数理解析研究所講究録
1523 巻 2006 年 63-69



# ON THE RESTRICTION OF AUTOMORPHIC FORMS ON AN ORTHOGONAL GROUP TO A SMALLER ORTHOGONAL GROUP AND THE GROSS-PRASAD CONJECTURE

ATSUSHI ICHINO AND TAMOTSU IKEDA

In this note, we announce a conjecture which expresses the restriction of automorphic forms on $\mathrm{SO}(n+1)$ to $\mathrm{SO}(n)$ in terms of special values of automorphic $L$-functions. Details will appear elsewhere.

## 1. THE GROSS-PRASAD CONJECTURE

Our conjecture can be regarded as a refinement of the global Gross-Prasad conjecture [4].

Let $k$ be a global field of characteristic not two and $\mathbb{A}$ the ring of adeles of $k$. Let $V_0$ and $V_1$ be quadratic spaces over $k$ of dimension $n$ and $n+1$, respectively. We assume that there exists an embedding $\iota : V_0 \to V_1$ over $k$. Let $G_i$ denote the special orthogonal group of $V_i$ for $i=0,1$. Then $\iota$ induces an embedding $\iota : G_0 \to G_1$ over $k$. Let $\pi_i \simeq \otimes_v \pi_{i,v}$ be an irreducible tempered cuspidal automorphic representation of $G_i(\mathbb{A})$ for $i=0,1$. We are interested in the integral

$$\langle \varphi_1|_{G_0}, \varphi_0 \rangle = \int_{G_0(k)\backslash G_0(\mathbb{A})} \varphi_1(\iota(g_0))\overline{\varphi_0(g_0)}\, dg_0$$

and consider an element $\ell$ in

$$\mathrm{Hom}_{G_0(\mathbb{A})}(\pi_1 \otimes \bar{\pi}_0, \mathbb{C})$$

defined by

$$\ell(\varphi_1 \otimes \varphi_0) = \langle \varphi_1|_{G_0}, \varphi_0 \rangle$$

for $\varphi_i \in \pi_i$. Here $dg_0$ is the Tamagawa measure on $G_0(\mathbb{A})$ and $\bar{\pi}_0$ is the complex conjugate of $\pi_0$. If $\ell \neq 0$, then it is obvious that

$$\mathrm{Hom}_{G_0(k_v)}(\pi_{1,v} \otimes \bar{\pi}_{0,v}, \mathbb{C}) \neq 0 \tag{1.1}$$

for all places $v$ of $k$. Conversely, assume that (1.1) holds for all $v$. The global Gross-Prasad conjecture [4] asserts that $\ell \neq 0$ if and only if

$$L(\tfrac{1}{2}, \pi_1 \boxtimes \pi_0) \neq 0.$$

Here $L(s, \pi_1 \boxtimes \pi_0)$ is the automorphic $L$-function associated to $\mathrm{st}_1 \boxtimes \mathrm{st}_0$, where $\mathrm{st}_i$ is the standard representation of the $L$-group ${}^L G_i$ of $G_i$ for $i =$

0, 1. Note that the existence of the local factor $L_v(s, \pi_{1,v} \boxtimes \pi_{0,v})$ and the meromorphic continuation of $L(s, \pi_1 \boxtimes \pi_0)$ have not been proved in general.

It is believed that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G_0(k_v)}(\pi_{1,v} \otimes \bar{\pi}_{0,v}, \mathbb{C}) \leq 1$$

for all places $v$ of $k$. Hence if $\ell \neq 0$, then there should be a constant $C$ such that

$$\ell = C \cdot \prod_v \ell_v,$$

where $\ell_v$ is a non-zero element in $\mathrm{Hom}_{G_0(k_v)}(\pi_{1,v} \otimes \bar{\pi}_{0,v}, \mathbb{C})$. One can guess this constant $C$ is related to the special value $L(\frac{1}{2}, \pi_1 \boxtimes \pi_0)$, but it does not work unless one knows how to choose $\ell_v$.

