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Nakanishi-Lautrup B-Field, Crossed Product & Duality*

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Abstract

By re-examining the Nakanishi-Lautrup formalism of abelian gauge theory, we clarify the following fact: while the longitudinal photons or unphysical Goldstone bosons in the Higgs mechanism are eliminated from the physical space of states in the usual formulation, this statement applies to the above modes *only in their particle forms.* In their non-particle forms, the former appears physically as the infrared Coulomb tails and the latter as the so-called "macroscopic wave functions" arising from the Cooper pairs, both of which play essential physical roles.

1 Nakanishi-Lautrup formalism and its basic ingredients

Before entering the discussion, we recapitulate the basic points of the Nakanishi-Lautrup formalism [1] relevant to us in the following form:

1. Second Noether theorem as the essence of local gauge invariance (see, for instance, pp.138-9 in [1]):

   Theorem 1 (Second Noether Theorem) *A Lagrangian density* $\mathcal{L} = \mathcal{L}(\varphi^A, \partial_\mu \varphi^A)$ *is invariant,* $\delta\mathcal{L} = 0$, *under an infinitesimal transformation,*

$$\delta\varphi^A = \sum_{\alpha=1}^{r} \left(G^A{}_\alpha \Lambda^\alpha(x) + T^{A\mu}{}_\alpha \partial_\mu \Lambda^\alpha(x) \right), \tag{1}$$

*Talk presented at a RIMS workshop, "Reseach on Quantum Field Theory" in May 2006 *involving arbitrary (C^2-class) functions $\Lambda^\alpha(x)$ ($\alpha = 1,\ldots,r$) iff the following three identities hold:*

$$\begin{aligned}
\partial_\mu \left(T^{A\mu}{}_\alpha \frac{\delta \mathcal{L}}{\delta \varphi^A} \right) &= G^A{}_\alpha \frac{\delta \mathcal{L}}{\delta \varphi^A} \text{ [: constraints]}, && (2)\\
\partial_\nu K^{\nu\mu}{}_\alpha + J^\mu{}_\alpha &= 0 \text{ [: Maxwell-type eqn]}, && (3)\\
K^{\mu\nu}{}_\alpha &= -K^{\nu\mu}{}_\alpha, && (4)
\end{aligned}$$

with $J^\mu{}_\alpha$ and $K^{\nu\mu}{}_\alpha$ defined by

$$\begin{aligned}
J^\mu{}_\alpha &\equiv G^A{}_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^A)} + T^{A\mu}{}_\alpha \frac{\delta \mathcal{L}}{\delta \varphi^A}, && (5)\\
K^{\nu\mu}{}_\alpha &\equiv T^{A\mu}{}_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi^A)}. && (6)
\end{aligned}$$

2. Constraints and gauge fixing: as Eq.(2) means the presence of constraints among the Euler-Lagrange equations of motion $\delta\mathcal{L}/\delta\varphi^A = 0$, we need to "solve" them to attain a non-degenerate dynamics. This can be done by introducing a gauge-fixing condition $F[A] = 0$ ($A_\mu(x)$: gauge potential) which changes the first-class constraints into the second-class.

3. Nakanishi-Lautrup formalism: in terms of the Nakanishi-Lautrup (NL for short) B-field $B(x)$, the Lorentz gauge condition $\partial_\mu A^\mu = 0$ can be generalized to the covariant linear gauges with such gauge-fixing terms as
$$\mathcal{L}_{GF} = B\partial A + \frac{\alpha}{2}B^2,$$
to be added to the gauge invariant Lagrangian density $\mathcal{L}$, which realizes a "manifestly-covariant" quantization:

   (a) ***Basic structure*** of NL formalism: the NL field $B(x)$ satisfies
   $$\begin{aligned}
   \partial A + \alpha B &= 0 \quad \text{(: gauge-fixing condition)}\\
   \Box B &= 0
   \end{aligned}$$
   and ***4-dimensional commutation relations:***
   $$\begin{aligned}
   [B(x), B(y)] &= 0,\\
   [B(x), A_\mu(y)] &= i\partial^x_\mu D(x-y),\\
   [B(x), \psi(y)] &= e\psi(y)D(x-y),
   \end{aligned}$$
   where $D(x-y)$ is the commutator function of a massless free field and ψ a matter field carrying a charge e (e.g., Dirac field).

