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CLASS ONE WHITTAKER FUNCTIONS ON REAL SEMISIMPLE LIE GROUPS

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Introduction. Maass wave form f is an automorphic form on the upper half plane $\mathfrak{H}=\{z=x+\sqrt{-1}y \mid y>0\}$ which is an eigenfunction of the Laplacian of $\mathfrak{H}$, that is,

$$-y^2\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\Big)f(z)=\Big(\frac{1}{4}-\nu^2\Big)f(z)$$

($\nu\in\mathbf{C}$). Then f has the Fourier expansion of the form

$$f(x+\sqrt{-1}y)=\sum_{n\neq 0}a_n\sqrt{y}K_\nu(2\pi|n|y)\,\exp(2\pi\sqrt{-1}nx)+ay^{\nu+1/2}+by^{-\nu+1/2}.$$

Here $K_\nu(z)$ is the K-Bessel function (= *class one Whittaker function* on $SL_2(\mathbf{R})$) and satisfies Bessel's differential equation

$$\Big[\Big(z\frac{d}{dz}\Big)^2-(z^2+\nu^2)\Big]K_\nu(z)=0.$$

When $\nu\notin\mathbf{Z}$, the fundamental solution of the above differential equation around $z=0$ is $\{I_\nu(z),I_{-\nu}(z)\}$ with

$$I_\nu(z)=\sum_{m=0}^{\infty}\frac{(z/2)^{2m+\nu}}{m!\,\Gamma(\nu+m+1)}$$

the I-Bessel function (= *fundamental Whittaker function* on $SL_2(\mathbf{R})$) and there is the relation

$$K_\nu(z)=\frac{\pi}{2\sin\nu\pi}\big(I_{-\nu}(z)-I_\nu(z)\big).$$

In this note we shall discuss the explicit formula of these special functions on higher rank groups, and its application to automorphic L-functions.

1. Whittaker functions for class one principal series representations

We recall the notion of Whittaker functions for class one principal series representations of real semisimple Lie groups. Our main reference is Hashizume's paper [3]. Let G be a real semisimple Lie group with finite center and $\mathfrak{g}$ its Lie algebra. Fix a maximal compact subgroup K of G and put $\mathfrak{k}=\mathrm{Lie}(K)$. Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ and θ the corresponding Cartan involution. For a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ and $\alpha\in\mathfrak{a}^*$, put $\mathfrak{g}_\alpha=\{X\in\mathfrak{g}\mid [H,X]=\alpha(H)X \text{ for all } H\in\mathfrak{a}\}$ and $\Delta=\Delta(\mathfrak{g},\mathfrak{a})$ the restricted root system. Denoted by Δ^+ the positive system in Δ and Π the set of simple roots. Then we have an Iwasawa decomposition $\mathfrak{g}=\mathfrak{n}\oplus\mathfrak{a}\oplus\mathfrak{k}$ with $\mathfrak{n}=\sum_{\alpha\in\Delta^+}\mathfrak{g}_\alpha$. Let $G=NAK$ be the Iwasawa decomposition corresponding to that of $\mathfrak{g}$. We denote by S the Weyl group of the root system Δ.

Let $P_0 = MAN$ be the minimal parabolic subgroup of G with $M = Z_K(A)$. For a linear form $\nu \in \mathfrak{a}_{\mathbf{C}}^* = \mathfrak{a}^* \otimes_{\mathbf{R}} \mathbf{C}$, define a character e^ν on A by $e^\nu(a) = \exp(\nu(\log a))$ $(a \in A)$. We call the induced representation

$$\pi_\nu = L^2\text{-}\mathrm{Ind}_{P_0}^G(1_M \otimes e^{\nu+\rho} \otimes 1_N)$$

the *class one principal series representation* of G. Here $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+} m_\alpha \alpha$ is the half sum of positive roots ($m_\alpha = \dim \mathfrak{g}_\alpha$).

Let $U(\mathfrak{g}_{\mathbf{C}})$ and $U(\mathfrak{a}_{\mathbf{C}})$ be the universal enveloping algebras of $\mathfrak{g}_{\mathbf{C}}$ and $\mathfrak{a}_{\mathbf{C}}$, the complexifications of $\mathfrak{g}$ and $\mathfrak{a}$ respectively. Set

$$U(\mathfrak{g}_{\mathbf{C}})^K = \{X \in U(\mathfrak{g}_{\mathbf{C}}) \mid \mathrm{Ad}(k)X = X \text{ for all } k \in K\}.$$

Let p be the projection $U(\mathfrak{g}_{\mathbf{C}}) \to U(\mathfrak{a}_{\mathbf{C}})$ along the decomposition

$$U(\mathfrak{g}_{\mathbf{C}}) = U(\mathfrak{a}_{\mathbf{C}}) \oplus (\mathfrak{n}U(\mathfrak{g}_{\mathbf{C}}) + U(\mathfrak{g}_{\mathbf{C}})\mathfrak{k}).$$

Define the automorphism γ of $U(\mathfrak{a}_{\mathbf{C}})$ by $\gamma(H) = H + \rho(H)$ for $H \in \mathfrak{a}_{\mathbf{C}}$. For $\nu \in \mathfrak{a}_{\mathbf{C}}^*$, define the algebra homomorphism $\chi_\nu : U(\mathfrak{g}_{\mathbf{C}})^K \to \mathbf{C}$ by

$$\chi_\nu(z) = \nu(\gamma \circ p(z))$$

for $z \in U(\mathfrak{g}_{\mathbf{C}})^K$.

