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A GEOMETRIC INTERPRETATION OF EICHLER'S BASIS PROBLEM FOR HILBERT MODULAR FORMS

MARC-HUBERT NICOLE

1. Introduction

This report is essentially a transcription of the talk given at the RIMS in Kyōto. We thus refer the reader to [19] and subsequent papers ([9], [20] and [21]) for details and the complete proofs of the theorems mentioned in the text.
We describe in this work a geometric interpretation of Eichler's Basis Problem for Hilbert modular forms (cf. [6]) in terms of abelian varieties with real multiplication in characteristic $p > 0$. Recall that a g-dimensional abelian variety A is said to have real multiplication (or RM for short) if it is equipped with the action of the ring of integers $\mathcal{O}_L$ of a totally real field L of dimension $[L:\mathbb{Q}] = g$. We start with some sketchy, partly historical remarks with the motivational goal of providing the geometric picture in dimension one before addressing our generalization in dimension g.

Let H be the class number of $B_{p,\infty}$ i.e., the number of left ideal classes of a maximal order in the rational quaternion algebra $B_{p,\infty}$ ramified at p and ∞. Let $I_i, I_j, 1 \leq i,j \leq H$ be left ideal classes representatives. Using the norm of the quaternion algebra, we can define:

$$Q_{ij}(x) := \mathrm{Norm}(x)/\mathrm{Norm}(I_j^{-1}I_i), \text{ for } x \in I_j^{-1}I_i,$$

i.e., a quadratic form of level p, discriminant p^2, with values in $\mathbb{N}$. Since the quaternion algebra $B_{p,\infty}$ is definite (i.e., ramified at the infinite place), the representation numbers $a(n) := |\{x | Q_{ij}(x) = n\}|$ are finite. The theta series

$$\theta_{ij}(z) := \sum_{n\in\mathbf{N}} a(n)q^n, \quad \text{for } q = e^{2\pi iz},$$

is a modular form of weight 2 for $\Gamma_0(p) := \{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) | \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) = \left(\begin{smallmatrix} * & * \\ 0 & *\end{smallmatrix}\right) \mod p\}$ by the Poisson summation formula. In 1954, Eichler ([5]) showed that the $H(H-1)$ cusp forms

$$\theta_{ij}(z) - \theta_{1j}(z), \quad 2 \leq i \leq H, 1 \leq j \leq H,$$

span the vector space $S_2(\Gamma_0(p))$ of cusp forms of weight 2 for the group $\Gamma_0(p)$. Hecke had originally conjectured in 1940 ([11, p. 884-885]) that $H-1$ differences of theta series (say, obtained from fixing the index j in the above formulation) would form a basis of $S_2(\Gamma_0(p))$, maybe inspired by the similarity of the explicit formulae for the class number (of a maximal order) of $B_{p,\infty}$ and for the dimension of $S_2(\Gamma_0(p))$ (see below Remark 1.2). In spite of this striking coincidence, Hecke's conjecture holds only for $p \leq 31$, and $p = 41, 47, 59, 71$ (cf. [23, Rmk. 2.16]). For further historical remarks on the Basis Problem, we refer to [23] and the references therein.
We now introduce some geometric notions.

Definition 1.1. *An elliptic curve E over $\overline{\mathbb{F}}_p$ is* supersingular *if $E[p](\overline{\mathbb{F}}_p) = 0$.*

In 1941, Deuring ([3]) determined, for E a supersingular elliptic curve over $\overline{\mathbb{F}}_p$, that $\mathrm{End}_{\overline{\mathbb{F}}_p}(E)$ is a maximal order in the quaternion algebra $B_{p,\infty}$ over $\mathbb{Q}$. It has been pointed out to me by Prof. Ernst Kani that in [4], Deuring indeed discussed the connection with Hecke's conjecture, albeit supposing wrongly that the latter held. Using the idea of $\mathfrak{A}$-transform of Serre as described in [28], one can show that there is a bijection between left ideal classes $[I_1], \ldots, [I_H]$ of $\mathrm{End}_{\overline{\mathbb{F}}_p}(E)$ and isomorphism classes of supersingular elliptic curves $E_1, \ldots, E_H$ over $\overline{\mathbb{F}}_p$, given functorially by the tensor map

$$[I] \mapsto [E \otimes_{\mathrm{End}(E)} I].$$

Remark 1.2. *From a modern point of view, the most natural geometrical context where supersingular elliptic curves arise is in the special fiber at p of the elliptic curve $X_0(p)$, consisting of two projective lines intersecting at supersingular points. By flatness of the model of $X_0(p)$ over* $\mathrm{Spec}(\mathbb{Z})$*, the number $|S|$ of supersingular points on $X_0(p)_{\overline{\mathbb{F}}_p}$ and the genus g of the Riemann surface $X_0(p)_{\mathbb{C}}$ are related by the formula $|S| = g + 1$ and once we identify modular forms and differential forms on $X_0(p)_{\mathbb{C}}$, this explains the similitary of the formulas for the dimension of $S_2(\Gamma_0(p))$, the number of supersingular points and thus the class number.*

It is not too hard to check that the norm form of $\phi \in \mathrm{End}(E)$ coming from the quaternion algebra corresponds to the degree of ϕ as an endomorphism. This holds more generally for ideals $\mathrm{Hom}(E_i, E_j)$ (i.e., isogenies $\phi : E_i \longrightarrow E_j$), and thus the above bijection can be strenghtened to include the quadratic module structure.
We are now in position to give the geometric interpretation of Eichler's original Basis Problem.