(b) B-field as ***the generator of local gauge transformations***: the charge given by

$$Q_\Lambda := \int B(x)\overleftrightarrow{\partial_0}\Lambda(x)d^3x; \quad \Box\Lambda = 0,$$

is conserved and generates an infinite-dimensional ***abelian*** Lie group $\mathcal{G}_B$ of local gauge transformations,

$$\begin{aligned} [-iQ_\Lambda, A_\mu(x)] &= \partial_\mu\Lambda(x), \\ [-iQ_\Lambda, \psi(x)] &= -ie\Lambda(x)\psi(x), \\ [Q_{\Lambda_1}, Q_{\Lambda_2}] &= 0, \end{aligned}$$

which do not change the gauge fixing condition $\partial A + \alpha B = 0$. The algebraic action τ_Λ of the group $\mathcal{G}$ of general local gauge transformations $\Lambda \in \mathcal{G}$ on quantum fields can be formulated as:

$$\begin{aligned} \tau_\Lambda(A_\mu(x)) &= A_\mu(x) + \partial_\mu\Lambda(x), \\ \tau_\Lambda(\psi(x)) &= \exp(-ie\Lambda(x))\psi(x), \\ \tau_{\Lambda_1}\circ\tau_{\Lambda_2} &= \tau_{\Lambda_2}\circ\tau_{\Lambda_1}. \end{aligned}$$

(c) Physical states and observables: let *physical states* Φ be specified by the subsidiary condition $B^{(+)}(x)\Phi = 0$ (called Gupt-Bleuler-Nakanishi-Lautrup condition, or GBNL condition, for short) and let $\mathcal{V}_{phys}$ denote the physical subspace spanned by them,

$$\Phi \in \mathcal{V}_{phys} \Longleftrightarrow B^{(+)}(x)\Phi = 0.$$

Corresponding to this, observables $A (= A^*)$ are defined by the condition,

$$A\mathcal{V}_{phys} \subset \mathcal{V}_{phys},$$

in terms of which the standard probabilistic interpretation of quantum theory is assured in the physical subspace $\mathcal{V}_{phys}$ so that i) the longitudinal photons A_L with negative "norms" are excluded from $\mathcal{V}_{phys}$ owing to $[B(x), A_L(y)] \neq 0$, and also the "scalar photons" B are invisible because of their null probabilities, as a result of which only transverse photons with two polarization modes remain in the physical world (kinematical "confinement") described by the Hilbert space $H_{phys} := \overline{\mathcal{V}_{phys}/\mathcal{V}_0}$ where $\mathcal{V}_0 := \mathcal{V}_{phys} \cap \mathcal{V}_{phys}^\perp$, and that
ii) in the Higgs phase with the global gauge symmetry broken spontaneously, the Goldstone bosons χ (which exist consistently with the Goldstone theorem) are excluded from the physical world as unphysical modes owing to $[B, \chi] \neq 0$ (as is consistent with such an informal expression that the Goldstone boson is "eaten" by the massive vector meson as the longitudinal component).

4. Some "elementary" questions regarded as "already settled":

   α) why should the ***gauge potential*** $A_\mu(x)$ be introduced?

   β) while Goldstone bosons are interpreted as "kinematically confined in the Higgs phase", aren't the ***Cooper pair condensates*** responsible for the ***superconductivity as a Higgs phenomenon*** nothing but the Goldstone modes surviving and even ***"visible" in the physical world*** in the form of ***"macroscopic wave functions"***? The longitudinal photons also seem to be ***"visible" as Coulomb tails*** in such ***macroscopic phenomena*** related with ***infrared divergence*** as spontaneous breakdown of Lorentz invariance in charged sectors or "infra-particles", etc. How should these points be properly understood?

   Before entering the detailed arguments, we note that the above points are interrelated closely with each other in the following way:

   α') for the microscopic description of the electric current j_μ (e.g., $= e\bar{\psi}\gamma_\mu\psi$), non-observable charged fields ψ are required;

   α'') to describe the minimal coupling $-j^\mu A_\mu$ of ψ with the electromagnetic field and the Aharonov-Bohm effect, the gauge potential A_μ is necessary.

   For these reasons α') and α''), it is usually believed that "the quantum-theoretical description of electromagnetic phenomena is impossible in terms of such gauge invariant observables only as the field strength $F_{\mu\nu}$ and the electric current j_μ".

On the basis of the ***NL B-field as the generator of*** (a subgroup $\mathcal{G}_B$) ***local gauge transformations***, we re-examine, in what follows, the above points, α) – α''), from the viewpoint of ***crossed products*** to describe the ***duality*** of groups and their actions as the mathematical basis of what I call "Micro-Macro duality" ([2, 3]).