Let η be a unitary character of N. Since $\mathfrak{n} = [\mathfrak{n}, \mathfrak{n}] \oplus \sum_{\alpha\in\Pi} \mathfrak{g}_\alpha$, η is determined by the restriction $\eta_\alpha := d\eta|_{\mathfrak{g}_\alpha}$ $(\alpha \in \Pi)$. The length $|\eta_\alpha|$ of η_α is defined as $|\eta_\alpha|^2 = \sum_{1\le i\le m(\alpha)} -d\eta(X_{\alpha,i})^2$ (note that $d\eta(X_{\alpha,i}) \in \sqrt{-1}\mathbf{R}$), where the root vector $X_{\alpha,i}$ is chosen as $B(X_{\alpha,i}, \theta X_{\alpha,j}) = -\delta_{i,j}$ $(1 \le i, j \le m(\alpha))$. Here $B(\ ,\ )$ is the Killing form on $\mathfrak{g}$. In this paper we assume that η is nondegenerate, that is, $\eta_\alpha \ne 0$ for all $\alpha \in \Pi$.

Definition 1.1. Denote by $\mathrm{Wh}(\nu, \eta)$ the space of smooth functions $w : G \to \mathbf{C}$ satisfying

- $w(ngk) = \eta(n)w(g)$ for all $n \in N$, $g \in G$ and $k \in K$,
- $Zw = \chi_\nu(Z)w$ for all $Z \in U(\mathfrak{g}_{\mathbf{C}})^K$.

Remark 1. Because of the Iwasawa decomposition $w \in \mathrm{Wh}(\nu, \eta)$ is determined by its restriction $w|_A$ to A, which we call *radial part* of w. Then w can be considered as n-variable function (n is the real rank of G).

1.1. **Fundamental Whittaker functions.** Hashizume [3] constructed the basis of the space $\mathrm{Wh}(\nu, \eta)$. Let $\langle\ ,\ \rangle$ be the inner product on $\mathfrak{a}_{\mathbf{C}}^*$ induced by the Killing form $B(\ ,\ )$. We denote by L the set of linear functions on $\mathfrak{a}_{\mathbf{C}}$ of the form $\sum_{\alpha\in\Pi} n_\alpha \alpha$ with $n_\alpha \in \mathbf{Z}_{\ge 0}$.

For each $\lambda \in L$, we can define the rational function c_λ on $\mathfrak{a}_{\mathbf{C}}^*$ as follows. Put $c_0(\nu) = 1$ and determine c_λ for $\lambda \in L\backslash\{0\}$ by

$$(\langle\lambda, \lambda\rangle + 2\langle\lambda, \nu\rangle)c_\lambda(\nu) = 2\sum_{\alpha\in\Pi} |\eta_\alpha|^2 c_{\lambda-2\alpha}(\nu),$$

inductively. It comes from that w is an eigenfunction of the Casimir element. Here we assumed that $\langle\lambda, \lambda\rangle + 2\langle\lambda, \nu\rangle \ne 0$ for all $\lambda \in L\backslash\{0\}$.

Definition 1.2. For $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ and unitary character η of N, define a series $M_{\nu,\eta}(a)$ on A by

$$M_{\nu,\eta}(a) = a^{\nu+\rho} \sum_{\lambda\in L} c_\lambda(\nu) a^\lambda \quad a \in A,$$

and extend it to the function on G by

$$M_{\nu,\eta}(g) = \eta(n(g)) M_{\nu,\eta}(a(g))$$

with $g = n(g)a(g)k(g)$ the Iwasawa decomposition of $g \in G$. We call $M_{\nu,\eta}$ the *fundamental Whittaker function* or the *secondary Whittaker function* on G.

Actually Hashizume proved the following.

Theorem 1.3. ([3, Theorem 5.4]) *If $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ is regular, then the set*

$$\{M_{s\nu,\eta}(g) \mid s \in W\}$$

forms a basis of $\mathrm{Wh}(\nu,\eta)$. *Here an element $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ is called regular if the following two conditions are satisfied.*

- $\langle\lambda,\lambda\rangle + 2\langle\lambda, s\nu\rangle \neq 0$ *for all* $\lambda \in L\backslash\{0\}$ *and* $s \in S$,
- $s\nu - t\nu \notin \{\sum_{\alpha\in\Pi} m_\alpha\alpha \mid m_\alpha \in \mathbf{Z}\}$ *for all* $s \neq t \in S$.