Proposition 1.3. *The theta series coming from the modules*

$$\mathrm{Hom}(E_i, E_j) \cong I_j^{-1} I_i$$

equipped with the quadratic degree map span *the rational vector space* $S_2(\Gamma_0(p))$.

It is worth pointing out that in 1982, Ohta ([22]) gave an explicit connection between the geometry of $X_0(p)$ in characteristic p and the basis problem modulo p. Further development of the geometric perspective can be found in Gross ([10]). As for recent work on the Eichler Basis Problem from this point of view, we cite [7] that establishes the integral version of the basis problem using deep methods and ideas of Mazur and Ribet on modular curves.

The remainder of the paper deals with the generalization of the above geometric interpretation to Hilbert modular forms using *superspecial* points (to be defined shortly) on a Hilbert moduli space. This Hilbert moduli space is an algebraic stack parametrizing principally polarized abelian varieties with RM. Note that this moduli space is the natural generalization of $X_0(1)$, not $X_0(p)$. In particular, we use very little information about the global geometry of the space (except maybe when p is ramified).

Terminology

We explain the meaning of two concepts that are identical for elliptic curves, but decisively different for higher dimensional abelian varieties. Let k be an algebraically closed field of characteristic $p > 0$.

Definition 1.4. *A abelian variety A over k of dimension g is* superspecial *if and only if $A \cong E^g$, for E some supersingular elliptic curve.*

In dimension $g \geq 2$, there is a unique superspecial abelian variety by the following theorem.

Theorem 1.5. (Deligne [25]) *Let E_1, E_2, E_3, E_4 be supersingular elliptic curves over $k = \overline{k}$. Then $E_1 \times E_2 \cong E_3 \times E_4$.*

In dimension one, we obtain the same objects if we replace the condition $A \cong E^g$ by the condition that $A \sim E^g$ i.e., that A is merely isogenous to E^g. In higher dimensions, this is false e.g., the isogeny class of E^g contains infinitely many isomorphism classes.

Definition 1.6. *A abelian variety A over k of dimension g is* supersingular *if and only if $A \sim E^g$, for E some supersingular elliptic curve. Equivalently, all the slopes of its Newton polygon are $\frac{1}{2}$.*

The same definitions apply of course to abelian varieties with additional structures. In the RM case, the superspecial condition yields finitely many isomorphism classes (in contrast with the supersingular condition). Also, it is a fact that the number of polarizations of fixed degree (e.g., principal polarizations) on an abelian variety is finite. In particular, the superspecial locus on the Hilbert moduli space i.e., the set of points whose underlying abelian variety is superspecial, is finite. On the other hand, the supersingular locus is positive dimensional for $g > 1$.

2. Superspecial Orders in $B_{p,\infty} \otimes L$

To fix notation, we recall some basic material about quaternion algebras.

2.1. Quaternion algebras.

Let L be any field.

Definition 2.1. *A quaternion algebra B over L is a central, simple algebra of rank 4 over L.*

If the char $L \neq 2$, the quaternion algebra B is given by a couple (c, d), where $c, d \in L \backslash \{0\}$, as the L-algebra of basis $1, i, j, k$, where $i, j \in B, k = ij$, and

$$i^2 = c, \quad j^2 = d, \quad ij = -ji.$$

A quaternion algebra is equipped with a canonical involutive L-endomorphism $b \mapsto \overline{b}$ called conjugation. The (reduced) norm of B is defined as $n(b) := b\overline{b}$, $b \in B$. Any field L admits over itself the quaternion algebra $M_2(L)$. For local fields (different than $\mathbb{C}$), there is only one more:

Theorem 2.2. *Let $L \neq \mathbb{C}$ be a local field. Then there exists a unique quaternion division algebra over L, up to isomorphism.*

Theorem 2.3. *Let B be a quaternion algebra over a number field L. Let v be a place of L. We denote $B_v := B \otimes_L L_v$. A place v is ramified if B_v is a division algebra. If $B_v \cong M_2(L_v)$, we say the place v is split.*

Theorem 2.4. *Let L be a number field. The number $|\mathbf{Ram}(B)|$ of ramified places is even. For any even set S of places, there exists a unique quaternion algebra B/L up to isomorphism such that $\mathbf{Ram}(B) = S$.*

Example 2.5. *The quaternion algebra $B_{p,\infty}$ over $\mathbb{Q}$ is ramified only at p and ∞ i.e., $B_{p,\infty} \otimes \mathbb{Q}_\ell \cong M_2(\mathbb{Q}_\ell)$ for $\ell \neq p, \infty$.*

In general, we denote by $B_{\nu_1, \cdots, \nu_{2m}}$ the quaternion algebra ramified at the places $\nu_1, \cdots, \nu_{2m}$.

2.2. **Orders.** Having recalled the rational theory of quaternion algebras, we now describe a certain class of orders of $B_{p,\infty} \otimes L$ arising from superspecial abelian varieties with real multiplication by $\mathcal{O}_L$, where L is a totally real field.

