathbb{C}$ be such that $\alpha_i \neq \alpha_j$ for $i \neq j$. Further we assume that $\alpha_j \neq 0$ for every j. Let $p \in C$ be such that $d(p) \geq 1$. Then the following conditions are equivalent.*

(i) *p has a leading term $p_{d(p)}$ such that $p_{d(p)} \in C_A$.*
(ii) *There exist $c_1, c_2 > 0$ such that $c_1 < |p/p_{d(p)}| < c_2$ on $\mathbb{C}^2 \setminus \Omega_{A,r}$ for some $r > 0$.*
(iii) *$|p| > 0$ on $\mathbb{C}^2 \setminus \Omega_{A,r}$ for some $r > 0$.*

5. Partial order in Hol($\mathbb{C}^2$)

Let $A = \{\alpha_j\}_{j=1}^k \subset \mathbb{C}$ be such that $\alpha_i \neq \alpha_j$ for $i \neq j$. For $f, g \in \mathrm{Hol}(\mathbb{C}^2)$, we write $f \preceq_A g$ if $|f| \leq M|g|$ on $\mathbb{C}^2 \setminus \Omega_{A,r}$ for some $M, r > 0$. Then $\mathrm{Hol}(\mathbb{C}^2)$ is a partially ordered set with $\preceq_A$. First we prove the following theorem.

Theorem 5.1. *Let $A = \{\alpha_j\}_{j=1}^k \subset \mathbb{C}$ be such that $\alpha_i \neq \alpha_j$ for $i \neq j$. Let $p, q \in \mathcal{C}(\mathbb{C}^2)$ be such that p, q do not have common factor. Then $q \preceq_A p$ if and only if $p_{d(p)} \in \mathcal{C}_A$, $p \ll p_{d(p)}$, and $q \ll p_{d(p)}$.*

To prove our theorem, we need some lemmas. In [2], Chen, Guo, and Hou proved the following.

Lemma 5.1. *Let $f, g \in \mathrm{Hol}(\mathbb{C})$. Then $|f(z)| \leq M|g(z)|$ on $\{|z| > r\}$ for some $r, M > 0$ if and only if there exist $p, g \in \mathcal{C}(\mathbb{C})$ with $d(q) \leq d(p)$ such that $f/g = q/p$.*

In [6], Guo proved the following.

Lemma 5.2. *Let $f(z, w)$ be in the Nevanlinna class on the polydisk $\mathbb{D}^2$. Suppose that the slice function $f_{(z,w)}(\lambda) = f(\lambda z, \lambda w)$ is rational in λ for almost all $(z, w) \in \mathbb{T}^2$. Then f is a rational function.*

Lemma 5.3. *Let $f = q/p$ be a rational function, where p and q have no common factor. If f is analytic in $\Omega \subset \mathbb{C}^2$, then $Z(p) \cap \Omega = \emptyset$.*

The following is the main theorem in this section.

Theorem 5.2. *Let $A = \{\alpha_j\}_{j=1}^k \subset \mathbb{C}$ be such that $\alpha_i \neq \alpha_j$ for $i \neq j$. Let $f, g \in \mathrm{Hol}(\mathbb{C}^2)$. Then $f \preceq_A g$ if and only if there exist $p, q \in \mathcal{C}$ such that $f/g = q/p$, $p_{d(p)} \in \mathcal{C}_A$, $p \ll p_{d(p)}$, and $q \ll p_{d(p)}$.*

6. Unitary transformations

In Sections 2-5, we studied the case $l_0 = 0$ in (1.1) In this section, we study the case $l_0 \neq 0$. Let $A = \{\alpha_j\}_{j=1}^k \subset \mathbb{C}$ be such that $\alpha_i \neq \alpha_j$ for $i \neq j$. Let $\tilde{\mathcal{C}}_A$ be the set of $p \in \mathcal{C}_h$ such that

$$p(z, w) = a w^{l_0} \prod_{j=1}^{k} (z - \alpha_j w)^{l_j}, \quad a \in \mathbb{C}$$

and

$$\tilde{\Omega}_{A,r} = \tilde{\Omega}_{(A,r)} = \{(z, w) \in \mathbb{C}^2; |w| < r\} \cup \Omega_{A,r}$$

for $r > 0$. For each $\alpha \in \mathbb{C}$ with $\alpha \neq 0$, let

$$A_\alpha = \left\{ \frac{\alpha_j - \alpha}{1 + \bar{\alpha}\alpha_j} \right\}_{j=1}^{k}$$

and

$$\tilde{A}_\alpha = \left\{ \frac{1}{\bar{\alpha}}, \frac{\alpha_j - \alpha}{1 + \bar{\alpha}\alpha_j} \right\}_{j=1}^k.$$

