Geometry of polysymbols

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Introduction

We discuss a multiple generalization of the classical tame symbol on a Riemann surface. Let us start to explain a motivation of our work which is coming from the analogies between knots and primes.

\[
\begin{align*}
\text{knot } K & : S^1 = K(\mathbb{Z}, 1) \hookrightarrow \mathbb{R}^3 & \iff \text{prime } \text{Spec}(\mathbb{F}_p) = K(\hat{\mathbb{Z}}, 1) \hookrightarrow \text{Spec}(\mathbb{Z}) \\
\text{double cover } Y & \rightarrow X = \mathbb{R}^3 \setminus K & \iff \text{Frobenius}
\end{align*}
\]

\[
\text{double } \acute{\text{e}tale cover } Y \rightarrow X = \text{Spec}(\mathbb{Z}[\frac{1}{p}])
\]

For a 2-component link \( K \cup L \), monodromy around \( L \) in \( \text{Gal}(Y/X) \) \iff \text{Frobenius over } q \text{ in } \text{Gal}(Y/X) = \text{Legendre symbol } \left( \frac{p}{q} \right)

where we set \( p^* := (-1)^{\frac{l-1}{2}} p \).

Now, let us consider the polynomial ring \( \mathbb{F}_l[X] \) (\( l \) being an odd prime) in place of \( \mathbb{Z} \). For monic irreducible polynomials \( f, g \in \mathbb{F}_l[X] \text{ s.t. } (f, g) = 1 \), we have:

\[
\left( \frac{f}{g} \right) = 1 \iff \exists h \in \mathbb{F}_l[X] \text{ s.t. } h^2 \equiv f \mod g \\
\iff f \mod g \in (\mathbb{F}_l[X]/(g))^2 \\
\iff \prod_{g(\beta) = 0} f(\beta) \in (\mathbb{F}_l^x)^2.
\]

where \( R(f, g) := \prod_{g(\beta) = 0} f(\beta) \prod_{f(\alpha) = g(\beta) = 0} (\beta - \alpha) \) is the resultant of \( f \) and \( g \). Hence we have

\[
\left( \frac{f}{g} \right) = R(f, g)^{\frac{l-1}{2}}.
\]
Note that the resultant $R(f, g)$ can be defined for any ground field $k$ and $f, g \in k[X]$. Summing up, we have the following analogies:

\[
\begin{array}{c}
\text{linking number} \\
\text{Legendre symbol} \\
\text{resultant}
\end{array}
\]

The symbol in the title means the tame symbol over $\mathbb{C}$ defined by

\[
\{f, g\}_x := (-1)^{\text{ord}_x(f)\text{ord}_x(g)} \frac{f^\text{ord}_x(g)}{g^\text{ord}_x(f)}
\]

for $f, g \in \mathbb{C}(X)$ and $x \in \mathbb{C} \cup \{\infty\}$. The connection with the resultant is given by

\[
R(f, g) = \prod_{g(\beta)=0} \{f, g\}_\beta
\]

for $f, g \in \mathbb{C}[X]$ and hence the tame symbol plays a local and more basic role like the Hilbert symbol.

Now, it is known in knot theory that there is a higher order generalization of the linking number for a link. For example, the Borromean ring $K_1 \cup K_2 \cup K_3$ has the linking numbers $\text{lk}(K_i, K_j) = 0$ for all $i \neq j$ and the triple linking number $\text{lk}(K_1, K_2, K_3) = \pm 1$:

There are known two ways to construct the higher linking numbers; 1) Massey's higher order cup products ([Ma]) 2) Milnor's invariants defined as unipotent monodromies ([Mi]). For $\mathbb{Z}$, both constructions are known ([Mo]). The following question was asked by M. Kapranov:

**Question.** Is there Massey-Milnor type construction for a multiple $\{f_1, \ldots, f_n\}_x$ (or $R(f_1, \ldots, f_n)$) under a certain condition?
Our result is to give the Massey type construction of a line bundle with holomorphic flat connection \(\langle f_1, \ldots, f_n \rangle\), called a polysymbol, on a Riemann surface so that \(\{f_1, \ldots, f_n\}_x\) is obtained as its holonomy along a loop encircling \(x\), and also to give a global geometric construction of \(\langle f_1, \ldots, f_n \rangle\).

In the following, we shall discuss only triple symbols, i.e., the case \(n = 3\), though we have similar results for any \(n\) ([MT]). Let us fix the notations:

- \(\overline{X} := \text{a closed Riemann surface,}\)
- \(f_1, f_2, f_3 \in \mathbb{C}(\overline{X}),\)
- \(X := \overline{X} \setminus \bigcup_{i=1}^{3}\text{Supp}(f_i).\)

1. **Massey product construction**

For \(p \in \mathbb{N}\), the Deligne complex is defined by

\[
\mathbb{Z}(p)_D := ((2\pi \sqrt{-1})^p \mathbb{Z} \rightarrow \mathcal{O}_X \rightarrow^{d} \Omega^1_X \rightarrow^{d} \cdots \rightarrow^{d} \Omega^{p-1}_X)
\]

which is quasi-isomorphic to

\[
(\mathcal{O}^\times_X \rightarrow^{\log} \Omega^1_X \rightarrow^{d} \cdots \rightarrow^{d} \Omega^{p-1}_X)[-1].
\]

