Uniform non-$\ell_1^n$-ness of direct sums of Banach spaces

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Abstract. This is a résumé of some recent results on the uniform non-$\ell_1^n$-ness of direct sums of Banach spaces. In particular we present those for the $\ell_1$- and $\ell_\infty$-sums as well.

1. Introduction

Since it was introduced in [24], the $\psi$-direct sum of Banach spaces have attracted a good deal of attention ([5, 6, 7, 13, 14, 19, 20, 17, 16, etc.]; see also [22, 23]). The aim of this note is to present a sequence of recent results on the uniform non-$\ell_1^n$-ness of direct sums of Banach spaces. Our starting point is Theorem 1 below concerning the uniform non-squareness by the authors ([14]). To treat the uniform non-$\ell_1^n$-ness is much more complicated than expected. The results presented here is almost taken from the recent paper of the present authors [16].

Let $\Psi$ be the family of all convex (continuous) functions $\psi$ on $[0,1]$ satisfying

$$\psi(0) = \psi(1) = 1 \text{ and } \max\{1-t, t\} \leq \psi(t) \leq 1 \quad (0 \leq t \leq 1).$$

For any $\psi \in \Psi$ define

$$\|(z, w)\|_\psi = \begin{cases} 
(\|z\| + |w|)\psi \left( \frac{|w|}{\|z\|+|w|} \right) & \text{if } (z, w) \neq (0,0), \\
0 & \text{if } (z, w) = (0,0).
\end{cases}$$

(1)
Then $\|\cdot\| = \|\cdot\|_\psi$ is an absolute normalized norm on $\mathbb{C}^2$ (that is, $\|((z, w))\| = \|(z, |w|)\|$ and $\|(1, 0)\| = \|(0, 1)\| = 1$) and satisfies

$$\psi(t) = \|(1-t, t)\| \quad (0 \leq t \leq 1).$$ (3)

Conversely for any absolute normalized norm $\|\cdot\|$ on $\mathbb{C}^2$ define a convex function $\psi \in \Psi$ by (3). Then $\|\cdot\| = \|\cdot\|_\psi$.

The $\ell_p$-norms $\|\cdot\|_p$ are such examples and for all absolute normalized norms $\|\cdot\|$ on $\mathbb{C}^2$ we have

$$\|\cdot\|_\infty \leq \|\cdot\| \leq \|\cdot\|_1$$ (4)

([2]). By (3) the convex functions corresponding to the $\ell_p$-norms are given by

$$\psi_p(t) := \begin{cases} \{(1-t)^p + t^p\}^{1/p} & \text{if } 1 \leq p < \infty, \\ \max\{1-t, t\} & \text{if } p = \infty. \end{cases}$$ (5)

Let $X$ and $Y$ be Banach spaces and let $\psi \in \Psi$. The $\psi$-direct sum $X \oplus_{\psi} Y$ of $X$ and $Y$ is the direct sum $X \oplus Y$ equipped with the norm

$$\|(x, y)\|_\psi = \|(\|x\|, \|y\|)\|_\psi,$$ (6)

where the $\|\cdot, \cdot\|_\psi$ term in the right hand side is the absolute normalized norm on $\mathbb{C}^2$ corresponding to the convex function $\psi$ ([24, 13]; see [21] for several examples). This extends the notion of the $\ell_p$-sum $X \oplus_p Y$.

A Banach space $X$ is said to be uniformly non-$\ell_1^n$ (cf. [1, 18]) provided there exists $\epsilon (0 < \epsilon < 1)$ such that for any $x_1, \ldots, x_n \in S_X$, the unit sphere of $X$, there exists an $n$-tuple of signs $\theta = (\theta_j)$ for which

$$\left\| \sum_{j=1}^{n} \theta_j x_j \right\| \leq n(1-\epsilon).$$ (7)

We may take $x_1, \ldots, x_n$ from the unit ball $B_X$ of $X$ in the definition. In case of $n = 2$ $X$ is called uniformly non-square ([12]; cf. [1, 18]).

As is well known ([3, 11]), if $X$ is uniformly non-$\ell_1^n$, then $X$ is uniformly non-$\ell_1^{n+1}$ for every $n \in \mathbb{N}$.