If we suitably fix η_α and take a coordinate on A by $y = (y_1, \ldots, y_n)$ ($n = \dim A$), the radial parts of fundamental Whittaker functions can be written as the form:

$$M_\nu^G(y) \equiv M_{\nu,\eta}^G(y) = y^{\nu+\rho} \sum_{\mathbf{m}=(m_1,\ldots,m_n)\in\mathbf{N}^n} c_{\mathbf{m}}^G(\nu)(\pi y_1)^{2m_1}\cdots(\pi y_n)^{2m_n} \qquad (c_{(0,\ldots,0)}^G(\nu) = 1),$$

where the recurrence relations satisfied by $c_{\mathbf{m}}(\nu) = c_{\mathbf{m}}^G(\nu)$ for $G = SL_{n+1}(\mathbf{R})$, $SO_{2n+1}(\mathbf{R})$, $Sp_n(\mathbf{R})$, $SO_{2n}(\mathbf{R})$ and $G_2(\mathbf{R})$ are

(A) $\{\sum_{i=1}^n m_i^2 - \sum_{i=1}^{n-1} m_i m_{i+1} + \sum_{i=1}^n (\nu_i - \nu_{i+1})m_i\}c_{\mathbf{m}}(\nu) = \sum_{i=1}^n c_{\mathbf{m}-\mathbf{e}_i}(\nu)$,

(B) $\{\sum_{i=1}^{n-1} m_i^2 + \frac{1}{2}m_n^2 - \sum_{i=1}^{n-1} m_i m_{i+1} + \sum_{i=1}^{n-1}(\nu_i - \nu_{i+1})m_i + \nu_n m_n\}c_{\mathbf{m}}(\nu)$
$= \sum_{i=1}^{n-1} c_{\mathbf{m}-\mathbf{e}_i}(\nu) + \frac{1}{2}c_{\mathbf{m}-\mathbf{e}_n}(\nu)$,

(C) $\{\sum_{i=1}^{n-1} m_i^2 + 2m_n^2 - \sum_{i=1}^{n-2} m_i m_{i+1} - 2m_{n-1}m_n + \sum_{i=1}^{n-1}(\nu_i - \nu_{i+1})m_i + 2\nu_n m_n\}c_{\mathbf{m}}(\nu)$
$= \sum_{i=1}^{n-1} c_{\mathbf{m}-\mathbf{e}_i}(\nu) + 2c_{\mathbf{m}-\mathbf{e}_n}(\nu)$,

(D) $\{\sum_{i=1}^n m_i^2 - \sum_{i=1}^{n-2} m_i m_{i+1} - m_{n-2}m_n + \sum_{i=1}^{n-1}(\nu_i - \nu_{i+1})m_i + (\nu_{n-1} + \nu_n)m_n\}c_{\mathbf{m}}(\nu)$
$= \sum_{i=1}^n c_{\mathbf{m}-\mathbf{e}_i}(\nu)$,

(G) $(m_1^2 + 3m_2^2 - 3m_1m_2 + \nu_1 m_1 + \nu_2 m_2)c_{m_1,m_2}(\nu) = c_{m_1-1,m_2}(\nu) + 3\,c_{m_1,m_2-1}(\nu)$.

Here $\mathbf{e}_i$ is the i-th standard basis in $\mathbf{R}^n$ and $\sum_{i=1}^{n+1}\nu_i = 0$ in (A).

1.2. **Jacquet integrals and class one Whittaker functions.** Jacquet [8] introduced the integral

$$W_{\nu,\eta}(g) = \int_N \eta^{-1}(n)a(s_0^{-1}ng)^{\nu+\rho}dn,$$

where s_0 is a longest element in S. It gives a unique moderate growth Whittaker function and it is known that as a function of ν, $W_{\nu,\eta}$ converges absolutely and uniformly on $\{\nu \in \mathfrak{a}_{\mathbf{C}}^* \mid \mathrm{Re}(\langle\nu,\alpha\rangle) > 0 \text{ for all } \alpha \in \Delta^+\}$ and can be continued to a meromorphic function. We call $W_{\nu,\eta}$ the *class one Whittaker function* on G.

Remark 2. Jacquet integral of course gives an integral representation of Whittaker function, however, it is not satisfactory form for our use, such as computation of gamma factors of automorphic L-functions. For example the radial part of Jacquet integral on $SL_3(\mathbf{R})$ is

$$y_1^{-\nu_1+1}y_2^{-\nu_1-\nu_2+1}\int_{\mathbf{R}^3}(1+n_1^2+n_3^2)^{(-\nu_1-2\nu_2-1)/2}\{1+n_2^2+(n_1n_2-n_3)^2\}^{(-\nu_1+\nu_2-1)/2}$$
$$\cdot\exp\{-2\pi\sqrt{-1}(c_1n_1+c_2n_2)\}dn_1dn_2dn_3,$$

where (nonzero) real numbers c_1, c_2 are parameters of η.

Now we recall the linear relation between $W_{\nu,\eta}(g)$ and $M_{s\nu,\eta}(g)$ $(s \in S)$.