Definition 2.6. *Let B be the quaternion algebra over $L_{\mathfrak{p}}$. Let $K = K_{\mathfrak{p}}$ be a quadratic extension of $L_{\mathfrak{p}}$ contained in B. Set*

$$R_v(K) = \mathcal{O}_K + P_B^{v-1},$$

for P_B the unique maximal ideal in $\mathcal{O}_B$ and $v = 1, 2, \ldots$.

Definition 2.7. *An order $\mathcal{O}$ is* superspecial *of level $\mathcal{P}$ dividing p, $\mathcal{P} = \prod_i \mathfrak{p}_i^{\alpha_i} \cdot \prod_j \mathfrak{q}_j^{\beta_j}$, for $\mathfrak{p} \in Ram(B_{p,\infty} \otimes L)$, $\mathfrak{q}_j \notin Ram(B_{p,\infty} \otimes L)$, if:*

- for $\alpha_i \geq 1$, there is an unramified quadratic extension $\mathcal{O}_K$ of $\mathcal{O}_{L_{\mathfrak{p}}}$ such that $\mathcal{O}_{\mathfrak{p}_i} = R_{\alpha_i}(K)$;
- for $\beta_j > 1$, if $f(\mathfrak{q}_j/p)$ is even, $\mathcal{O}_{q_j}$ contains a split quadratic extension; if $f(\mathfrak{q}_j/p)$ is odd, there is an unramified quadratic extension $\mathcal{O}_K$ such that $\mathcal{O}_{\mathfrak{q}_j} \cong \left\{ \begin{pmatrix} \alpha & \beta^\sigma \\ \pi_{\mathfrak{q}_j}^{\beta_j}\beta & \alpha^\sigma \end{pmatrix}, \alpha, \beta \in \mathcal{O}_K \right\}$, for σ the involution on K, $\pi_{\mathfrak{q}_j}$ a uniformizer in $\mathcal{O}_{L_{\mathfrak{q}_j}}$;
- for any other finite prime $\mathfrak{l}$, $\mathcal{O}_{\mathfrak{l}}$ contains a split extension (i.e., $\mathcal{O}_{L_{\mathfrak{l}}} \oplus \mathcal{O}_{L_{\mathfrak{l}}}$).

We will explain later on how superspecial orders arise as endomorphism orders $\mathrm{End}_{\mathcal{O}_L}(A)$ of superspecial abelian varieties A with RM.

Example 2.8. *Let p be unramified. Then a superspecial order of level p is an Eichler order i.e., the intersection of two maximal orders (not necessarily distinct). This follows from the facts that p is squarefree and being Eichler is a local property.*

Remark 2.9. *(For experts) In general, superspecial orders are Bass orders, but they are not special orders (cf.* [13], [14]*) e.g., the superspecial order of level $\mathfrak{p}^2$ for $g = 2$ is not special.*

3. The Basis Problem for Hilbert Modular Forms

We explain the derivation of a particular case of the Basis Problem for Hilbert modular forms from the Jacquet-Langlands correspondence i.e., we show that theta series coming from ideals of an Eichler order of level p in $B_{p,\infty} \otimes L$ span the space of Hilbert modular newforms of weight two for $\Gamma_0(p)$ (and trivial character).
The Jacquet-Langlands correspondence ([17, Thm. 16.1]) establishes, for any totally definite quaternion algebra B, a Hecke-equivariant injection $\pi \mapsto JL(\pi)$ from the set of classes of automorphic representations $\pi = \otimes_v \pi_v$ of $G_B(\mathbb{A}) = (B \otimes_L \mathbb{A})^\times$ with the set of classes of automorphic representations of $\mathrm{GL}_2(\mathbb{A})$. The image of the map is the set of cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ that are discrete series (i.e., special or supercuspidal at a finite place) at all ramified places of B. Imposing that the representation is of the discrete series at infinite places means that it is holomorphic of weight $k \geq 2$. The key fact that we use is that the representation $\pi_{\mathfrak{p}}$ corresponding to a newform at a prime $\mathfrak{p}$ whose exponent is odd in the level is necessarily in the discrete series, since the conductor at $\mathfrak{p}$ is not a square (see [8, Proof of Prop. 5.21, p. 95; Table 4.20, p. 73]). Recall that a prime $\mathfrak{p}$ dividing p is ramified in $B_{p,\infty} \otimes L$ if and only if $[L_{\mathfrak{p}} : \mathbb{Q}_p]$ is odd. It is necessary for this to happen that the exponent α of $\mathfrak{p}^\alpha$ occuring in the prime decomposition of p is odd. Thus for level exactly p, only odd exponents occur for ramified primes, thence

the local representation $\pi_\mathfrak{p}$ of any cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ of level p occurs in the discrete series at $\mathfrak{p}$ for any ramified place p of $B_{p,\infty} \otimes L$. In brief, in the case of level exactly equal to p, the Jacquet-Langlands correspondence implies that all cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ arise as quaternionic representations on the adelic group associated to the quaternion algebra $B_{p,\infty} \otimes L$.
Recall that for p unramified, superspecial orders of level p are Eichler orders. We derive from the above representation-theoretic argument that the corresponding space of Hilbert newforms of weight 2 and level (p) is spanned by theta series by translating in classical terms the fact that the Jacquet-Langlands correspondence is a theta correspondence (cf. also [12], [8]).