2. Uniform non-$\ell_1^n$-ness of $X \oplus_{\psi} Y$, $\psi \neq \psi_1, \psi_\infty$

The following result by the authors [14] is our starting point.
Theorem 1 (Kato-Saito-Tamura [14]). Let $X$ and $Y$ be Banach spaces and $\psi \in \Psi$. Then the following are equivalent.

(i) $X \oplus_{\psi} Y$ is uniformly non-square.

(ii) $X$ and $Y$ are uniformly non-square and $\psi \neq \psi_1, \psi_\infty$.

To treat the uniform non-$\ell_1^n$-ness is much more complicated than expected. Indeed we need to prepare several lemmas, though we skip to mention them.

Theorem 2. Let $X$ and $Y$ be Banach spaces and let $\psi \in \Psi, \psi \neq \psi_1, \psi_\infty$. Then the following are equivalent.

(i) $X \oplus_{\psi} Y$ is uniformly non-$\ell_1^n$.

(ii) $X$ and $Y$ are uniformly non-$\ell_1^n$.

Theorem 2 does not answer the following question: Let $X$ and $Y$ be uniformly non-$\ell_1^n$. Is it possible for $X \oplus_{\psi} Y$ to be uniformly non-$\ell_1^n$ with $\psi = \psi_1$ or $\psi = \psi_\infty$? The next theorem will give an answer.

Theorem 3. Let $X$ and $Y$ be Banach spaces and let $\psi \in \Psi$. Assume that neither $X$ nor $Y$ is uniformly non-$\ell_1^{n-1}$. Then the following are equivalent.

(i) $X \oplus_{\psi} Y$ is uniformly non-$\ell_1^n$.

(ii) $X$ and $Y$ are uniformly non-$\ell_1^n$ and $\psi \neq \psi_1, \psi_\infty$.

Theorem 3 includes Theorem 1 as the case $n = 2$.

Remark 1. In Theorem 3 we can not remove the condition that neither $X$ nor $Y$ is uniformly non-$\ell_1^{n-1}$ ([16, Section 6]).

3. The $\ell_1$- and $\ell_\infty$-sums

Theorem 4. Let $X$ and $Y$ be Banach spaces. The following are equivalent.

(i) $X \oplus_1 Y$ is uniformly non-$\ell_1^n$.

(ii) There exist positive integers $n_1$ and $n_2$ with $n_1 + n_2 = n - 1$ such that $X$ is uniformly non-$\ell_1^{n_1 + 1}$ and $Y$ is uniformly non-$\ell_1^{n_2 + 1}$.

According to Theorem 1 the uniform non-squareness of $X$ and $Y$ is not inherited to the $\ell_1$-sum $X \oplus_1 Y$, whereas we have the following result as the case $n = 3$ of Theorem 4.
Theorem 5. Let $X$ and $Y$ be Banach spaces. Then the following are equivalent.

(i) $X \oplus_1 Y$ is uniformly non-$\ell_3^1$.

(ii) $X$ and $Y$ are uniformly non-square.

For the $\ell_\infty$-sum we obtain the following.

Theorem 6. Let $X_1, \ldots, X_m$ be uniformly non-square Banach spaces. Then $(X_1 \oplus \cdots \oplus X_m)_\infty$ is uniformly non-$\ell^n_1$ if and only if $m < 2^{n-1}$.

According to Theorem 5 the $\ell_1$-sum $X \oplus_1 Y$ is uniformly non-$\ell_3^1$ if and only if $X$ and $Y$ are uniformly non-square. On the other hand for the $\ell_\infty$-sum, by Theorem 6, if $X$ and $Y$ are uniformly non-square, then $X \oplus_\infty Y$ is uniformly non-$\ell_3^1$, whereas the converse is not true ([16, Remark 5.5]). Instead we obtain the following result which is interesting in contrast with the $\ell_1$-sum case.

Theorem 7. Let $X$, $Y$ and $Z$ be Banach spaces. Then the following are equivalent.

(i) $(X \oplus Y \oplus Z)_\infty$ is uniformly non-$\ell_3^1$.

(ii) $X$, $Y$ and $Z$ are uniformly non-square.

References