For $\alpha \in \mathbb{C}$, let

$$\begin{pmatrix} z \\ w \end{pmatrix} = U_\alpha \begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{\sqrt{1+|\alpha|^2}} \begin{pmatrix} 1 & \alpha \\ -\bar{\alpha} & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

Then U_α is a unitary transformation on $\mathbb{C}^2$. If $p \in \mathcal{C}$, then $p \circ U_\alpha$ is a polynomial in variables u and v. It is not difficult to show the following.

Lemma 6.1.

(i) $U_\alpha^{-1} = U_{-\alpha}$.
(ii) $d(p) = d(p \circ U_\alpha)$ *for* $p \in \mathcal{C}$.
(iii) *Let* $p \in \mathcal{C}_h$ *and* $q \in \mathcal{C}$. *If* $q \ll p$, *then* $q \circ U_\alpha \ll p \circ U_\alpha$.
(iv) *Let* $p \in \mathcal{C}$. *Then* p *has a leading term* $p_{d(p)}$ *if and only if* $p \circ U_\alpha$ *has a leading term* $(p \circ U_\alpha)_{d(p)}$.
(v) *If* $\alpha \neq 0$, *then*

$$U_\alpha^{-1}(\{(z,w); |w| < r\}) = \Omega_{\left(\frac{1}{\bar{\alpha}}, \frac{r\sqrt{1+|\alpha|^2}}{|\alpha|}\right)}.$$

(vi) *If* $\bar{\alpha}\beta \neq -1$, *then*

$$U_\alpha^{-1}(\Omega_{\beta,r}) = \Omega_{\left(\frac{\beta-\alpha}{1+\bar{\alpha}\beta}, \frac{r\sqrt{1+|\alpha|^2}}{|1+\bar{\alpha}\beta|}\right)}.$$

(vii) *If* $\bar{\alpha}\alpha_j \neq -1$ *for every* $1 \leq j \leq k$, *then* $p \in \mathcal{C}_A$ *if and only if* $p \circ U_\alpha \in \mathcal{C}_{A_\alpha}$, *and* $p \in \tilde{\mathcal{C}}_A$ *if and only if* $p \circ U_\alpha \in \mathcal{C}_{\tilde{A}_\alpha}$.
(viii) $\mathcal{C}_{\tilde{A}_\alpha} \circ U_{-\alpha} = \tilde{\mathcal{C}}_A$.

Applying Lemma 6.1, we give generalizations of results proved in the previous sections. The following is a generalization of Theorem 2.1.

Theorem 6.1. *Let* $p \in \tilde{\mathcal{C}}_A$ *be such that*

$$p(z,w) = w^{l_0} \prod_{j=1}^k (z - \alpha_j w)^{l_j}.$$

Let $q \in \mathcal{C}$ *and* $q = \sum_{i=0}^{d(q)} q_i$ *be the homogeneous expansion of* q *with* $d(q_i) = i$ *if* $q_i \neq 0$. *Then* $q \ll p$ *if and only if* $d(q) \leq d(p)$ *and*

$$q_{d(p)-i} = q' w^{(l_0-i)_+} \prod_{j=1}^k (z - \alpha_j w)^{(l_j-i)_+}$$

for every i with $0 \leq i < \max_{0 \leq j \leq k} l_j$ and $q' \in C_h$.

Corollary 6.1. *Let $p \in \tilde{C}_A$ be such that*

$$p(z,w) = w^{l_0} \prod_{j=1}^{k} (z - \alpha_j w)^{l_j}$$

and $l_j \leq 1$ for every $0 \leq j \leq k$. Then $p_1 \ll p$ for every $p_1 \in C$ with $d(p_1) < d(p)$.