Theorem 3.1. *Let p unramified. Let $S_2(\Gamma_0(p),1)^{new}$ be the subspace of newforms of the vector space of Hilbert modular forms of weight two, level p. Then $S_2(\Gamma_0(p),1)^{new}$ is spanned by theta series coming from left ideals of an Eichler order of level p in the quaternion algebra $B_{p,\infty} \otimes L$.*

For more general orders, the Jacquet-Langlands correspondence imposes a non-trivial hypothesis on the exponents arising in the level.

Conjecture 3.2. *Let $p\mathcal{O}_L = \mathfrak{p}^g$. Let $0 \leq j \leq [g/2]$. If $[L:\mathbb{Q}]$ is odd, suppose that $g-j$ is odd. Then the theta series attached to the (locally principal) left ideals of a superspecial order of level $\mathfrak{p}^{g-j}$ span the vector space of Hilbert modular newforms of level $\mathfrak{p}^{g-j}$.*

Remark 3.3. *For $g = 2$ and level $\mathfrak{p}$, Conjecture 3.2 holds since the underlying order is also an Eichler order.*

4. Geometric interpretation

In this section, we explain the origin of the concept of a superspecial order (cf. Definition 2.7) and we give a geometric interpretation of the quadratic modules giving rise to theta series. Note that the result referred to in the title of this paper is proved under the hypothesis that the narrow class number $h^+(L) = 1$.

Theorem 4.1. *For any superspecial abelian variety A with RM by $\mathcal{O}_L$, the endomorphism order $\mathrm{End}_{\mathcal{O}_L}(A)$ is a superspecial order.*

Proof. (Sketch for p unramified)
Let A be an abelian variety defined over $\overline{\mathbb{F}}_p$. For a rational prime $\ell \neq p$, we let $T_\ell(A) = \varprojlim A[\ell^n]$ i.e., the Tate module at ℓ. At $\ell = p$, let $\mathbb{D}(A)$ be the Dieudonné module (cf. Section 5 for details). Then we have the presumably well-known RM version of Tate's theorem (where the finite field k is such that A_1, A_2 and all $\mathcal{O}_L$-homomorphisms are defined over it):

Theorem 4.2. *Let A_1, A_2 be two supersingular abelian varieties with RM by $\mathcal{O}_L$. Then for $\ell \neq p$,*

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{O}_L,k}(A_1,A_2)\otimes\mathbb{Z}_\ell &\cong \mathrm{Hom}_{\mathcal{O}_L\otimes\mathbb{Z}_\ell}(T_\ell(A_1),T_\ell(A_2))\\
&\cong M_2(\mathcal{O}_L\otimes\mathbb{Z}_\ell),\\
\mathrm{Hom}_{\mathcal{O}_L,k}(A_1,A_2)\otimes\mathbb{Z}_p &\cong \mathrm{Hom}_{\mathcal{O}_L\otimes W(k)[F,V]}(\mathbb{D}(A_2),\mathbb{D}(A_1)),
\end{aligned}$$

where the homomorphisms respect the $\mathcal{O}_L$-structures.

Since local deformation theory decomposes according to primes, p unramified, implies there is a unique isomorphism class of Dieudonné module $\mathbb{D}$ with RM by reduction to the inert case. We can thus pick any point that we like to compute the discriminant of the order e.g., the superspecial abelian variety $E \otimes_{\mathbb{Z}} \mathcal{O}_L$. Since

$$\mathrm{End}_{\mathcal{O}_L}(E \otimes_{\mathbb{Z}} \mathcal{O}_L) = \mathrm{End}(E) \otimes_{\mathbb{Z}} \mathcal{O}_L,$$

we find that it is $p\mathcal{O}_L$, since the discriminant of the order $\mathrm{End}(E)$ is p, since it is maximal in $B_{p,\infty}$.

□

Theorem 4.3. *Let $h^+(L) = 1$. Fix a (principally polarized) superspecial abelian variety A_0 with RM by $\mathcal{O}_L$, with Dieudonné module $\mathbb{D}(A_0)$. There is a bijection between principally polarized superspecial abelian varieties A with RM by $\mathcal{O}_L$ such that $\mathbb{D}(A) \cong \mathbb{D}(A_0)$ (as $\mathcal{O}_L \otimes W(k)$-modules) and locally principal left ideal classes of the order* $\mathrm{End}_{\mathcal{O}_L}(A_0)$.

This bijection essentially follows from the tensor construction matching to an ideal I the abelian variety $A_0 \otimes_{\mathrm{End}_{\mathcal{O}_L}(A_0)} I$. In particular, the modules $\mathrm{Hom}_{\mathcal{O}_L}(A_i, A_0)$, as i varies, run through all left ideal classes of $\mathrm{End}_{\mathcal{O}_L}(A_0)$.

Remark 4.4. (Class and type numbers) *For p unramified, since there is a unique superspecial Dieudonné module, the class number of a superspecial order of level p is the number of superspecial points on the Hilbert moduli space. Given that superspecial abelian varieties* with RM *are also defined over $\mathbb{F}_{p^2}$, the geometric interpretation of the type number can be studied in a way similar to* [16], *where principally polarized superspecial abelian varieties were considered (i.e., the Siegel case); see* [9].

Theorem 4.5. *Let $h^+(L) = 1$. Let $\mathcal{P} = p$ if p is unramified in $\mathcal{O}_L$ and $\mathcal{P} \in \{\mathfrak{p}^g, \dots, \mathfrak{p}^{g-[g/2]}\}$ if $p = \mathfrak{p}^g$ is totally ramified in $\mathcal{O}_L$. Then for any superspecial order $\mathcal{O}$ of level $\mathcal{P}$, there exists a superspecial abelian variety A with RM by $\mathcal{O}_L$ such that* $\mathrm{End}_{\mathcal{O}_L}(A) \cong \mathcal{O}$.