Theorem 1.4. ([3, Theorem 7.8]) *Let $c(\nu)$ be the Harish Chandra c-function:*

$$\begin{aligned} c(\nu) &:= \int_N a(s_0^{-1}n)^{\nu+\rho}dn \\ &= \prod_{\alpha\in\Delta_0^+} 2^{(m_\alpha-m_{2\alpha})/2}\left(\frac{\pi}{\langle\alpha,\alpha\rangle}\right)^{(m_\alpha+m_{2\alpha})/2} \frac{\Gamma(\nu_\alpha)\Gamma(\frac{1}{2}(\nu_\alpha+\frac{m_\alpha}{2}))}{\Gamma(\nu_\alpha+\frac{m_\alpha}{2})\Gamma(\frac{1}{2}(\nu_\alpha+\frac{m_\alpha}{2}+m_{2\alpha}))}. \end{aligned}$$

Here $\nu_\alpha = \langle\nu,\alpha\rangle/\langle\alpha,\alpha\rangle$ and $\Delta_0^+ = \{\alpha \in \Delta^+ \mid \frac{1}{2}\alpha \notin \Delta\}$. For $s \in S$, we define $\gamma(s;\nu,\eta)$ as follows. For a simple reflection $s = s_\alpha$ $(\alpha \in \Pi)$, put

$$\gamma(s;\nu,\eta) := \left(\frac{|\eta_\alpha|}{2\sqrt{2\langle\alpha,\alpha\rangle}}\right)^{2\nu_\alpha} \frac{\Gamma(\frac{1}{2}(-\nu_\alpha+\frac{m_\alpha}{2}+1))\Gamma(\frac{1}{2}(-\nu_\alpha+\frac{m_\alpha}{2}+m_{2\alpha}))}{\Gamma(\frac{1}{2}(\nu_\alpha+\frac{m_\alpha}{2}+1))\Gamma(\frac{1}{2}(-\nu_\alpha+\frac{m_\alpha}{2}+m_{2\alpha}))}$$

and extend it by

$$\gamma(s_\alpha s;\nu,\eta) = \gamma(s;\nu,\eta)\gamma(s_\alpha;s\nu,\eta),$$

for $l(s_\alpha s) = l(s)+1$ where $l(s)$ is the length of s. Then, if ν is regular,

$$W_{\nu,\eta}(g) = \sum_{s\in W} \gamma(s_0 s;\nu,\eta)c(s_0 s\nu)M_{s\nu,\eta}(g).$$

2. The case of $SL_n(\mathbf{R})$

Bump [1] and Vinogradov and Tahtajan [15] studied Whittaker functions on $SL_3(\mathbf{R})$. The Jacquet integral is evaluated to derive integral representation involving K-Bessel functions and the recurrence relation (A) is solved. Extending these studies, Stade [10] discovered integral representation of class one Whittaker function on $SL_n(\mathbf{R})$, which is recursive relation between $W_\nu^{SL_n(\mathbf{R})}(y)$ and $W_\nu^{SL_{n-2}(\mathbf{R})}(y)$. An analogue for the fundamental Whittaker function has conjectured by Stade [12] and verified in [6]. Recently, Stade and the author [7] find new inductive relations between Whittaker functions on $SL_n(\mathbf{R})$ and $SL_{n-1}(\mathbf{R})$, and these formulas seem to be *natural* expressions.

Theorem 2.1. ([7]) *The solution of the recurrence relation (A) can be written as*

$$c_{\mathbf{m}}^{SL_{n+1}(\mathbf{R})}(\nu) = \sum_{\{k_1,\dots,k_{n-1}\}} \frac{c_{(k_1,\dots,k_{n-1})}^{SL_n(\mathbf{R})}(\tilde{\nu})}{\prod_{i=1}^n\{(m_i-k_i)!\,(\nu_i-\nu_{n+1}+1)_{m_i-k_{i-1}}\}},$$

where the indices k_i run through such that $0 \le k_i \le m_i$ $(1 \le i \le n-1)$ and $\tilde{\nu} = (\nu_2 + \nu_1/n, \dots, \nu_{n+1}+\nu_1/n)$ for $\nu = (\nu_1,\dots,\nu_{n+1})$ $(\sum_{i=1}^{n+1}\nu_i = 0)$. Here $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol.

Remark 3. Bump [1] expressed $c_{(m_1,m_2)}^{SL_3(\mathbf{R})}(\nu)$ as a ratio of Gamma functions and Stade [11] wrote $c_{(m_1,m_2,m_3)}^{SL_4(\mathbf{R})}(\nu)$ in terms of generalized hypergeometric series ${}_4F_3(1)$. We first arrive at the above result from these formulas by using Gauss' formula for ${}_2F_1(1)$.

Theorem 2.2. ([7]) *Let $W_\nu^{SL_{n+1}(\mathbf{R})}(y) = y^\rho \widetilde{W}_\nu^{SL_{n+1}(\mathbf{R})}(y)$ be the radial part of the class one Whittaker function on $SL_{n+1}(\mathbf{R})$. Up to the constant multiple, we have the following* *inductive relation:*

$$\widetilde{W}_{\nu}^{SL_{n+1}(\mathbf{R})}(y) = \int_{t_1,\dots,t_n=0}^{\infty} \prod_{i=1}^{n} \exp\Big\{-(\pi y_i)^2 t_i - \frac{1}{t_i}\Big\} \cdot \widetilde{W}_{\tilde{\nu}}^{SL_n(\mathbf{R})}\Big(y_2\sqrt{\frac{t_2}{t_1}},\dots,y_n\sqrt{\frac{t_n}{t_{n-1}}}\Big)$$
$$\cdot \Big(\prod_{i=1}^{n} y_i^{\frac{2(n-1)}{n}} t_i\Big)^{\nu_{n+1}} \prod_{i=1}^{n}\frac{dt_i}{t_i}.$$