The conclusions drawn from the analysis can be summarized in advance as follows:

A) the gauge-dependent unobservable matter field ψ in α') ***need not*** be introduced [: by the ***"Behind-the-Moon" argument*** in the context of ***Micro-Macro duality***], because its essential role can be seen simply in creating charged *states* from chargeless *states* which can be taken care of by the gauge-invariant bilinear forms of ψ;

B) in sharp contrast to the microscopic contexts focusing on ***"particle modes"***, we have, at such macroscopic levels as β), the ***Coulomb tails*** or ***Cooper pairs*** as infinitely accumulated ***longitudinal photons*** or ***unphysical" Goldstone modes***, respectively, in the physically ***visible*** form of ***"non-particle condensates"***. While the essential contents of the former can be reduced, because of A), to the gauge-invariant structure described by $F_{\mu\nu}$, j_μ, the physical reasons for the gauge structure of A_μ to be required behind the gauge-invariant $F_{\mu\nu}$ should now be found in this sort of macroscopic physical effects (mathematically realized at the level of representations and states), contrary to the standard belief [: to be described by a co-action and duality in a crossed product].

C) the minimal coupling term $-j^\mu A_\mu$ in $\alpha) = \alpha''$) can also be reformulated into such an expression as involving only $F_{\mu\nu}$, j_μ in combination with a classical variable A^c_L in the appropriate contexts of "macro-ization" processes (like the cases of Coulomb tails and of AB effects), where the presence of *classical* A^c_L does not require any indefinite inner product!

2 ψ from j_μ by "Behind-the-Moon" argument in Micro-Macro duality & crossed product

"Behind-the-Moon" argument in Micro-Macro duality provides the affirmative answer to the question "***Can gauge-dependent quantities and structures be described solely in terms of gauge invariant quantities?***"

A) ⟪charged fields ψ need not be introduced⟫ since they can be recovered from gauge invariants:
The physical role played by the charged fields ψ in QED is essentially to describe such state changes as changing the charges carried by the states (e.g. from a chargeless state to a charged state) in terms of ***state vectors***. For this purpose, charged fields ψ or certain unitary operators V_ψ: $\Psi_2 = V_\psi \Psi_1$ derived from ψ are necessary, either of which, V_ψ or ψ, is not gauge-invariant observables. When we describe the same process of state change $\Psi_1 \xrightarrow{V_\psi} \Psi_2$ in terms of ***expectation functionals***, however, this is equivalent to transforming observables A into $V_\psi^* A V_\psi$ [i.e. Heisenberg picture]:

$$\omega_{\Psi_2}(A) = \langle \Psi_2, A\Psi_2 \rangle = \langle V_\psi \Psi_1, AV_\psi \Psi_1 \rangle = \langle \Psi_1, V_\psi^* AV_\psi \Psi_1 \rangle = \omega_{\Psi_1}(V_\psi^* AV_\psi).$$

In contrast to the gauge-non-invariant treatment of V_ψ acting on state vectors, such a change $A \longmapsto V_\psi^* AV_\psi$ is meaningful as such an action on the ***observable algebra*** $\mathfrak{A}$ that a ***gauge-invariant observable*** A is transformed into ***another gauge-invariant observable*** $V_\psi^* AV_\psi$. This is just an important ***change of the vocabulary due to the level change of description.***

Moreover, if the "square-root" of this operation $V_\psi^*(-)V_\psi$ is somehow extracted, then charged sectors can directly be described also in the state vector space, at which point one of the essential roles of the "crossed product" can be found. This derivation $j_\mu \Longrightarrow \psi$ of a charged field ψ from the chargeless current j_μ provides, at the same time, the affirmative answer to the question as to how ***fermions*** can be described in terms of ***bosonic*** quantities.

The essence of the problem here can be seen as follows:

1) if one wants to treat everything in terms of state vectors, the use of such gauge-non-invariant unobservables as ψ is inevitable;

2) in view of the complementary roles played by the (algebra of) physical observables and by the states (understood as expectation functionals) in duality, however, it is enough to restrict the physical quantities to be measured to those belonging to the gauge-invariant observable algebra $\mathfrak{A}$, in terms of which all the remaining aspects can be described as the ***changes of states and representations*** of $\mathfrak{A}$ (according to the charge configurations);