To remedy this, instead of $\ell$, we consider an element $L$ in

$$\mathrm{Hom}_{G_0(\mathbb{A}) \times G_0(\mathbb{A})}((\pi_1 \boxtimes \bar{\pi}_1) \otimes (\bar{\pi}_0 \boxtimes \pi_0), \mathbb{C})$$

defined by

$$L((\varphi_1 \boxtimes \varphi_1') \otimes (\varphi_0 \boxtimes \varphi_0')) = \langle \varphi_1|_{G_0}, \varphi_0 \rangle \overline{\langle \varphi_1'|_{G_0}, \varphi_0' \rangle}$$

for $\varphi_i, \varphi_i' \in \pi_i$. As an analogue of $L$, we will define an element $L_v$ in

$$\mathrm{Hom}_{G_0(k_v) \times G_0(k_v)}((\pi_{1,v} \boxtimes \bar{\pi}_{1,v}) \otimes (\bar{\pi}_{0,v} \boxtimes \pi_{0,v}), \mathbb{C}) \tag{1.2}$$

as follows. Put

$$\langle \varphi_i, \varphi_i' \rangle = \int_{G_i(k) \backslash G_i(\mathbb{A})} \varphi_i(g_i) \overline{\varphi_i'(g_i)} \, dg_i$$

for $\varphi_i, \varphi_i' \in \pi_i$, where $dg_i$ is the Tamagawa measure on $G_i(\mathbb{A})$. Then there exists an invariant hermitian inner product $\langle \cdot, \cdot \rangle_v$ on $\pi_{i,v}$ such that

$$\langle \varphi_i, \varphi_i' \rangle = \prod_v \langle \varphi_{i,v}, \varphi_{i,v}' \rangle_v \tag{1.3}$$

and such that $\langle \varphi_{i,v}, \varphi_{i,v}' \rangle_v = 1$ for almost all $v$. We also write

$$dg_0 = \prod_v dg_{0,v}, \tag{1.4}$$

where $dg_{0,v}$ is a Haar measure on $G_0(k_v)$. We may assume that the volume of a hyperspecial maximal compact subgroup of $G_0(k_v)$ is equal to one for almost all $v$. Put

$$\begin{aligned} &I_v((\varphi_{1,v} \boxtimes \varphi_{1,v}') \otimes (\varphi_{0,v} \boxtimes \varphi_{0,v}')) \\ &= \int_{G_0(k_v)} \langle \pi_{1,v}(\iota(g_{0,v}))\varphi_{1,v}, \varphi_{1,v}' \rangle_v \overline{\langle \pi_{0,v}(g_{0,v})\varphi_{0,v}, \varphi_{0,v}' \rangle_v} \, dg_{0,v} \end{aligned}$$

for $\varphi_{i,v}, \varphi'_{i,v} \in \pi_{i,v}$. Using Harish-Chandra's estimate of matrix coefficients, we can prove that this integral is absolutely convergent under the assumption that $\pi_{i,v}$ is tempered. Hence $I_v$ defines an element in (1.2).

To relate $L$ and $I_v$, we must do the unramified calculation. Put

$$\Delta_{G_1} = \begin{cases} \zeta(2)\zeta(4)\cdots\zeta(2m) & \text{if } \dim V_1 = 2m+1, \\ \zeta(2)\zeta(4)\cdots\zeta(2m-2)L(m,\chi) & \text{if } \dim V_1 = 2m, \end{cases}$$

where $\chi$ is the quadratic character of $\mathbb{A}^\times/k^\times$ associated to the discriminant of $V_1$. We also put

$$\mathcal{P}_{\pi_1,\pi_0}(s) = \frac{L(s, \pi_1 \boxtimes \pi_0)}{L(s+\frac{1}{2}, \pi_1, \mathrm{Ad})L(s+\frac{1}{2}, \pi_0, \mathrm{Ad})},$$

where Ad is the adjoint representation of ${}^LG_i$ on the Lie algebra of the dual group $\hat{G}_i$ of $G_i$. Let $\Delta_{G_{1,v}}$ and $\mathcal{P}_{\pi_{1,v},\pi_{0,v}}(s)$ be the local components of $\Delta_{G_1}$ and $\mathcal{P}_{\pi_1,\pi_0}(s)$, respectively. Then

$$I_v((\varphi_{1,v} \boxtimes \varphi'_{1,v}) \otimes (\varphi_{0,v} \boxtimes \varphi'_{0,v})) = \Delta_{G_1,v} \cdot \mathcal{P}_{\pi_{1,v},\pi_{0,v}}(\tfrac{1}{2})$$

for almost all $v$. The proof of this formula is based on the unramified calculation by Ginzburg, Piatetski-Shapiro, and Rallis [3] and an explicit formula for Shintani functions by Kato, Murase, and Sugano [8]. Put

$$L_v = \Delta_{G_1,v}^{-1} \cdot \mathcal{P}_{\pi_{1,v},\pi_{0,v}}(\tfrac{1}{2})^{-1} \cdot I_v.$$

Then $L_v$ is an element in (1.2) such that

$$L_v((\varphi_{1,v} \boxtimes \varphi'_{1,v}) \otimes (\varphi_{0,v} \boxtimes \varphi'_{0,v})) = 1$$

for almost all $v$.