Remark 4. The proof is based on the formula of Stade [10, Theorem 2.1] (and its Mellin-Barnes type analogue [13]):

$$\widetilde{W}_{\nu}^{SL_{n+1}(\mathbf{R})}(y) = \int_{t_1,\dots,t_{n-1}=0}^{\infty} \prod_{i=1}^{n} y_i^{(n+1-2i)(\nu_1+\nu_{n+1})/(n-1)} K_{\nu_1-\nu_{n+1}}\Big(2\pi y_i\sqrt{(1+t_{i-1})(1+t_i^{-1})}\Big)$$
$$\cdot \widetilde{W}_{\mu}^{SL_{n-1}(\mathbf{R})}\Big(y_2\sqrt{\frac{t_1}{t_2}},\dots,y_{n-1}\sqrt{\frac{t_{n-2}}{t_{n-1}}}\Big)\prod_{i=1}^{n-1} t_i^{-(n+1)(\nu_1+\nu_{n+1})/2(n-1)} \prod_{i=1}^{n-1}\frac{dt_i}{t_i}$$

with $\mu = (\nu_2 + (\nu_1+\nu_{n+1})/(n-1),\dots,\nu_n + (\nu_1+\nu_{n+1})/(n-1))$.

3. The case of $SO_{2n+1}(\mathbf{R})$

We can solve the recurrence relation (B):

Theorem 3.1. *Put $\tilde{\nu} = (\nu_1,\dots,\nu_{n-1})$ for $\nu = (\nu_1,\dots,\nu_n)$. Then*

$$c_{\mathbf{m}}^{SO_{2n+1}(\mathbf{R})}(\nu) = \sum_{\substack{\{l_1,\dots,l_{n-1}\}\\ \{k_1,\dots,k_{n-1}\}}} \frac{c_{(k_1,\dots,k_{n-1})}^{SO_{2n-1}(\mathbf{R})}(\tilde{\nu})}{\prod_{i=1}^{n-1}(m_i-l_i)!\cdot(m_n-k_{n-1})!\prod_{i=1}^{n-1}(l_i-k_i)!}$$
$$\cdot\frac{1}{\prod_{i=1}^{n}(\nu_i+\nu_n+1)_{m_i-l_{i-1}}\prod_{i=1}^{n-1}(\nu_i-\nu_n+1)_{l_i-k_{i-1}}},$$

where the indices k_i, l_i run through such that $0 \le k_i \le l_i \le m_i$ $(1 \le i \le n-1)$ and $0 \le k_{n-1} \le m_n$ and we promise $k_0 = l_0 = 0$.

The recursive integral representation of class one Whittaker function is the following:

Theorem 3.2. *Let $W_{\nu}^{SO_{2n+1}(\mathbf{R})}(y) = y^{\rho}\widetilde{W}_{\nu}^{SO_{2n+1}(\mathbf{R})}(y)$ be the radial part of the class one Whittaker function on $SO_{2n+1}(\mathbf{R})$. Up to the constant multiple, we have the following inductive relation:*

$$\widetilde{W}_{\nu}^{SO_{2n+1}(\mathbf{R})}(y) = \int_{t_1,\dots,t_n=0}^{\infty}\int_{u_1,\dots,u_{n-1}=0}^{\infty} \prod_{i=1}^{n}\exp\Big\{-(\pi y_i)^2 t_i - \frac{1}{t_i}\Big\}\prod_{i=1}^{n-1}\exp\Big\{-(\pi y_i)^2 u_i - \frac{t_i}{t_{i+1}}\frac{1}{u_i}\Big\}$$
$$\cdot\widetilde{W}_{\tilde{\nu}}^{SO_{2n-1}(\mathbf{R})}\Big(y_2\sqrt{\frac{t_1u_2}{t_2u_1}},\dots,y_{n-1}\sqrt{\frac{t_{n-2}u_{n-1}}{t_{n-1}u_{n-2}}},y_n\sqrt{\frac{t_{n-1}}{u_{n-1}}}\Big)$$
$$\cdot\Big\{\Big(\prod_{i=1}^{n}y_i\Big)\Big(\prod_{i=1}^{n-1}\sqrt{t_iu_i}\Big)t_n\Big\}^{\nu_n}\prod_{i=1}^{n}\frac{dt_i}{t_i}\prod_{i=1}^{n-1}\frac{du_i}{u_i}$$
$$= c\int_{u_1,\dots,u_{n-1}=0}^{\infty}\prod_{i=1}^{n}K_{\nu_n}\Big(2\pi y_i\sqrt{(1+u_{i-1})(1+u_i^{-1})}\Big)$$
$$\cdot\widetilde{W}_{\tilde{\nu}}^{SO_{2n-1}(\mathbf{R})}\Big(y_2\sqrt{\frac{u_1}{u_2}},\dots,y_{n-1}\sqrt{\frac{u_{n-2}}{u_{n-1}}},y_n\sqrt{u_{n-1}}\Big)\prod_{i=1}^{n-1}\frac{du_i}{u_i}.$$

Remark 5. We can prove it by using Theorems 1.4 and 3.1. We remark that there is an interesting relation between Whittaker function for certain generalized principal series representations of $Sp_3(\mathbf{R})$ and $W_\nu^{SO_5(\mathbf{R})}$. Roughly speaking, in the first expression in Theorem 3.2 with $n = 3$, the inner integrals with respect to u_1, u_2 give the mentioned Whittaker function on $Sp_3(\mathbf{R})$. See [4] for the details.