Corollary 6.2. *Let $p \in \tilde{C}_A$ be such that*

$$p(z,w) = w^{l_0} \prod_{j=1}^{k} (z - \alpha_j w)^{l_j}.$$

Then $p_1 \ll p$ for every $p_1 \in C$ with $\deg p_1 < \deg p - \max_{0 \leq j \leq k} l_j$.

The following theorem shows the most easy way to check whether $q \ll p$ or not.

Theorem 6.2. *Let $p \in \tilde{C}_A$ be such that*

$$p(z,w) = w^{l_0} \prod_{j=1}^{k} (z - \alpha_j w)^{l_j}$$

and let $q \in C$. Then $q \ll p$ if and only if $d(q) \leq d(p)$ and the following three conditions hold.

(i) *$d_z(q) \leq d(p) - l_0$.*
(ii) *If $\alpha_{j_0} = 0$ for some $j_0, 1 \leq j_0 \leq k$, then $d_w(q) \leq d(p) - l_{j_0}$.*
(iii) *Suppose that $\alpha_m \neq 0, 1 \leq m \leq k$. Then $d_v(q \circ U_{\alpha_m}) \leq d(p) - l_m$.*

The following is a generalization of Theorem 3.1.

Theorem 6.3. *Let $p \in \tilde{C}_A$ be such that $d(p) \geq 1$ and*

$$p_{d(p)}(z,w) = w^{l_0} \prod_{j=1}^{k} (z - \alpha_j w)^{l_j}.$$

Then $|p| > 0$ *on* $\mathbb{C}^2 \setminus \tilde{\Omega}_{A,r}$ *for some* $r > 0$ *if and only if there exists* $r > 0$ *such that*

$$\limsup_{|w|\to\infty} \frac{\sup_{z\in D^1_{(\alpha_j,r)}(w)} |p(z,w)|}{|w|^{d(p)-l_j}} < \infty$$

for every $1 \le j \le k$, *and*

$$\limsup_{|z|\to\infty} \frac{\sup_{|w|<r} |p(z,w)|}{|z|^{d(p)-l_0}} < \infty.$$

The following is a generalization of Theorem 4.1 and [8, Proposition 2.4].

Theorem 6.4. *Let* $p \in C$ *be such that* $d(p) \ge 1$. *Then the following conditions are equivalent.*

(i) p *has a leading term* $p_{d(p)}$ *such that* $p_{d(p)} \in \tilde{C}_A$.

(ii) *There exist* $c_1, c_2 > 0$ *such that* $c_1 < |p/p_{d(p)}| < c_2$ *on* $\mathbb{C}^2 \setminus \tilde{\Omega}_{A,r}$ *for some* $r > 0$.

(iii) $|p| > 0$ *on* $\mathbb{C}^2 \setminus \tilde{\Omega}_{A,r}$ *for some* $r > 0$.

For $f, g \in \mathrm{Hol}(\mathbb{C}^2)$, we write $f \preceq_{\tilde{A}} g$ if $|f| \le M|g|$ on $\mathbb{C}^2 \setminus \tilde{\Omega}_{A,r}$ for some $M, r > 0$. The following is a generalization of Theorem 5.1.

Theorem 6.5. *Let* $p, q \in C$ *be such that* p, q *do not have common factor. Then* $q \preceq_{\tilde{A}} p$ *if and only if* $p_{d(p)} \in \tilde{C}_A$, $p \ll p_{d(p)}$, *and* $q \ll p_{d(p)}$.

The following is a generalization of Theorem 5.2 ang [8, Theorem 2.5].

Theorem 6.6. *Let* $f, g \in \mathrm{Hol}(\mathbb{C}^2)$. *Then* $f \preceq_{\tilde{A}} g$ *if and only if there exist* $p, q \in C$ *such that* $f/g = q/p$, $p_{d(p)} \in \tilde{C}_A$, $p \ll p_{d(p)}$, *and* $q \ll p_{d(p)}$.

