数理解析研究所講究録
1519 巻 2006 年 103-110

A NOTE ON COMPOSITION OPERATORS IN SEVERAL VARIABLES

HYUNGWOON KOO

ABSTRACT. In this article we survey some recent progress on the boundedness and the compactness of composition operators on Bergman or Hardy spaces on the unit ball or the unit polydisc. Also, we raise several relevant questions.

1. INTRODUCTION

For a smooth domain $\Omega\subset \mathbf{C}^{n}$, we use $H(\Omega)$ to denote the space of holomorphic functions in Ω. Most of this article is confined to three domains: the open unit disc in $\mathbf{C}$,

$$D=\{z\in \mathbf{C}:|z|<1\},$$

the open unit ball in $\mathbf{C}^{n}$

$$B^{n}=\{z=(z_{1}, \ldots, z_{n})\in \mathbf{C}^{n}:\sum_{j=1}^{n}|z_{j}|^{2}<1\}$$

and the open unit polydisc in $\mathbf{C}^{n}$

$$D^{n}=\{z=(z_{1}, \ldots, z_{n})\in \mathbf{C}^{n}:|z_{1}|<1, \ldots, |z_{n}|<1\}.$$

If we do not specify Ω, then Ω is either disc, ball or polydisc.

Bergman and Hardy spaces on the unit ball

For $0<p<\infty$ and $\alpha>-1$, the weighted Bergman space $A_{\alpha}^{p}(B^{n})$ is the space of all $f\in H(B^{n})$ for which

$$||f||_{A_{\alpha}^{p}}^{p}=\int_{B^{n}}|f(z)|^{p}(1-|z|^{2})^{\alpha}\,dV(z)<\infty,$$

2000 *Mathematics Subject Classification.* Primary 47B33, Secondary 30D55, 46E15.

Key words and phrases. Composition operator, ball, polydisc, several variables, Hardy space, Bergman space.

This research was partially supported by KRF-2004-015-C00019.

where dV is normalized volume measure on B^n. Also, for $0 < p < \infty$, the Hardy space $H^p(B^n)$ is the space of all $g \in H(B^n)$ for which

$$||g||_{H^p}^p = \sup_{0<r<1} \int_{\partial B^n} |g(r\zeta)|^p d\sigma(\zeta) < \infty$$

where $d\sigma$ is normalized surface measure on ∂B^n. If $g \in H^p(B^n)$, then the radial limit $g(\zeta) = \lim_{r\to 1^-} g(r\zeta)$ exists for almost all $\zeta \in \partial B^n$ and

$$||g||_{H^p}^p = \int_{\partial B^n} |g(\zeta)|^p d\sigma(\zeta).$$

Bergman and Hardy spaces on the unit Polydisc

For $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space $A_\alpha^p(D^n)$ is the space of all $f \in H(D^n)$ for which

$$||f||_{A_\alpha^p}^p = \int_{D^n} |f(z)|^p \left(\prod_{i=1}^{n} (1 - |z_i|^2)^\alpha \right) dV(z) < \infty,$$

where dV is normalized volume measure on D^n. Also, for $0 < p < \infty$, the Hardy space $H^p(B^n)$ is the space of all $g \in H(D^n)$ for which

$$||g||_{H^p}^p = \sup_{0<r<1} \int_{T^n} |g(r\zeta)|^p d\sigma(\zeta) < \infty$$

where $T^n = \{z \in \mathbf{C}^n : |z_1| = \cdots = |z_n| = 1\}$ and $d\sigma$ is normalized surface measure on T^n. If $g \in H^p(D^n)$, then the radial limit $g(\zeta) = \lim_{r\to 1^-} g(r\zeta)$ exists for almost all $\zeta \in T^n$ and

$$||g||_{H^p}^p = \int_{T^n} |g(\zeta)|^p d\sigma(\zeta).$$

We will often use the following notation to allow unified statements:

$$A_{-1}^p(\Omega) = H^p(\Omega).$$

Let φ be a vector-valued holomorphic function from $\Omega^m \subset \mathbf{C}^m$ to $\Omega^n \subset \mathbf{C}^n$ for some positive integers n and m. That is,

$$\varphi = (\varphi_1, ..., \varphi_n) : \Omega^m \to \Omega^n$$

where each φ_j is holomorphic on Ω^m. Then φ induces the composition operator C_φ, defined on $H(\Omega^n)$ by

$$C_\varphi f = f \circ \varphi.$$

Boundedness and compactness for the disc

Composition operators on the function spaces on the unit disc have long been studied. Many beautiful theories have been developed on the unit disc case, but for several variables not much is known for corresponding results to the disc case. Here, we introduce the boundedness and the compactness criteria on the unit disc.

It is a well known consequence of Littlewood's Subordination Principle that every composition operator C_φ is bounded on each of the spaces $A^p_\alpha(B^1)$, $p > 0$, $\alpha \geq -1$; see for example [CM].