Proof. (Sketch) This essentially follows from the bijection between principally polarized superspecial abelian varieties with RM and projective, left ideal classes of a superspecial order in $B_{p,\infty} \otimes L$. Indeed, all superspecial orders of fixed level are locally isomorphic, and the set of right orders of a complete set of representatives of left, projective ideal classes of any superspecial order of level $\mathcal{P}$ represents all isomorphism classes of superspecial orders of level $\mathcal{P}$. □

Example 4.6. *Let $g = 2$. Let A be a superspecial abelian surface with RM by $\mathcal{O}_L$.*

p	$\mathrm{End}_{\mathcal{O}_L}(A)$	$B_{p,\infty} \otimes L$
inert: $p = \mathfrak{p}$	*Eichler of level* p	B_{∞_1,∞_2}
split: $p = \mathfrak{p} \cdot \bar{\mathfrak{p}}$	*maximal*	$B_{\mathfrak{p},\bar{\mathfrak{p}},\infty_1,\infty_2}$
ramified: $p = \mathfrak{p}^2$	• *Eichler of level* $\mathfrak{p}$ • *superspecial of level* $\mathfrak{p}^2$	B_{∞_1,∞_2}

So far, we provided a geometric interpretation of projective modules of superspecial orders as modules of $\mathcal{O}_L$-isogenies $\mathrm{Hom}_{\mathcal{O}_L}(A_i, A_j)$. We now explain how these latter modules can be also be given a quadratic module structure in a natural way by using the geometry of the abelian varieties. For (A_1, λ_1), (A_2, λ_2), two principally polarized superspecial abelian varieties and $\phi \in \mathrm{Hom}_{\mathcal{O}_L}(A_1, A_2)$, define

$$||\phi||_{\mathcal{O}_L} = ||\phi||_{\mathcal{O}_L} := \lambda_1^{-1} \circ \phi^t \circ \lambda_2 \circ \phi, \qquad \begin{tikzcd} A_2^t \arrow[d, "\phi^t"'] & A_2 \arrow[l, "\lambda_2"'] \\ A_1^t \arrow[r, "\lambda_1^{-1}"'] & A_1 \arrow[u, "\phi"'] \end{tikzcd} .$$

The application $|| - ||_{\mathcal{O}_L}$ is an $\mathcal{O}_L$-integral quadratic form:

$$|| - ||_{\mathcal{O}_L} : \mathrm{Hom}_{\mathcal{O}_L}(A_1, A_2) \longrightarrow \mathrm{End}_{\mathcal{O}_L}(A_1)^{R=1} = \mathcal{O}_L.$$

The only non-trivial fact that needs to be checked is that it indeed takes values in $\mathcal{O}_L$. This holds because the formula $\lambda_1^{-1} \circ \phi^t \circ \lambda_2 \circ \phi$ is stable under the Rosati involution, which is simply the canonical involution of the totally definite quaternion algebra $B_{p,\infty} \otimes L$.

The theta series

$$\Theta(\mathrm{Hom}_{\mathcal{O}_L}(A_1, A_2)) := \sum_{\mathcal{O}_L \ni \nu \gg 0 \text{ or } \nu = 0} a_\nu q^\nu,$$

where $a_\nu = |\{\phi \in \mathrm{Hom}_{\mathcal{O}_L}(A_1, A_2) \text{ such that } ||\phi||_{\mathcal{O}_L} = \nu\}|$, is the q-expansion of a Hilbert modular form of parallel weight 2 for the group $\Gamma_0((p)) \subset \mathrm{SL}_2(\mathcal{O}_L)$.

We can now state the geometric interpretation of Eichler's Basis Problem for Hilbert modular forms:

Theorem 4.7. *Let $h^+(L) = 1$, and p be unramified. Let $(A_i, \iota_i, \lambda_i)$ run through the superspecial points on the Hilbert moduli space. The theta series coming from the quadratic modules*

$$(\mathrm{Hom}_{\mathcal{O}_L}(A_i, A_j), || - ||_{\mathcal{O}_L})$$

span *the vector space $S_2^{new}(\Gamma_0((p)))$ of Hilbert modular newforms.*

Remark 4.8. *From the theory of newforms, it follows that if $S_2^{new}(\Gamma_0((p)))$ is spanned by theta series of level p, then $S_2(\Gamma_0((p)))$ is spanned by theta series of level dividing p and their translates. Since we used up all superspecial points on the Hilbert moduli space, a geometric origin for those extra theta series has to be found elsewhere. Indeed, we can cook up suitable (superspecial) points with bigger endomorphism orders (e.g., for p inert, $g = 2$, the quaternion algebra B_{∞_1,∞_2} is unramified at any finite prime: in particular, it is thus possible to construct modular forms of level 1 from abelian varieties in characteristic p), albeit these exotic abelian varieties cannot be found on any familiar moduli space; see* [9]. *In the other direction, adding prime-to-p level structure allows to increase correspondingly the level of the endomorphism order.*