**Conjecture A.** *There exists an integer $\beta$ such that*

$$L = 2^\beta \cdot \Delta_{G_1} \cdot \mathcal{P}_{\pi_1,\pi_0}(\tfrac{1}{2}) \cdot \prod_v L_v.$$

We will state Conjecture A in a different way to apply it to the study of periods of automorphic forms. We drop the assumptions (1.3) and (1.4) and choose an arbitrary invariant hermitian inner product $\langle \cdot, \cdot \rangle_v$ on $\pi_{i,v}$ and an arbitrary Haar measure $dg_{0,v}$ on $G_0(k_v)$. Let $C_0$ be a positive real number such that

$$dg_0 = C_0 \cdot \prod_v dg_{0,v}.$$

Put

$$\alpha_v(\varphi_{1,v}, \varphi_{0,v}) = L_v((\varphi_{1,v} \boxtimes \varphi_{1,v}) \otimes (\varphi_{0,v} \boxtimes \varphi_{0,v}))$$

for $\varphi_{i,v} \in \pi_{i,v}$. Then Conjecture A is equivalent to the following.

**Conjecture A′.** *There exists an integer $\beta$ such that*

$$\frac{|\langle \varphi_1|_{G_0}, \varphi_0\rangle|^2}{\|\varphi_1\|^2 \cdot \|\varphi_0\|^2} = 2^\beta \cdot C_0 \cdot \Delta_{G_1} \cdot \mathcal{P}_{\pi_1,\pi_0}(\tfrac{1}{2}) \cdot \prod_v \frac{\alpha_v(\varphi_{1,v}, \varphi_{0,v})}{\|\varphi_{1,v}\|_v^2 \cdot \|\varphi_{0,v}\|_v^2}$$

*for* $\varphi_i = \otimes_v \varphi_{i,v} \in \pi_i$.

## 2. The Arthur conjecture

It is more delicate to determine the quantity $2^\beta$ in Conjecture A. We believe that this quantity $2^\beta$ is related to the Arthur conjecture [1].

Let $\mathcal{L}$ be the hypothetical Langlands group of $k$. Then there should exist a one-to-one correspondence between the set of irreducible cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$ and the set of equivalence classes of irreducible representations of $\mathcal{L}$ of dimension $n$. The Arthur conjecture [1] asserts that there exists a unique equivalence class of homomorphisms

$$\psi_i : \mathcal{L} \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^L G_i$$

associated to $\pi_i$. Note that the restriction of $\psi_i$ to $\mathrm{SL}_2(\mathbb{C})$ is expected to be trivial since $\pi_i$ is tempered. Let $\mathcal{S}_{\psi_i}$ be the centralizer of the image of $\psi_i$ in $\hat{G}_i$. Then $\mathcal{S}_{\psi_i}$ is expected to be a finite group since $\pi_i$ is square integrable. Moreover the order of $\mathcal{S}_{\psi_i}$ should be a power of two since $G_i$ is a special orthogonal group.

**Conjecture B.** *The quantity $2^\beta$ in Conjecture A is equal to*

$$\frac{1}{|\mathcal{S}_{\psi_1}| \cdot |\mathcal{S}_{\psi_0}|}.$$

## 3. Examples

Let $n = 2$. Then there exist a quaternion algebra $B$ over $k$ and a quadratic extension $K$ of $k$ which splits $B$ such that $G_0 = \ker \mathrm{N}_{K/k}$ and $G_1 = B^\times/\mathbb{G}_m$. Note that

$$\mathcal{P}_{\pi_1,\pi_0}(\tfrac{1}{2}) = \frac{L(\frac{1}{2}, \pi_{1,K} \otimes \omega)}{L(1, \pi_1, \mathrm{Ad}) L(1, \chi_{K/k})}.$$