Combine with Theorems 6.3 and 6.4, we have the following.

Corollary 6.3. *Let* $p \in C$ *be such that* $d(p) \ge 1$. *Then the following conditions are equivalent.*

(i) p *has a leading term* $p_{d(p)}$ *such that* $p_{d(p)} \in \tilde{C}_A$.

(ii) $|p| > 0$ *on* $\mathbb{C}^2 \setminus \tilde{\Omega}_{A,r}$ *for some* $r > 0$.

(iii) $p_{d(p)} \in \tilde{C}_A$ *and there exist* $c_1, c_2 > 0$ *such that* $c_1 \leq |p/p_{d(p)}| \leq c_2$ *on* $\mathbb{C}^2 \setminus \tilde{\Omega}_{A,r}$ *for some* $r > 0$.

(iv) $p_{d(p)} = aw^{l_0}(z - \alpha_1 w)^{l_1}(z - \alpha_2 w)^{l_2} \cdots (z - \alpha_k w)^{l_k}$, $a \neq 0$, *and there exists* $r > 0$ *such that*

$$\limsup_{|w|\to\infty} \frac{\sup_{z\in D^1_{(\alpha_j,r)}(w)} |p(z,w)|}{|w|^{d(p)-l_j}} < \infty$$

for every $1 \leq j \leq k$, *and*

$$\limsup_{|z|\to\infty} \frac{\sup_{|w|<r} |p(z,w)|}{|z|^{d(p)-l_0}} < \infty.$$

7. Quasi-invariant subspaces

Let $p \in \mathcal{C}$. If $p \ll p_{d(p)}$ holds, we say that p has a leading term $p_{d(p)}$. It is not known whether $[p] = \overline{p\mathcal{C}}$ is quasi-invariant for every $p \in \mathcal{C}$. It is known that if $p \in \mathcal{C}_h$, then $[p]$ is quasi-invariant, see [1, Propoisition 5.5.1]. In Theorem 4.4 of [8], Guo and Hou proved that if $p \in \mathcal{C}$ has a leading term $z^m w^n$, then $[p]$ is quasi-invariant. The following is the main theorem in this section.

Theorem 7.1. *Let* $p \in \mathcal{C}$ *be having a leading term* $p_{d(p)}$. *Then we have the following.*

(i) $[p]/p = [p_{d(p)}]/p_{d(p)}$.

(ii) $[p]$ *is quasi-invariant.*

(iii) $[p] = \{pf \in L^2_a(\mathbb{C}^2); f \in \mathrm{Hol}(\mathbb{C}^2)\} = \{pf \in L^2_a(\mathbb{C}^2); f \in L^2_a(\mathbb{C}^2)\}$.

Recall that

$$\Omega_{\alpha,r} = \{(z,w) \in \mathbb{C}^2; |z - \alpha w| < r\}$$

for $\alpha \in \mathbb{C}$ and $r > 0$. Then $\Omega_{0,r} = \{(z,w); |z| < r\}$. For $R \geq 0$, let

$$\Omega_{0,r,R} = \Omega_{(0,r,R)} = \{(z,w) \in \mathbb{C}^2; |z| < r, |w| \geq R\}.$$

Note that $\Omega_{0,r,0} = \Omega_{0,r}$. For a subset Ω of $\mathbb{C}^2$, let

$$\|f\|^2_\Omega = \int_\Omega |f(z,w)|^2 e^{-\frac{|z|^2+|w|^2}{2}} dA(z,w)/(2\pi)^2.$$

As the proof of Theorem 3.1 in [8], we have the following.

Lemma 7.1. *Let $r_1, r_2, r_3 > 0$ and $r_1 < r_2$. Then there exists a constant $C > 0$, depends on r_1, r_2, and r_3, such that*

$$\|f\|_{\Omega_{0,r_2}} \leq C\|f\|_{(\Omega_{(0,r_2,r_3)} \setminus \Omega_{(0,r_1,r_3)})}$$

for every $f \in \mathrm{Hol}(\mathbb{C}^2)$.