Theorem 1.1. *If $\varphi : D \to D$ is holomorphic, then*

$$C_\varphi : A^p_\alpha(D) \to A^p_\alpha(D)$$

for all $p > 0$ and $\alpha \geq -1$.

This result does not extend to the case that $m = n > 1$, where even such a simple function as $\varphi(z_1, z_2) = (2z_1 z_2, 0)$ is known to induce an unbounded composition operator on $H^p(B^2)$; see section 3.5 in [CM]. Also, for the polydisc case, $\varphi(z_1, z_2) = (z_1, z_1)$ is known to induce an unbounded composition operator on $H^p(D^2)$; see [SZ2].

Also, the compactness criteria on Bergman or Hardy spaces is well-known for the disc case. Bergman space case is easier and the criteria is the non-existence of the finite angular derivative. See [MS] or [CM] for a proof.

Theorem 1.2. *Let $\alpha > -1$. C_φ is compact on $A^p_\alpha(D)$ if and only if φ has no finite angular derivative.*

The Hardy space case is much more complicated and the criteria is given in terms of the Nevanlinna counting function. The Nevanlinna counting function is defined as

$$N_\varphi(w) = \sum_{z_j \in \varphi^{-1}(w)} \log(1/|z_j|).$$

For the following compactness criteria, see [Sh] or [CM].

Theorem 1.3. *C_φ is compact on $H^p(D)$ if and only if*

$$\lim_{|w| \to 1} \frac{N_\varphi(w)}{\log(1/|w|)} = 0.$$

2. Boundedness

In this section, we discuss the boundedness of a composition operator. More precisely, given $A^p_\alpha(\Omega)$ we are looking for a weighted space $A^p_\beta(\Omega)$ such that $C_\varphi : A^p_\alpha(\Omega) \to A^p_\beta(\Omega)$ is bounded for any holomorphic map $\varphi : \Omega \to \Omega$. The polydisc case is complete solved by Stessin and Zhu([SZ2]), but the unit ball case is still open.

Ball

For the ball case, $\varphi(z_1, z_2) = (2z_1z_2, 0)$ is known to induce an unbounded composition operator on $H^p(B^2)$; see section 3.5 in [CM]. So, we need to find a natural target space. The following result says $A^p_{n+\alpha-1}(B^n)$ is a natural target space for $C_\varphi(A^p_\alpha(B^n))$.

Theorem 2.1. *Let n and m be positive integers, and let $\alpha \geq -1$. Let φ be a vector-valued holomorphic function from B^m to B^n. Then C_φ maps $A^p_\alpha(B^n)$ boundedly into $A^p_{n+\alpha-1}(B^m)$:*

$$C_\varphi : A^p_\alpha(B^n) \to A^p_{\alpha+n-1}(B^m).$$

Moreover, there is a constant C independent of φ such that

$$||C_\varphi|| \leq C\left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{\frac{n+\alpha+1}{p}} \cdots$$

This result was proved for $\alpha = -1$ and $m = n$ by B. MacCluer and P. Mercer in [MM], and subsequently extended to $\alpha > -1$ and $m = n$ by J. Cima and P. Mercer in [CiMe]. For $m \neq n$, this is proved in [KS1] and [SZ1].

When $m = n = 1$ the choice $\varphi(z) = z$ (which makes C_φ the identity operator) shows it is sharp in the sense that the target space can not be replaced by a smaller Bergman or Hardy space. Moreover result is sharp when either $(n, \alpha) = (1, -1)$ or $m = 1$. See, [KS1] for details. For any other cases, we do not know whether the target space is sharp. Here, we state the important simple case of the optimal target space problem.

Question 2.2. *Is there holomorphic $\varphi : B^n \to B^n$ with the following property ? :*

$$C_\varphi : A^p_\alpha(B^n) \not\to A^p_{\alpha+n-1-\epsilon}(B^n)$$

for any $\epsilon > 0$.

In other words, is the target space in Theorem 2.1 sharp ? One might expect that an inner function would be an example, but due to the bad behavior of inner functions near the boundary it looks technically impossible to calculate Carleson measure. Carleson measure is hard to calculate but for general symbol map we do not have any other tools at hands to use. See [CM] for the Carleson measure characterization of the boundedness of a composition operator from a Bergman(or Hardy) space to another.

Polydisc

For the polydisc case, $\varphi(z_1, z_2) = (z_1, z_1)$ is known to induce an unbounded composition operator on $A^p_\alpha(D^2)$; see [SZ2]. So, we need to find a natural target space again like the ball case. The following result says $A^p_{n(\alpha+2)-2}(D^n)$ is the natural target space for $C_\varphi(A^p_\alpha(D^n))$. See [SZ2] for a proof.