Here $\pi_{1,K}$ is the base change of $\pi_1$ to $\mathrm{GL}_2(\mathbb{A}_K)$, $\omega$ is the character of $\mathbb{A}_K^\times/K^\times$ with $\omega|_{\mathbb{A}^\times} = 1$ determined by $\pi_0$, and $\chi_{K/k}$ is the quadratic character of $\mathbb{A}^\times/k^\times$ associated to $K/k$. In this case, Conjecture A is proved by Waldspurger [9]. We remark that $\beta = -2$ and $|\mathcal{S}_{\psi_i}| = 2$ for $i = 0, 1$.

Let $n = 3$ and $k = \mathbb{Q}$. We assume that $G_0 = \mathrm{PGL}_2$ and $G_1 = \mathrm{G}(\mathrm{SL}_2 \times \mathrm{SL}_2)/\mathbb{G}_m$, where

$$\mathrm{G}(\mathrm{SL}_2 \times \mathrm{SL}_2) = \{(g_1, g_2) \in \mathrm{GL}_2 \times \mathrm{GL}_2 \mid \det(g_1) = \det(g_2)\}.$$

Let $\kappa_j$ be an even positive integer for $j = 1, 2, 3$ such that $\kappa_1 + \kappa_2 = \kappa_3$ and $f_j \in S_{\kappa_j}(\mathrm{SL}_2(\mathbb{Z}))$ a normalized Hecke eigenform. Then $f_1 \otimes f_2$ and $f_3$ define $\pi_1$ and $\pi_0$, respectively. Note that

$$\mathcal{P}_{\pi_1,\pi_0}(\tfrac{1}{2}) = \frac{L(\frac{1}{2}, \sigma_1 \times \sigma_2 \times \sigma_3)}{\prod_{j=1}^{3} L(1, \sigma_j, \mathrm{Ad})}.$$

Here $\sigma_j$ is the irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ determined by $f_j$. We consider the integral

$$\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}} f_1(\tau) f_2(\tau) \overline{f_3(\tau)}\, \mathrm{Im}(\tau)^{\kappa_3-2}\, d\tau.$$

In this case, Conjecture A follows from the result of Harris and Kudla [5]. We remark that $\beta = -2$ and $|S_{\psi_i}| = 2$ for $i = 0, 1$. Also, using the result of Watson [10], we can prove Conjecture A for more general cases.

Let $n = 4$. Let $k$ be a global field of characteristic greater than two and of genus one. We assume that $G_0 = \mathrm{G}(\mathrm{SL}_2 \times \mathrm{SL}_2)/\mathbb{G}_m$ and $G_1 = \mathrm{PGSp}_2$. Let $\pi_1$ be an endoscopic lift of $\sigma_1 \otimes \sigma_2$ and $\pi_0 = \tau_1 \otimes \tau_2$. Here $\sigma_1, \sigma_2, \tau_1$, and $\tau_2$ are irreducible unramified cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ with trivial central characters such that $\sigma_1 \not\simeq \sigma_2$. Note that

$$\mathcal{P}_{\pi_1,\pi_0}(\tfrac{1}{2}) = \frac{L(\frac{1}{2}, \sigma_1 \times \tau_1 \times \tau_2) L(\frac{1}{2}, \sigma_2 \times \tau_1 \times \tau_2)}{L(1, \sigma_1 \times \sigma_2) \prod_{j=1}^{2} L(1, \sigma_j, \mathrm{Ad}) L(1, \tau_j, \mathrm{Ad})}.$$

In this case, Conjectures A and B are compatible with an analogue of the result of Böcherer, Furusawa, and Schulze-Pillot [2]. We remark that $|S_{\psi_1}| = 4$ and $|S_{\psi_0}| = 2, 4, 8$.

## 4. The non-tempered case

When $\pi_i$ is only square integrable, we have not proved that the integral $\ell$ is convergent. What is worse is that the integral $I_v$ is divergent in general. Nevertheless we expect that Conjecture A can be extended for square integrable automorphic representations under the assumption that (1.1) holds for all $v$. Note that Conjecture B does not hold. We will provide some examples.