5. Classification up to isomorphism of Dieudonné modules over totally ramified Witt vectors

Let A be a superspecial abelian variety with RM by $\mathcal{O}_L$ over a perfect field k. When p is ramified in $\mathcal{O}_L$, it is not true that the order $\mathrm{End}_{\mathcal{O}_L}(A)$ always has level p. This is related to the number of isomorphism classes of superspecial Dieudonné modules with RM being in general greater than one (in contrast with the unramified case, cf. the proof of Theorem 4.1). Dieudonné modules arise in geometry in a way relevant to us as the first crystalline cohomology group $H^1_{cris}(A/W(k))$ of an abelian variety $A_{/k}$. Since this construction is functorial, additional structure (such as real multiplication) carry over from A to the Dieudonné module. In this section, we thus sketch the classification up to isomorphism of Dieudonné modules over *totally ramified* Witt vectors (our proof in [19] follows Manin ([18]) mutatis mutandis).

Let k be algebraically closed, and let $\mathfrak{F}$ be a totally ramified extension of $\mathbb{Q}_p$. The Witt vectors $W(k)$ is a complete discrete valuation ring in characteristic zero with residue field k i.e., $W(k)/pW(k) \cong k$. Let K be the fraction field of $W(k)$. Denote by $K_{\mathfrak{F}} := K \cdot \mathfrak{F}$ the compositum of K and $\mathfrak{F}$, with ring of integers $W_{\mathfrak{F}}$. The main tools that appear in Manin's classification are two finiteness theorems and some algebro-geometric classifying spaces. The key idea behind the finiteness theorems is the concept of a *special* module (due to Remark 5.6, we refer the reader to [19, Def. 1.3.11] for the definition); a crucial fact is that every Dieudonné module has a unique maximal special submodule, of finite colength.

Definition 5.1. *A Dieudonné module $\mathbb{D}$ is a left $W_{\mathfrak{F}}[F,V]$-module free of finite rank over $W_{\mathfrak{F}}$ with the condition that $\mathbb{D}/F\mathbb{D}$ has finite length.*

Definition 5.2. *Two Dieudonné modules $\mathbb{D}_1, \mathbb{D}_2$ are isogenous if there is an injective homomorphism $\phi : \mathbb{D}_1 \hookrightarrow \mathbb{D}_2$ such that $\mathbb{D}_2/\phi(\mathbb{D}_1)$ has finite length over $W_{\mathfrak{F}}$. If $\mathbb{D}_1$ is isogenous to $\mathbb{D}_2$, we write: $\mathbb{D}_1 \sim \mathbb{D}_2$.*

Theorem 5.3. *(First Finiteness Theorem) Let $\mathbb{D}$ be a Dieudonné module. There exists only a finite number of non-isomorphic special modules isogenous to $\mathbb{D}$.*

Theorem 5.4. *(Second Finiteness Theorem) Let $\mathbb{D}$ be a Dieudonné module. The module $\mathbb{D}$ has a maximal special submodule $\mathbb{D}_0$. The length $[\mathbb{D} : \mathbb{D}_0]$ is bounded uniformly in the isogeny class of $\mathbb{D}$.*

Theorem 5.5. *(Classification Theorem) Let k be an algebraically closed field. A Dieudonné module $\mathbb{D}$ is determined by the following collection of invariants:*

- *the Newton polygon slopes of $\mathbb{D}$;*
- *the maximal special submodule $\mathbb{D}_0 \subset \mathbb{D}$ (parametrized by discrete invariants);*
- *a $\Gamma(\mathbb{D}_0, H)$-orbit of a point corresponding to $\mathbb{D}$ in a constructible algebraic set $A(\mathbb{D}_0, H)$, where H is a nonnegative integer that depends only on the slopes; $A(\mathbb{D}_0, H)$ and $\Gamma(\mathbb{D}_0, H)$ depend only on $\mathbb{D}_0$ and H, and $\Gamma(\mathbb{D}_0, H)$ is* a finite *group.*

Two Dieudonné modules are isomorphic if and only if all these invariants coincide.

Recall that a supersingular Dieudonné module is a Dieudonné module whose Newton polygon slopes are $\frac{1}{2}$.

Remark 5.6. *A supersingular Dieudonné module is superspecial if and only if it is special.*

Corollary 5.7. *The number of isomorphism classes of superspecial Dieudonné modules with RM by $\mathcal{O}_L$ of rank 2 over a totally ramified prime $p = \mathfrak{p}^g$ is:* $\left[\frac{g}{2}\right] + 1$.

Remark 5.8. *This explains why in Theorem 4.5, the levels only take values in the set $\{\mathfrak{p}^g, \ldots, \mathfrak{p}^{g-[g/2]}\}$ when p is totally ramified.*

5.1. Application of Manin's theory to a truncation conjecture of Traverso. The first application deals only with the case $\mathfrak{F} = \mathbb{Q}_p$ i.e., Manin's original version.

Conjecture 5.9. ([27, Conj. 4]) *Let k be an algebraically closed field of characteristic p, and let $g \in \mathbb{N}$. Suppose that $\mathbb{D}$ is a Dieudonné module of height $2g$. Then $\mathbb{D}$ is uniquely determined up to isomorphism by $\mathbb{D}/p^g\mathbb{D}$ i.e., its truncation modulo p^g.*

Theorem 5.10. *Traverso's conjecture holds for supersingular Dieudonné modules.*

See [21] for a proof of this result and a principally polarized version (with the same bound).