4. The case of $Sp_n(\mathbf{R})$ and $SO_{2n}(\mathbf{R})$

As for the fundamental Whittaker function on $SO_{2n}(\mathbf{R})$, we have an inductive relation similar to the previous sections, on the other hand we have not yet obtained such formula for $Sp_n(\mathbf{R})$. However there is a relation between Whittaker functions on $Sp_n(\mathbf{R})$ and $SO_{2n}(\mathbf{R})$ in view of Fourier-Whittaker coefficients of symplectic-orthogonal theta lifting [5].

Theorem 4.1. *(i) Put $\tilde{\nu} = (\nu_1, \ldots, \nu_{n-1})$ for $\nu = (\nu_1, \ldots, \nu_n)$. Then the solution of the recurrence relation (D) is*

$$c_{\mathbf{m}}^{SO_{2n}(\mathbf{R})}(\nu) = \sum_{\substack{\{l_1,\ldots,l_{n-1}\} \\ \{k_1,\ldots,k_{n-1}\}}} \frac{c_{(k_1,\ldots,k_{n-1})}^{SO_{2n-2}(\mathbf{R})}(\tilde{\nu})}{\prod_{i=1}^{n-3}(m_i - l_i)! \cdot (m_{n-2} - l_{n-2} - l_{n-1})(m_{n-1} - k_{n-1})!\,(m_n - l_{n-1})!}$$
$$\cdot \frac{1}{\prod_{i=1}^{n-1}(l_i - k_i)!} \cdot \frac{1}{\prod_{i=1}^{n-1}(\nu_i - \nu_n + 1)_{m_i - l_{i-1}}(\nu_{n-1} + \nu_n + 1)_{m_n - k_{n-2}}}$$
$$\cdot \frac{1}{\prod_{i=1}^{n-3}(\nu_i + \nu_n + 1)_{l_i - k_{i-1}}(\nu_{n-2} + \nu_n + 1)_{l_{n-2} + l_{n-1} - k_{n-3}}},$$

where the indices k_i, l_i run through such that $0 \le k_i \le l_i$ $(1 \le i \le n-1)$, $0 \le k_{n-1} \le m_{n-1}$, $0 \le l_i \le m_i$ $(1 \le i \le n-3)$, $0 \le l_{n-2} + l_{n-1} \le m_{n-2}$ and $0 \le n_{n-1} \le m_n$ and we promise $k_0 = n_0 = 0$.
(ii) We have the relation

$$c_{\mathbf{m}}^{Sp_n(\mathbf{R})}(\nu) = \sum_{\{k_1,\ldots,k_n\}} \frac{c_{(k_1,\ldots,k_n)}^{SO_{2n}(\mathbf{R})}(\nu)}{\prod_{i=1}^{n-2}(m_i - k_i)! \cdot (m_{n-1} - k_{n-1} - k_n)!\,(m_n - k_n)! \prod_{i=1}^{n}(\nu_i + 1)_{m_i - k_{i-1}}},$$

where the indices k_i run through such that $0 \le k_i \le m_i$ $(1 \le i \le n-2, i = n)$ and $0 \le k_{n-1} + k_n \le m_{n-1}$ and we promise $k_0 = 0$.

5. The case of $G_2(\mathbf{R})$

Theorem 5.1. *The solution of the recurrence relation (G) is*

$$c_{(m_1,m_2)}^{G_2(\mathbf{R})}(\nu) = \sum_{\substack{0 \le n_1 + n_2 \le m_1 \\ 0 \le n_4 \le n_3 \le n_2 \le m_2}} \frac{1}{(m_1 - n_1 - n_2)!\,(m_2 - n_2)!\,n_1!\,(n_2 - n_3)!\,(n_3 - n_4)!\,n_4!}$$
$$\cdot \frac{1}{(\nu_1 + \nu_2 + 1)_{m_1 - n_3}(\nu_2 + 1)_{m_2 - n_1}(\nu_1 + 1)_{n_1 - n_4}}$$
$$\cdot \frac{1}{(\nu_1 + 2\nu_2 + 1)_{n_2}(2\nu_1 + 3\nu_2 + 1)_{n_3}(\nu_1 + 3\nu_2 + 1)_{n_4}}.$$

Moreover, the radial part of the fundamental Whittaker function $M^{G_2(\mathbf{R})}_{(\nu_1,\nu_2)}(y)$ is related to that of $SL_3(\mathbf{R})$:

$$M^{G_2(\mathbf{R})}_{(\nu_1,\nu_2)}(y) = y_1^4 y_2^2 \sum_{k_1,k_2=0}^{\infty} (\pi^3 y_1^2 y_2)^{2(k_1+k_2+2\nu_1+3\nu_2)/3} c^{SL_3(\mathbf{R})}_{(k_1,k_2)}(\nu_1+\nu_2, \nu_2, -\nu_1-2\nu_2)$$
$$\cdot M^{SL_3(\mathbf{R})}_{((k_1+k_2+2\nu_1+3\nu_2)/3,\,(-2k_1+k_2-\nu_1)/3,\,(k_1-2k_2-\nu_1-3\nu_2)/3)}(y).$$

In view of the relation between $M^{G_2(\mathbf{R})}_\nu(y)$ and $M^{SL_3(\mathbf{R})}_\nu(y)$, we have the following.