Theorem 2.3. *Let $0 < p$ and $-1 \leq \alpha$, then*

$$C_\varphi : A^p_\alpha(D^n) \to A^p_{n(\alpha+2)-2}(D^m).$$

Moreover, the weight $n(\alpha+2)-2$ is the best possible.

Unlike the ball case, this theorem completely solves the optimal target space problem for the polydisc compositions.

3. COMPACTNESS

In this section, we discuss the compactness of a composition operator. In Section 2, given $A^p_\alpha(\Omega)$ we found(or found a candidate for) a weighted space $A^p_\beta(\Omega)$ such that $C_\varphi : A^p_\alpha(\Omega) \to A^p_\beta(\Omega)$ is bounded for any holomorphic map $\varphi : \Omega \to \Omega$. In this section, we discuss the compactness criteria for the operator $C_\varphi : A^p_\alpha(\Omega) \to A^p_\beta(\Omega)$. As in the boundedness case, the problem is complete solved by Stessin and Zhu([SZ2]) for the polydisc case, but the unit ball case is still open.

Ball

For the Bergman space on the unit ball, we have the following result by Zhu. See [Z].

Theorem 3.1. *Let $p > 0$ and $\alpha > 0$. If C_φ is bounded on $A^q_\beta(B^n)$ for some $-1 < \beta < \alpha$, then C_φ is compact on $A^p_\alpha(B^n)$ if and only if*

$$\lim_{|z|\to 1^-} \frac{1-|z|^2}{1-|\varphi(z)|^2} = 0.$$

As is stated in [Z], the boundedness condition on $A^q_\beta(B^n)$ is only needed in the necessity part. I.e., if the operator is compact then the above limit is zero(the non-existence of finite angular derivatives, by Julia-Caratheodory theorem in B^n([CM])) for any holomorphic map φ. Note that the compactness criteria on the unit ball is very similar to the disc case. On the other hand, the natural target space for $C_\varphi(A^p_\alpha(B^n))$ is $A^p_{\alpha+n-1}(B^n)$. So, it would be very interesting to know the compactness criteria for this natural target space for the boundedness.

Question 3.2. *Characterize the compactness of the operator*

$$C_\varphi : A^p_\alpha(B^n) \to A^p_{\alpha+n-1}(B^n).$$

For Bergman spaces on the unit disc($n = 1$), the compactness criteria of the C_φ above is the non-existence of finite angular derivatives, Theorem 1.2. For the unit ball case, Theorem 3.1 says if $C_\varphi : A^p_\alpha(B^n) \to A^p_\alpha(B^n)$ is compact, then φ has no finite angular derivative at any point of ∂B^n (also see, Propositon 1 of [Me]). It is proved in Proposition 2 of [Me] that $C_\varphi : A^p_\alpha(B^n) \to A^p_{\alpha+n-1}(B^n)$ is always compact, but some part of the proof seems to be unclear.

Polydisc

For the polydisc case, the compactness criteria for the natural target space is completely solved by Stessin and Zhu([SZ2]).

Theorem 3.3. *Let* $0 < p$ *and* $-1 \leq \alpha$, *then*

$$C_\varphi : A^p_\alpha(D^n) \to A^p_{n(\alpha+2)-2}(D^m)$$

is compact if and only if

$$\lim_{z \to \partial D^n} \prod_{j=1}^{n} \left(\frac{1-|z_j|^2}{1-|\varphi_j(z)|^2} \right) = 0.$$

4. Boundedness into the same space on the unit ball

In the previous two chapters, we discussed the boundedness or the compactness of the composition operators from one space to another. In this section, we discuss the composition operator from one space on the unit ball into itself.

There is a characterization using the Carleson measure for this case(see [CM]), but the Carleson measure criteria is very hard to verify. For example $C_\varphi : A^p_\alpha(B^2) \to A^p_{\alpha+1/4}(B^2)$ when $\varphi(z) = (z_1 - z_2^2/2, 0)$, and the weight $(\alpha + 1/4)$ is the best possible(see [KS2]). But this fact is very hard to verify using the Carleson measure criteria. Actually, I do not know a proof of this which uses the Carleson measure criteria.

Other than the Carleson measure characterization, there is a very nice criteria by Wogen when the symbol map φ is sufficiently smooth. Let $\varphi : B^n \to B^n$ and we say that Condition W is satisfied if

$$\partial_\zeta \varphi_\zeta(\eta) \neq \left|\partial_{\zeta^\perp}\partial_{\zeta^\perp}\varphi_\zeta(\eta)\right|$$

for all ζ, η with

$$\varphi(\zeta) = \eta \in \partial B^n.$$

Here, $\varphi_\zeta(z) =< \varphi(z), \zeta >$. For a proof of the following theorem, due to Wogen, see [CM] or [W1].