5.2. Application to Hilbert moduli spaces over totally ramified primes. In this section, we explain that the stratification of the Hilbert moduli space over a totally ramified prime $p = \mathfrak{p}^g$ introduced by Andreatta-Goren in [1] coincides with the stratification suggested by the decomposition of the moduli spaces à la Manin, at least on the supersingular stratum.
We recall briefly the definition of the stratification of [1]. Let p be a totally ramified prime. Let A/k be a polarized abelian variety with RM, defined over a field k of characteristic p. Fix an isomorphism $\mathcal{O}_L \otimes_{\mathbb{Z}} k \cong k[T]/(T^g)$. One knows that $H^1_{dR}(A)$ is a free $k[T]/(T^g)$-module of rank 2, and there are two generators α and β such that:

$$H^1(A, \mathcal{O}_A) = (T^i)\alpha + (T^j)\beta, i \geq j, i + j = g.$$

The index $j = j(A)$ is called the singularity index. For perspective, recall the short exact sequence:

$$0 \longrightarrow H^0(A, \Omega^1_A) \longrightarrow H^1_{dR}(A) \longrightarrow H^1(A, \mathcal{O}_A) \longrightarrow 0.$$

These modules are Dieudonné modules of group schemes, and we rewrite this exact sequence as:

$$0 \longrightarrow (k, \mathrm{Fr}^{-1}) \otimes_k \mathbb{D}(\mathrm{Ker}(\mathrm{Fr})) \longrightarrow \mathbb{D}(A[p]) \longrightarrow \mathbb{D}(\mathrm{Ker}(\mathrm{Ver})) \longrightarrow 0.$$

The slope $n = n(A)$ is defined by the relation $j(A) + n(A) = a(A)$, where $a(A)$ is the a-number of the abelian variety. The subsets $\mathfrak{W}_{(j,n)}$ parameterizing abelian varieties with singularity index j and slope n are quasi-affine, locally closed and form a stratification ([1, Thm. 10.1], [2, §6.1]). Note that for any Dieudonné module $\mathbb{D}$ with RM of rank 2, we can define abstractly $j(\mathbb{D})$ and $n(\mathbb{D})$ without any reference to abelian varieties e.g., $j(\mathbb{D}) = j$ is the integer such that

$$T^i\alpha + T^j\beta = \mathrm{Ker}(V : \mathbb{D}/p\mathbb{D} \longrightarrow \mathbb{D}/p\mathbb{D}), i \geq j,$$

for α, β some generators of $\mathbb{D}$. The slope is $n(\mathbb{D}) := a(\mathbb{D}) - j(\mathbb{D})$.

Consider the supersingular Newton stratum. It decomposes in $\Big(([g/2] + 1) \cdot ([g/2] + 2)/2\Big)$ strata $\{\mathfrak{W}_{(j,n)}\}_{j,n}$ indexed by the type (j, n), for $n/g \geq 1/2$. For a fixed superspecial module $\mathbb{D}_c$, denote by $\mathfrak{M}^d_c$ the component classifying modules of index $(0, d)$ over the special module $\mathbb{D}_c$.

Conjecture 5.11. *Define $\mathfrak{N}^d_c$ as the strata on the Hilbert moduli space such that for $\underline{A} \in \mathfrak{N}^d_c$, the Dieudonné module $\mathbb{D}(\underline{A})$ of $\underline{A}$ belongs to $\mathfrak{M}^d_c$. Then the stratification induced by the components $\mathfrak{M}^d_c$ coincide with the slope stratification $\{\mathfrak{W}_{(j,n)}\}_{j,n}$ i.e.,*

$$\mathfrak{N}^d_c = \mathfrak{W}_{(c-d, g-c)}.$$

Theorem 5.12. ([19, Section 1.6]) *Let $\mathbb{D}$ be a Dieudonné module, with $\mathbb{D}_c$ its maximal (super)special submodule.*

- *The slope $n(\mathbb{D})$ of $\mathbb{D}$ depends only on the maximal special submodule $\mathbb{D}_c$.*
- *The a-number of $\mathbb{D}$ depends only on the index $d(\mathbb{D}_c, \mathbb{D})$ of $\mathbb{D}$ over its maximal special module $\mathbb{D}_c$:*

$$a(\mathbb{D}) = a(\mathbb{D}_c) - d(\mathbb{D}_c, \mathbb{D}).$$

Remark 5.13. *In view of Remark 5.6, Theorem 5.12 is equivalent to the statement of Conjecture 5.11 for the* supersingular *stratum.*

6. Open question

Representation-theoretic statements may in propitious circumstances be given a geometric version in terms of Shimura varieties e.g., the Ribet Exact Sequence ([24, Thm. 4.1]) can be viewed as a geometric Jacquet-Langlands correspondence, comparing Hecke modules supported on the supersingular loci on one hand of a Shimura curve (associated to the quaternion algebra B_{pq} ramified at p and q only) and on the other hand of the modular curve $X_0(pq)$ (associated to GL_2). Work in progress of the author concerns the generalization of that result of Ribet to higher-dimensional (quaternionic) Shimura varieties.
Besides the Jacquet-Langlands correspondence, the most compelling theme to us is base change. In its simplest terms, it boils down to the following question:

Question 6.1. *Is there a natural geometric construction of the Doi-Naganuma lift in terms of Hilbert modular surfaces* in characteristic *p?*

In particular, the construction sketched in this paper provides only modular forms with trivial quadratic character. Note that in characteristic zero, this question has been studied by Hirzebruch and Zagier ([15]).