Theorem 5.2. *Let $W^{G_2(\mathbf{R})}_\nu(y) = y_1^5 y_2^3 \widetilde{W}^{G_2(\mathbf{R})}_\nu(y)$ and $W^{SL_3(\mathbf{R})}_\nu(y) = y_1 y_2 \widetilde{W}^{SL_3(\mathbf{R})}_\nu(y)$. Then up to the constant multiple,*

$$\widetilde{W}^{G_2(\mathbf{R})}_{(\nu_1,\nu_2)}(y) = \int_0^\infty\int_0^\infty\int_0^\infty \exp\left\{-(\pi y_1)^2 t_1 - (\pi y_2)^2 t_2 - (\pi y_2)^2 t_3 - \frac{1}{t_1} - \frac{1}{t_2} - \frac{1}{t_3}\frac{t_2}{t_1}\right\}$$
$$\cdot \widetilde{W}^{SL_3(\mathbf{R})}_{(\nu_1+\nu_2,\nu_2,-\nu_1-2\nu_2)}\left(y_1 y_2\sqrt{t_1 t_3},\, y_1\sqrt{\frac{t_2}{t_3}}\right)\frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3}.$$

6. Application to the computation of gamma factors of automorphic L-functions

As an application of the explicit formula of the class one Whittaker functions, we can compute the gamma factors of automorphic L-functions attached to generic cuspidal representations $\pi = \otimes' \pi_v$ whose infinite type π_∞ is isomorphic to class one principal series representation.

6.1. L-functions for $GL_n \times GL_k$.

The theory of integral representations of automorphic L-function $L(s, \pi \times \pi')$ for $GL_n \times GL_k$ has developed by Jacquet, Shalika and Piatetski-Shapiro. The functional equation and poles of the global L-function are known, however, it is worth evaluating the archimedean zeta integrals directly. When $\pi_\infty \cong \pi_\nu$ and $\pi'_\infty \cong \pi_\mu$, the archimedean zeta integrals for $GL_n \times GL_{n+1}$ and $GL_{n+1} \times GL_{n+1}$ become

$$Z^\infty_{1,n}(s) = \int_{(\mathbf{R}^\times)^n} \widetilde{W}^{SL_n(\mathbf{R})}_\nu(y_1,\ldots,y_{n-1})\widetilde{W}^{SL_{n+1}(\mathbf{R})}_\mu(y_1,\ldots,y_n)\cdot(y_1 y_2^2\cdots y_n^n)^s \prod_{i=1}^n \frac{dy_i}{y_i},$$
$$Z^\infty_{2,n}(s) = \int_{(\mathbf{R}^\times)^n} \widetilde{W}^{SL_{n+1}(\mathbf{R})}_\nu(y_1,\ldots,y_n)\widetilde{W}^{SL_{n+1}(\mathbf{R})}_\mu(y_1,\ldots,y_n)\cdot(y_1 y_2^2\cdots y_n^n)^s \prod_{i=1}^n \frac{dy_i}{y_i}.$$

Stade [13], [14] has evaluated these integrals by using the Mellin-Barnes type integral representations of class one Whittaker functions and (generalized) Barnes' lemma. For $(s_1,\ldots,s_n) \in \mathbf{C}^n$, let $V^G_\nu(s_1,\ldots,s_n)$ be the (multiple) Mellin transform of the radial part of the (ρ-shifted) class one Whittaker function $\widetilde{W}^G_\nu$:

$$V^G_\nu(s_1,\ldots,s_n) = \int_{(\mathbf{R}_{\geq 0})^n} \widetilde{W}^G_\nu(y_1,\ldots,y_n)\prod_{i=1}^n y_i^{s_i}\prod_{i=1}^n \frac{dy_i}{y_i}.$$

Then

$$Z_{1,n}^{\infty}(s) = \int_{s_1,\ldots,s_{n-1}} V_{\nu}^{SL_n(\mathbf{R})}(s_1, s_2, \ldots, s_{n-1}) \cdot V_{\mu}^{SL_{n+1}(\mathbf{R})}(s - s_1, 2s - s_2, \ldots, (n-1)s - s_{n-1}, ns)\, ds_1 \cdots ds_{n-1},$$

$$Z_{2,n}^{\infty}(s) = \int_{s_1,\ldots,s_n} V_{\nu}^{SL_{n+1}(\mathbf{R})}(s_1, s_2, \ldots, s_n) V_{\mu}^{SL_{n+1}(\mathbf{R})}(s - s_1, 2s - s_2, \ldots, ns - s_n)\, ds_1 \cdots ds_n.$$

Here the path of integration in each s_i being a vertical line in the complex plane which is taken to separate the poles integrand appropriately ([13],[14]). Stade's computation is based on Barnes' first lemma:

$$\frac{1}{2\pi\sqrt{-1}} \int_{-\sqrt{-1}\infty}^{\sqrt{-1}\infty} \Gamma(a+s)\Gamma(b+s)\Gamma(c-s)\Gamma(d-s)\, ds = \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)},$$

where the path of integration is curved, if necessary, to ensure that the poles of $\Gamma(c-s)\Gamma(d-s)$ lie on the right of the path and of $\Gamma(a+s)\Gamma(b+s)$ on the left.