The Deligne cohomology is then defined by the hypercohomology groups

\[
H^q(X, \mathbb{Z}(p)_D) = H^{q-1}(X, \mathcal{O}^\times_X \rightarrow^{\log} \Omega^1_X \rightarrow^{d} \cdots \rightarrow^{d} \Omega^{p-1}_X), \quad q \geq 1.
\]

We compute Deligne cohomology groups in terms of Čech cohomology and so we take an open cover \(U = \bigcup_a U_a\) of \(X\). (For the properties of the Deligne cohomology, we refer to [Br], [EV]).

(1) For \(p = q = 1\), we have

\[
H^0(X, \mathcal{O}^\times_X) \simeq H^1(X, \mathbb{Z}(1)_D)
\]

\[
f \leftrightarrow [(2\pi \sqrt{-1}n_{ab}, (\log f)_a)]
\]

where \((\log f)_a\) means a branch of \(\log f\) on \(U_a\) with \((\log f)_b - (\log f)_a = 2\pi \sqrt{-1}n_{ab}\) on \(U_{ab} = U_a \cap U_b\).

Let \(\text{Pic}^\nabla(X)\) denote the group of isomorphism classes of line bundles with holomorphic (flat) connection on \(X\).
(2) For \( p = q = 2 \), we have

\[
\mathbb{H}^2(X, \mathbb{Z}(2)_D) \cong \mathbb{H}^1(X, \mathcal{O}_X^\times \log \Omega_X^1) \cong \text{Pic}^\nabla(X) \\
[(2\pi i)^2 n_{abc}, (\log f)_{ab}, \Omega_a)] \leftrightarrow [(\exp \frac{1}{2\pi i} (\log f)_{ab}, \frac{1}{2\pi i} \Omega_a)] \\
[(\xi_{ab}, \omega_a)] \leftrightarrow [(L, \nabla)]
\]

where \( \xi_{ab} \) gives the transition function of a \( \mathbb{C} \)-line bundle \( L \) on \( U_{ab} \) and local 1-forms \( \omega_a \)'s define a connection \( \nabla \) on \( L \).

The Deligne complexes have the product \( \mathbb{Z}(p)_D \times \mathbb{Z}(q)_D \to \mathbb{Z}(p+q)_D \) which induces the cup product

\[
\mathbb{H}^q(X, \mathbb{Z}(p)_D) \times \mathbb{H}^q(X, \mathbb{Z}(q)_D) \to \mathbb{H}^{q+q'}(X, \mathbb{Z}(p+q')_D).
\]

By (1) and (2), \( f_1 \cup f_2 \) determines the isomorphism class of line bundles with holomorphic connection on \( X \), which we denote by \( \langle f_1, f_2 \rangle \). Deligne ([D]) interpreted the tame symbol \( \{f_1, f_2\}_x \) as the holonomy of \( \langle f_1, f_2 \rangle \) along a loop \( l \) based at \( x_0 \) and encircling \( x \):

\[
\{f_1, f_2\}_x = \exp \frac{1}{2\pi i} \left( \int_l \log f_1 \frac{df_2}{f_2} - \log f_2(x_0) \int_l \frac{df_1}{f_1} \right).
\]

Now, assume that \( f_1 \cup f_2 = f_2 \cup f_3 = 0 \) in the following so that

\[
\alpha_1 \cup \alpha_2 = \partial \alpha_{12}, \quad \alpha_2 \cup \alpha_3 = \partial \alpha_{23}, \quad \exists \alpha_{ij} \in C^1(U, \mathbb{Z}(2)_D) \\
\Leftrightarrow (\log f_1)_a \frac{df_2}{f_2} = \frac{df_{12}}{f_{12}}, \quad (\log f_2)_a \frac{df_3}{f_3} = \frac{df_{23}}{f_{23}}, \quad \exists f_{ij} \in H^0(X, \mathcal{O}_X^\times) \\
+ \text{some equations}
\]

We fix branches \( (\log f_{ij})_a \)'s on \( U_a \)'s so that we have a unique cohomology class of the Massey product \( [\alpha_1 \cup \alpha_{23} + \alpha_{12} \cup \alpha_3] \in \mathbb{H}^2(X, \mathbb{Z}(3)_D) \). We then define \( \langle f_1, f_2, f_3 \rangle \) as the corresponding isomorphism class of line bundles with holomorphic connection under the isomorphisms:

\[
\mathbb{H}^2(X, \mathbb{Z}(3)_D) \simeq \mathbb{H}^2(X, \mathbb{Z}(2)_D) \simeq \text{Pic}^\nabla(X).
\]

Let \( x_0 \in X \) be a base point and put \( a_i := \frac{1}{2\pi i} \log f_i(x_0) \), \( a_{ij} := \frac{1}{(2\pi i)^2} \log f_{ij}(x_0) \) and \( \omega_i := \frac{1}{(2\pi i)^2} \frac{df_i}{f_i} \).