Recall that $U_\alpha, \alpha \in \mathbb{C}$, are unitary transformations of $\mathbb{C}^2$;

$$\begin{pmatrix} z \\ w \end{pmatrix} = U_\alpha \begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{\sqrt{1+|\alpha|^2}} \begin{pmatrix} 1 & \alpha \\ -\bar{\alpha} & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

It is easy to see the following.

Lemma 7.2.

(i) $\|f\|_\Omega = \|f \circ U_\alpha\|_{U_\alpha^{-1}\Omega} = \|f \circ U_\alpha^{-1}\|_{U_\alpha \Omega}$ *for $\Omega \subset \mathbb{C}^2$ and $f \in \mathrm{Hol}(\mathbb{C}^2)$.*

(ii) $U_\alpha(\Omega_{\alpha,r}) = \Omega_{(0,r/\sqrt{1+|\alpha|^2})}$.

(iii) $\Omega_{\left(0,\frac{r}{\sqrt{1+|\alpha|^2}},t\right)} \subset U_\alpha(\Omega_{\alpha,r} \setminus B_t)$, *where $B_t = \{(z,w); |z|^2 + |w|^2 < t\}$.*

Lemma 7.3. —it For $\alpha \in \mathbb{C}, r_2 > r_1 > 0$, and $t > 0$, there exists a constant $C > 0$, depends on r_1, r_2, and t, such that

$$\|f\|_{\Omega_{\alpha,r_2}} \leq C\|f\|_{(\Omega_{\alpha,r_2} \setminus (\Omega_{\alpha,r_1} \cup B_t))}$$

for every $f \in \mathrm{Hol}(\mathbb{C}^2)$.

Recall that $A = \{\alpha_j\}_{j=1}^k \subset \mathbb{C}$, $\alpha_i \neq \alpha_j$ for $i \neq j$, and

$$\Omega_{A,r} = \bigcup_{j=1}^k \Omega_{\alpha_j,r} \quad \text{and} \quad \tilde{\Omega}_{A,r} = \{(z,w); |w| < r\} \cup \Omega_{A,r}$$

for $r > 0$. The following lemma is not difficult to prove.

Lemma 7.4. *For $\alpha \in \mathbb{C}$ and $r_1 > r > 0$, there exists a large $t > 0$ such that*

$$(\Omega_{\alpha_i,r_1} \setminus (\Omega_{\alpha_i,r} \cup B_t)) \cap (\Omega_{\alpha_j,r_1} \setminus (\Omega_{\alpha_j,r} \cup B_t)) = \emptyset$$

for $i \neq j$ and

$$\Omega_{\alpha_j,r_1} \setminus (\Omega_{\alpha_j,r} \cup B_t) \subset \mathbb{C}^2 \setminus \Omega_{A,r}$$

for every $1 \leq j \leq k$.

Proposition 7.1. *Let $A = \{\alpha_j\}_{j=1}^k \subset \mathbb{C}$ be such that $\alpha_i \neq \alpha_j$ for $i \neq j$. Then for each $r > 0$, there exists a constant $C > 0$, depends on r, such that $C\|f\| \leq \|f\|_{\mathbb{C}^2 \setminus \Omega_{A,r}}$ for every $f \in \mathrm{Hol}(\mathbb{C}^2)$.*

Corollary 7.1. *Let $A = \{\alpha_j\}_{j=1}^k \subset \mathbb{C}$ be such that $\alpha_i \neq \alpha_j$ for $i \neq j$. Then for each $r > 0$, there exists a constant $C > 0$, depends on r, such that $C\|f\| \leq \|f\|_{\mathbb{C}^2 \setminus \tilde{\Omega}_{A,r}} \leq \|f\|$ for every $f \in \mathrm{Hol}(\mathbb{C}^2)$.*

The following is a generalization of [8, Theorem 3.1].

Corollary 7.2. *Let $f, g \in \mathrm{Hol}(\mathbb{C}^2)$. If $f \preceq_{\tilde{A}} g$ and $g \in L_a^2(\mathbb{C}^2)$, then $f \in L_a^2(\mathbb{C}^2)$.*

By Corollary 6.1 and Theorem 7.1, we have the following.