Let $n = 4$ and $k = \mathbb{Q}$. We assume that $G_0 = \mathrm{G}(\mathrm{SL}_2 \times \mathrm{SL}_2)/\mathbb{G}_m$ and $G_1 = \mathrm{PGSp}_2$. Let $\kappa$ be an odd positive integer. Let $f \in S_{2\kappa}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform and $\mathcal{F} \in S_{\kappa+1}(\mathrm{Sp}_2(\mathbb{Z}))$ a Saito-Kurokawa lift of $f$. Let $g \in S_{\kappa+1}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform. Then $\mathcal{F}$ and $g \otimes g$ define $\pi_1$ and $\pi_0$, respectively. Note that

$$\mathcal{P}_{\pi_1,\pi_0}(\tfrac{1}{2}) = \frac{L(\frac{1}{2}, \mathrm{Ad}(\sigma) \times \tau)}{\zeta(2) L(\frac{3}{2}, \tau) L(1, \tau, \mathrm{Ad})}.$$

Here $\sigma$ and $\tau$ are the irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ determined by $g$ and $f$, respectively. We consider the integral

$$\int_{\mathrm{SL}_2(\mathbf{Z})\backslash\mathfrak{H}}\int_{\mathrm{SL}_2(\mathbf{Z})\backslash\mathfrak{H}} \mathcal{F}\left(\begin{pmatrix}\tau_1 & 0\\ 0 & \tau_2\end{pmatrix}\right)\overline{g(\tau_1)g(\tau_2)}\,\mathrm{Im}(\tau_1)^{\kappa-1}\,\mathrm{Im}(\tau_2)^{\kappa-1}\,d\tau_1\,d\tau_2.$$

In this case, Conjecture A is compatible with the result of [6]. We remark that $\beta = -2$ and $|S_{\psi_i}| = 2$ for $i = 0, 1$.

Let $n = 5$ and $k = \mathbb{Q}$. Let $K = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field with discriminant $-D$ and $\mathcal{O}_K$ the ring of integers of $K$. We assume that $G_0 = \mathrm{Sp}_2/\{\pm 1\}$ and $G_1 = \mathrm{SU}(2,2)/\{\pm 1\}$. Let $\kappa$ be an even positive integer. Let $g \in S_{\kappa-1}(\Gamma_0(D), \chi_{K/k})$ be a primitive form and $\mathcal{G} \in S_\kappa(\Gamma_K^{(2)})$ the hermitian Maass lift of $g$, where $\Gamma_K^{(2)} = \mathrm{SU}(2,2)(\mathbb{Q}) \cap \mathrm{GL}_4(\mathcal{O}_K)$. We may assume that $\mathcal{G} \neq 0$. Let $f \in S_{2\kappa-2}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform and $\mathcal{F} \in S_\kappa(\mathrm{Sp}_2(\mathbb{Z}))$ a Saito-Kurokawa lift of $f$. Then $\mathcal{G}$ and $\mathcal{F}$ define $\pi_1$ and $\pi_0$, respectively. Note that

$$\mathcal{P}_{\pi_1,\pi_0}(\tfrac{1}{2}) = -\frac{L(\frac{1}{2}, \mathrm{Sym}^2(\tau)\times\sigma)L(\frac{3}{2},\sigma)}{L(2,\mathrm{Sym}^2(\tau))L(1,\tau,\mathrm{Ad})L(1,\sigma,\mathrm{Ad})}.$$

Here $\sigma$ and $\tau$ are the irreducible cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ determined by $f$ and $g$, respectively. We consider the integral

$$\int_{\mathrm{Sp}_2(\mathbf{Z})\backslash\mathfrak{H}_2} \mathcal{G}(Z)\overline{\mathcal{F}(Z)}\det(\mathrm{Im}(Z))^{\kappa-3}\,dZ.$$

In this case, Conjecture A is compatible with the result of [7]. We remark that $\beta = -2$ and $|S_{\psi_i}| = 2$ for $i = 0, 1$.

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DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, OSAKA CITY UNIVERSITY, 3-3-138 SUGIMOTO, SUMIYOSHI-KU, OSAKA 558-8585, JAPAN
*E-mail address*: `ichino@sci.osaka-cu.ac.jp`

GRADUATE SCHOOL OF MATHEMATICS, KYOTO UNIVERSITY, KITASHIRAKAWA, KYOTO 606-8502, JAPAN
*E-mail address*: `ikeda@math.kyoto-u.ac.jp`