By the way, we can give a little modified proof of Stade's result by using a formula in the previous sections. That is, in view of the recursive relation

$$V_{\nu}^{SL_{n+1}(\mathbf{R})}(s_1, \ldots, s_n) = \int_{t_1,\ldots,t_{n-1}} \prod_{i=1}^{n} \left\{ \Gamma\Big(s_i + t_i - \frac{i}{n}\nu_1\Big) \Gamma\Big(s_i + t_{i-1} + \frac{n+1-i}{n}\nu_1\Big) \right\} \cdot V_{\tilde{\nu}}^{SL_n(\mathbf{R})}(-t_1, \ldots, -t_{n-1})\, dt_1 \cdots dt_{n-1},$$

we can see that the evaluation of $Z_{1,n}^{\infty}$ and $Z_{2,n}^{\infty}$ is reduced to that of $Z_{2,n-1}^{\infty}$ and $Z_{1,n}^{\infty}$, respectively. Thus these computation is reduced to that of $Z_{2,1}^{\infty}$, which is equivalent to Barnes' first lemma. The result is as follows:

Theorem 6.1. ([13], [14]) *Let* $\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2)$. *Then, up to constant multiple,*

$$Z_{1,n}^{\infty}(s) = \prod_{\substack{1 \le i \le n \\ 1 \le j \le n+1}} \Gamma_{\mathbf{R}}(s + \nu_i + \mu_j),$$

$$Z_{2,n}^{\infty}(s) = \frac{\prod_{1 \le i,j \le n+1} \Gamma_{\mathbf{R}}(s + \nu_i + \mu_j)}{\Gamma_{\mathbf{R}}((n+1)s)}.$$

Remark 6. The numerators coincide with the gamma factors defined from the Langlands parameters of π_ν and π_μ, and the denominator in $Z_{2,n}^{\infty}(s)$ is the normalizing factor of Eisenstein series used in the construction of the global zeta integral.

6.2. ***L*-functions for $SO_{2n+1} \times GL_k$.** Gelbart and Piatetski-Shapiro [2] constructed zeta integrals representing the (standard) L-functions for $G \times GL_n$ for classical group G. As is the case with $GL_n \times GL_k$, when $G = SO_{2n+1}$, the archimedean zeta integrals we want to compute are

$$Z_{1,n}^{\infty}(s) = \int_{(\mathbf{R}^{\times})^n} \widetilde{W}_{(\nu_1,\ldots,\nu_n)}^{SO_{2n+1}(\mathbf{R})}(y_1, \ldots, y_n) \widetilde{W}_{(\mu_1,\ldots,\mu_n)}^{SL_n(\mathbf{R})}(y_1, \ldots, y_{n-1}) \cdot (y_1 y_2^2 \cdots y_n^n)^s \prod_{i=1}^{n} \frac{dy_i}{y_i},$$

$$Z_{2,n}^{\infty}(s) = \int_{(\mathbf{R}^{\times})^n} \widetilde{W}_{(\nu_1,\ldots,\nu_n)}^{SO_{2n+1}(\mathbf{R})}(y_1, \ldots, y_n) \widetilde{W}_{(\mu_1,\ldots,\mu_{n+1})}^{SL_{n+1}(\mathbf{R})}(y_1, \ldots, y_n) \cdot (y_1 y_2^2 \cdots y_n^n)^s \prod_{i=1}^{n} \frac{dy_i}{y_i}.$$

It is natural to expect the following:

Conjecture Up to constant multiple,

$$Z_{1,n}^{\infty} = \frac{\prod_{i=1}^{n}\prod_{j=1}^{n}\Gamma_{\mathbf{R}}(s+\nu_i+\mu_j)}{\prod_{1\le i<j\le n}\Gamma_{\mathbf{R}}(2s+\mu_i+\mu_j)},$$

$$Z_{2,n}^{\infty}(s) = \frac{\prod_{i=1}^{n}\prod_{j=1}^{n+1}\Gamma_{\mathbf{R}}(s+\nu_i+\mu_j)}{\prod_{1\le i<j\le n+1}\Gamma_{\mathbf{R}}(2s+\mu_i+\mu_j)}.$$

Remark 7. From the inductive relation for class one Whittaker functions on $SO_{2n+1}(\mathbf{R})$, we can see that the evaluation of $Z_{1,n}^{\infty}(s)$ is reduced to that of $Z_{2,n-1}^{\infty}(s)$. At the present, the conjecture is true for $Z_{1,n}$ with $n = 2$ (Niwa [9]), 3, 4 and $Z_{2,n}$ with $n = 2$, 3.

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