Corollary 7.3. *Let $p \in \mathcal{C}$. If $p_{d(p)} = aw(z - \alpha_1 w) \cdots (z - \alpha_k w)$ and $\alpha_i \neq \alpha_j$ for $i \neq j$, then $[p]$ is quasi-invariant.*

Corollary 7.4. *Let $p \in \mathcal{C}$ be having a leading term $p_{d(p)}$. Then there exists a similar module map T from $[p_{d(p)}]$ onto $[p]$ such that $Tf = (pf)/p_{d(p)}$ for $f \in [p_{d(p)}]$.*

8. Quasi-similarity

If $p \in \mathcal{C}$ is a polynomial with the leading term $p_{d(p)}$, then by Corollary 7.4 $[p]$ and $[p_{d(p)}]$ are similar. The following is the main theorem in this section and a generalization of [8, Theorem 4.7].

Theorem 8.1. *Let M be a quasi-invariant subspace of $L_a^2(\mathbb{C}^2)$. Let $p \in \mathcal{C}_h$ be a homogeneous polynomial. Then $[p]$ and M are quasi-similar if and only if $M = [q]$ for some $q \in \mathcal{C}$ having the leading term p.*

To prove this, we need some lemmas. The following is proved by Guo and Hou [8, Lemma 4.6].

Lemma 8.1. *Let M_1, M_2 be quasi-invariant subspaces of $L_a^2(\mathbb{C}^2)$. Let T be a quasi-module map from M_1 to M_2. Suppose that $p \in \mathcal{C} \cap M_1$ and $p \neq 0$. Let $q = Tp$. Then q is a polynomial, $d_z(q) \leq d_z(p)$, and $d_w(q) \leq d_w(p)$.*

Similarly, we have the following.

Lemma 8.2. *Let M_1, M_2 be quasi-invariant subspaces of $L^2_a(\mathbb{C}^2)$. Let $T : M_1 \to M_2$ be a quasi-module map. Suppose that $p \in \mathcal{C} \cap M_1$ and $p \neq 0$. Let $q = Tp$. Then q is a polynomial and $d(q) \leq d(p)$.*

The following lemma is obvious. To clear our argument, we give here.

Lemma 8.3. *Let $p \in \mathcal{C}_h$ and $q \in \mathcal{C}$. Then q has a leading term $ap, a \in \mathbb{C}, a \neq 0$, if and only if $d(q) = d(p)$ and $q \ll p$.*

For $\alpha \in \mathbb{C}$, let

$$\begin{pmatrix} z \\ w \end{pmatrix} = U_\alpha \begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{\sqrt{1+|\alpha|^2}} \begin{pmatrix} 1 & \alpha \\ -\bar{\alpha} & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

and we use the same notations as in Section 6. Let M_1, M_2 be quasi-invariant subspaces of $L^2_a(\mathbb{C}^2)$ and

$$M_i \circ U_\alpha = \left\{ f \circ U_\alpha \begin{pmatrix} u \\ v \end{pmatrix} ; f \in M_i \right\}.$$

Then it is not difficult to see that $M_i \circ U_\alpha$ is quasi-invariant in variables u and v. Let C_{U_α} be the unitary operator from M_i onto $M_i \circ U_\alpha$ defined by

$$(C_{U_\alpha} f) \begin{pmatrix} u \\ v \end{pmatrix} = f \circ U_\alpha \begin{pmatrix} u \\ v \end{pmatrix}.$$

Let $T : M_1 \to M_2$ be a quasi-module map. Then we have a map

$$M_1 \circ U_\alpha \ni g(u,v) \to C_{U_\alpha} T C^{-1}_{U_\alpha} \in M_2 \circ U_\alpha.$$

It is also not difficult to see that $C_{U_\alpha} T C^{-1}_{U_\alpha}$ is a quasi-module map.

Corollary 8.1. *Let M be a quasi-invariant subspace, and let $p \in \mathcal{C}$ be having a leading term $p_{d(p)}$. Then the following conditions are equivalent.*

(i) *M is similar to $[p]$.*
(ii) *M is quasi-similar to $[p]$.*
(iii) *$M = [q]$ for some $q \in \mathcal{C}$ having a leading term $p_{d(p)}$.*

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