**Theorem 1.** For \( [l] \in \pi_1(X, x_0) \), the holonomy of \( \langle f_1, f_2, f_3 \rangle \) along \( l \) is

\[
\exp(2\pi i m_{123}(l)),
\]
where \( m_{123}(l) \) is given by

\[
m_{123}(l) = \int l \omega_1 \omega_2 \omega_3 + a_1 \int \omega_2 \omega_3 + a_{12} \int \omega_3 - \int \omega_1 a_{23} - a_1 \int \omega_2 a_3 + \int \omega_1 a_{2} a_3.
\]

Here \( \int \omega_{i_1} \cdots \omega_{i_k} \) denotes the iterated integral ([C])

\[
\int_{0 \leq t_1 < \cdots < t_k \leq 1} F_1(t_1) \cdots F_k(t_k) dt_1 \cdots dt_k, \quad l^*(\omega_{i_j}) = F_j(t_j) dt_j.
\]

For \( x \in \overline{X} \), we define \( \{f_1, f_2, f_3\}_x \) by the holonomy of \( \langle f_1, f_2, f_3 \rangle \) along a loop encircling \( x \).

**Theorem 2.** (Reciprocity) \( \prod_{x \in \overline{X}} \{f_1, f_2, f_3\}_x = 1. \)

2. Global geometric construction

We give a global geometric construction of \( \langle f_1, f_2, f_3 \rangle \), which generalizes those by S. Bloch ([Bl]) and R. Hain ([H]) for \( \langle f_1, f_2 \rangle \).

We set:

\[
N_4(R) := \{ \begin{pmatrix} 1 & x_1 & x_{12} & x_{123} \\ 0 & 1 & x_2 & x_{23} \\ 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} | x_* \in R \} \quad (R : \text{a ring}),
\]

\[
C_4 := \{ \begin{pmatrix} 1 & 0 & 0 & z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} | z \in \mathbb{C} \},
\]

\[
P := N_4(\mathbb{Z}) \setminus N_4(\mathbb{C}), \quad B := N_4(\mathbb{Z}) \setminus N_4(\mathbb{C})/C_4.
\]

The projection \( P \to B \) is a principal \( \mathbb{C}^\times \)-bundle. The 1-form

\[
\theta := dx_{123} - x_{12}dx_3 - x_{1}d_{23} + x_1x_2dx_3 = (1,4)-\text{entry of } x^{-1}dx
\]
is left $N_4(\mathbb{Z})$-invariant, right $\mathbb{C}^\times$-invariant and a Maurer-Cartan form along fibers so that it boils down to a connection form on $P$. Fixing a base point $x_0 \in X$ and $N_4(\mathbb{Z})AC_4 \in \mathcal{B}$ where

$$A := \begin{pmatrix} 1 & a_1 & a_{12} & a_{123} \\ 0 & 1 & a_2 & a_{23} \\ 0 & 0 & 1 & a_3 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

we define a holomorphic map

$$T(f_1, f_2, f_3) : X \to B$$

by

$$T(f_1, f_2, f_3) := N_4(\mathbb{Z})A \begin{pmatrix} 1 & \int_{\gamma_x} \omega_1 & \int_{\gamma_x} \omega_1 \omega_2 & \int_{\gamma_x} \omega_1 \omega_2 \omega_3 \\ 0 & 1 & \int_{\gamma_x} \omega_2 & \int_{\gamma_x} \omega_2 \omega_3 \\ 0 & 0 & 1 & \int_{\gamma_x} \omega_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} C_4$$

where $\gamma_x$ is a path from $x_0$ to $x$. The map $T(f_1, f_2, f_3)$ is shown to be independent of the choice of $\gamma_x$. By computing the holonomy of $T(f_1, f_2, f_3)^*(P, \theta)$, we have

**Theorem 3.** $\langle f_1, f_2, f_3 \rangle = \text{the isomorphism class of } T(f_1, f_2, f_3)^*(P, \theta)$.

**Remark.** We expect that $\langle f_1, f_2, f_3 \rangle$ would be an obstruction to a variation of the mixed Hodge structure $V$ on $X$ so that the weight filtration $V \supset W_3 \supset W_2 \supset W_1 \supset 0$ satisfies the properties:

(i) $V/W_3 = \mathbb{Z}, W_3/W_2 = \mathbb{Z}(1), W_2/W_1 = \mathbb{Z}(2), W_1 = \mathbb{Z}(3),$

(ii) $V/W_1$ and $W_3$ are classified respectively by

$$T(f_1, f_2) : X \to N_3(\mathbb{Z})\backslash N_3(\mathbb{C})/C_3$$

$$T(f_2, f_3) : X \to N_3(\mathbb{Z})\backslash N_3(\mathbb{C})/C_3,$$

where $T(f_i, f_{i+1})$ $(i = 1, 2)$ is defined in a similar manner to $T(f_1, f_2, f_3)$. 
References