Theorem 4.1. *Suppose $\varphi \in C^3(\overline{B^n})$ and let $0 < p$ and $-1 \leq \alpha$, then*

$$C_\varphi : A^p_\alpha(B^n) \to A^p_\alpha(B^n)$$

if and only if Condition W is satisfied.

Wogen proved this when $\alpha = -1$, i.e., for the Hardy space. This is generalized to the strictly pseudo-convex domains by [MM] and for the weighted Bergman spaces($\alpha > -1$) on the unit ball by [KS2]. It is very interesting that there is a polynomial map $\varphi : B^2 \to B^2$ which is of degree 3 and one to one on $\overline{B^2}$ such that C_φ is not bounded on $H^2(B^n)$. See [W1] or [CM]. Meanwhile, we have the following result by Wogen([W2]).

Theorem 4.2. *If $\varphi : B^n \to B^n$ is biholomorphic, $\varphi \in C^3(\overline{B^n})$ and $\varphi(B^n)$ is convex, then C_φ is bounded on $H^2(B^n)$.*

Next, we discuss what happens when C_φ is not bounded on Bergman spaces, assuming the symbol is very smooth. In this case, there is a jump phenomena as the following result shows. See [KS2] for a proof.

Theorem 4.3. *Let $\alpha \geq -1$, $p > 0$ and $\varphi : B^n \to B^n$ be a holomorphic function on B^n of class C^4 on $\overline{B^n}$. If $0 < \epsilon < 1/4$ and $C_\varphi : A^p_\alpha(B^n) \to A^p_{\alpha+\epsilon}(B^n)$, then $C_\varphi : A^p_\alpha(B^n) \to A^p_\alpha(B^n)$. Moreover, this fails for $\epsilon = 1/4$.*

One natural question rises from this result.

Question 4.4. *Let $\varphi : B^n \to B^n$ be a holomorphic function on B^n which is sufficiently smooth on $\overline{B^n}$. When C_φ is not bounded on $H^2(B^n)$, what is the criteria for*

$$C_\varphi : A^p_\alpha(B^n) \to A^p_{\alpha+1/4}(B^n)?$$

For the compactness we do not have any result other than Theorem 3.1. For the Hardy space, even we do not have a result similar to Theorem 3.1.

Question 4.5. *Characterize the compactness of the operator*

$$C_\varphi : A^p_\alpha(B^n) \to A^p_\alpha(B^n).$$

Note that when $\varphi \in C^3(\overline{B^n})$, by Theorem 4.1 and Theorem 3.1 it is easy to see that $C_\varphi : A^p_\alpha(B^n) \to A^p_\alpha(B^n)$ is compact if and only if $\overline{\varphi(B^n)} \subset B^n$.

References

[CiMe] J. Cima and P. Mercer, *Composition operators between Bergman spaces on convex domains in* $\mathbf{C}^n$, J. Operator Theory, 33(2) (1995), 363-369.

[CM] C. Cowen and B. MaCluer, *Composition operators on spaces of analytic functions*, CRC Press, New York, 1995. Amer. J. Math., 107 (1985), 85–111

[KS1] H. Koo and W. Smith, *composition operators between Bergman spaces of functions of several variables*, Comtemp. Math., 393(2006), 123-132.

[KS2] H. Koo and W. Smith, *Composition operators induced by smooth self-maps of the unit ball in* C^N, to appear in J.Math. Anal. Appl.

[Me] P. Mercer, *Compact composition operators between weighted Bergman sapces on convex domains in* $\mathbf{C}^n$, Integr. Equ. Oper. Theory 31(1998) 482-488.

[MS] B. MacCluer and J.H. Shaprio, *Angular derivatives and compact composition operators on the Hardy and Bergman spaces*, Can. J. Math., XXXVIII(4)(1986), 878-906.

[MM] B. MacCluer and P. Mercer, *Composition operators between Hardy and Weighted Bergman spaces on convex domains in* $\mathbf{C}^n$, Proc. Amer. Math. Soc., 123(7) (1995), 2093-2102.

[Sh] J.H. Shaprio, *The essential norm of a composition operator*, Annals of Math. 125(1987), 375-404.

[SZ1] M. Stessin and K. Zhu, *composition operator on embedded disks*, preprint.

[SZ2] M. Stessin and K. Zhu, *composition operator induced by symbols defined on a polydisc*, preprint.

[W1] W. R. Wogen, *The smooth mappings which preserve the Hardy space* $H^2_{B_n}$, Operator Theory: Advances Appl. 35(1988), 249-267.

[W2] W. Wogen, *On geometric properties of smooth maps which preserve* $H^2(B_n)$, preprint.

[Z] K. Zhu, *Compact composition operators on Bergman spacers of the unit ball*, preprint.

Department of Mathematics, Korea University, Seoul 136–713, Korea

E-mail address: koohw@korea.ac.kr