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References

[1] Andreatta, F., Goren, E.Z., Geometry of Hilbert Modular Varieties over Totally Ramified Primes, *Int. Math. Res. Not.*, 2003, no. **33**, 1786–1835.
[2] Andreatta, F., Goren, E.Z., Hilbert modular varieties of low dimension. *Geometric aspects of Dwork theory*, Vol. **I**, 113–175, Walter de Gruyter, Berlin, 2004.
[3] Deuring, M., Die Typen der Multiplikatorenringe elliptischer Funktionenkörper, *Abh. Math. Sem. Hansischen Univ.* **14**, (1941). 197–272.
[4] Deuring, M., Die Anzahl der Typen von Maximalordnungen einer definiten Quaternionenalgebra mit primer Grundzahl, *Jber. Deutsch. Math. Verein.*, **54**, (1950). 24–41.
[5] Eichler, M., Über die Darstellbarkeit von Modulformen durch Thetareihen, *J. Reine Angew. Math.*, **195**, (1955), 156–171.
[6] Eichler, M., The basis problem for modular forms and the traces of the Hecke operators, *Modular functions of one variable, I*, (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), pp. 75–151, Lecture Notes in Math., Vol. **320**, Springer, Berlin, 1973.
[7] Emerton, M., Supersingular elliptic curves, theta series and weight two modular forms, *J. Amer. Math. Soc.*, **15**, (2002), no. 3, 671–714.
[8] Gelbart, S., *Automorphic forms on adele groups,* Annals of Mathematics Studies, No. **83**, Princeton University Press, Princeton, N.J., University of Tokyo Press, Tokyo, 1975. x+267 pp.

[9] Goren, E.Z., Nicole, M.-H., Superspecial abelian varieties, quaternion algebras and Hilbert modular forms, in preparation.

[10] Gross, B., Heights and the special values of L-series, Number theory (Montréal, Qué., 1985), 115–187, *CMS Conf. Proc.*, **7**, Amer. Math. Soc., Providence, RI, 1987.

[11] Hecke, E., *Mathematische Werke,* Herausgegeben im Auftrage der Akademie der Wissenschaften zu Göttingen Vandenhoeck und Ruprecht, Göttingen, 1959, 955 pp.

[12] Hida, H., The integral basis problem of Eichler, *Int. Math. Res. Not.*, 2005, no. **34**, 2101–2122.

[13] Hijikata, H.; Pizer, A.; Shemanske, T., Orders in quaternion algebras, *J. Reine Angew. Math.*, **394**, (1989), 59–106.

[14] Hijikata, H.; Pizer, A.; Shemanske, T., *The basis problem for modular forms on* $\Gamma_0(N)$, Mem. Amer. Math. Soc. **82**, (1989), no. 418, vi+159 pp.

[15] Hirzebruch, F.; Zagier, D., Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus, *Invent. Math.*, **36**, (1976), 57–113.

[16] Ibukiyama, T.; Katsura, T., On the field of definition of superspecial polarized abelian varieties and type numbers, *Compositio Math.* **91**, (1994), no. 1, 37–46.

[17] Jacquet, H.; Langlands, R. P., *Automorphic forms on* GL(2), Lecture Notes in Mathematics, Vol. **114**. Springer-Verlag, Berlin-New York, 1970. vii+548 pp.

[18] Manin, Y., Theory of commutative formal groups over fields of finite characteristic, *Uspehi Mat. Nauk,* **18**, 1963, no. 6, (114), 3–90.

[19] Nicole, M.-H., *Superspecial abelian varieties, theta series and the Jacquet-Langlands correspondence*, Ph.D. thesis, McGill University, 2005.

[20] Nicole, M.-H., The Eichler Basis Problem and Dieudonné Modules over Totally Ramified Witt Vectors, in preparation.

[21] Nicole, M.-H., Vasiu, A., Minimal truncations of supersingular p-divisible groups, submitted.

[22] Ohta, M., On theta series mod p, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, **28**, (1981), no. 3, 679–686 (1982).

[23] Pizer, A., An algorithm for computing modular forms on $\Gamma_0(N)$, *J. Algebra,* **64**, (1980), no. 2, 340–390.

[24] Ribet, K. A., On modular representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ arising from modular forms, *Invent. Math.*, **100**, (1990), no. 2, 431–476.

[25] Shioda, T., Supersingular $K3$ surfaces, *Algebraic geometry,* (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978), pp. 564–591, Lecture Notes in Math., **732**, Springer, Berlin, 1979.

[26] Tate, J., Endomorphisms of abelian varieties over finite fields, *Invent. Math.* **2**, 1966, 134–144.

[27] Traverso, C., Specializations of Barsotti–Tate groups, *Symposia Mathematica* **XXIV** (Sympos., INDAM, Rome, 1979), pp. 1–21, Acad. Press, London-New York, 1981.

[28] Waterhouse, W., Abelian varieties over finite fields, *Ann. Sci. Ecole Norm. Sup.* (4), **2**, 1969, 521–560.

Marc-Hubert Nicole, Email: nicole@ms.u-tokyo.ac.jp
Address: University of Tokyo, Department of Mathematical Sciences,
Komaba, 153-8914, Tokyo, Japan.