3) to go back from 2) to 1), we need to recover the field algebra $\mathfrak{F}$ of gauge-dependent quantities from the gauge-invariant observable algebra $\mathfrak{A}$. The mathematical mechanism for solving such an "inverse problem" can be found in the Galois extension based on the crossed product of $\mathfrak{A}$ with the ***co-actions*** $\hat{\tau}$ of the group duals $\hat{G}$ and/or $\widehat{\mathcal{G}_B}$ given by the character groups, respectively, of the global gauge group $G = U(1)$ and the corresponding infinite-dimensional group $\mathcal{G}_B = \{e^{iB_\Lambda}; B_\Lambda = \int B(x)\overleftrightarrow{\partial_0}\Lambda(x)$ with $\Box\Lambda = 0\}$ of local gauge transformations: $\mathfrak{F} \overset{??}{=} \mathfrak{A} \rtimes_{\hat{\tau}} \widehat{G}$ or $\mathfrak{F} \overset{??}{=} \mathfrak{A} \rtimes_{\hat{\tau}} \widehat{\mathcal{G}_B}$. The essence of such a crossed product as $\mathfrak{A} \rtimes_{\hat{\tau}} \widehat{G}$ is just a composite algebra containing both $\mathfrak{A}$ and $\widehat{G}$ preserving such a commutation relation as $(A_1, \gamma_1)\cdot(A_2, \gamma_2) = (A_1 \hat{\tau}_{\gamma_1}(A_2), \gamma_1\gamma_2)$ for $A_1, A_2 \in \mathfrak{A}$, $\gamma_1, \gamma_2 \in \widehat{G}$. While the elements $A \in \mathfrak{A} = \mathfrak{F}^G$ are invariant under G, the second component γ in (A, γ) is transformed by G, according to which the behaviours of the field algebra $\mathfrak{F}$ is recovered by $\mathfrak{A} \rtimes_{\hat{\tau}} \widehat{G}$. In this way, the former choice $\mathfrak{F} = \mathfrak{A} \rtimes_{\hat{\tau}} \widehat{G}$ satisfactorily explains the mathematical mechanism for recovering the matter field ψ from the gauge invariant j_μ by the above "Behind-the-Moon" argument. If the latter choice $\mathfrak{F} \overset{??}{=} \mathfrak{A} \rtimes_{\hat{\tau}} \widehat{\mathcal{G}_B}$ is necessary (to control the relation between A_μ and $F_{\mu\nu}$), the problem becomes difficult and is not completely solved yet, because of the mathematical difficulty caused by the infinite-dimensionality of the group $\mathcal{G}_B$ of local gauge transformations. If we take into account properly the level differences between the relevant microscopic and macroscopic aspects, however, we can avoid such a technical difficulty as above related to the infinite-dimensional $\mathcal{G}_B$ as seen below.

3 From gauge-invariant $F_{\mu\nu}$ to gauge potential A_μ?

If it were necessary to recover the microscopic quantum gauge field A_μ from the gauge-invariant field strength $F_{\mu\nu}$, the problem would be mathematically difficult as mentioned above. As we saw in Sec.1, however, we can eventually avoid to treat such unphysical modes as the longitudinal photon A_L or the unphysical Goldstone boson χ as far as their *microscopic particle-excitation* *modes* are concerned. On the contrary, it is just on the *macroscopic side* that we actually need the gauge dependent $\hat{A}_\mu$, and hence, the co-action of $\widehat{\mathcal{G}_B}$ should be provided by the *macroscopic classical* field A_L^c, according to which we can show that

B) β): the longitudinal photons and Goldstone mode in the Higgs phase are "physical" in macroscopic non-particle modes!

To see the relevant logical structure, it is crucial to distinguish between two versions of gauge transformations, one in the algebraic version and the other at the operator level, and to understand the contrast of different roles played by the quantum and classical components in the longitudinal photon $\hat{A}_L = A_L^q + A_L^c$:

(1) under the *algebraic* gauge transformation τ_Λ, (both quantum and classical components of) the longitudinal modes A_L and the Cooper pair χ are non-invariant:

$$\tau_\Lambda(A_L) = (\hat{A}_\mu + \partial_\mu\Lambda)_L \neq A_L.$$

Because of the 4-dimensional commutation relation, $[B(x), A_\mu(y)] = i\partial_\mu^x D(x-y)$, mentioned at the beginning, A_L is a ***dual*** quantity $\in \widehat{\mathcal{G}_B}$ satisfying the canonical commutation relation:

$$[iQ_\Lambda, A_L(x)] = -\Lambda(x)1,$$

with the abelian group $\mathcal{G}_B$ (of local gauge transformations fixing the gauge condition).

$\Longrightarrow$ Essentially by the Fourier duality (in an infinite-dimensional Heisenberg group), A_μ can be recovered from the gauge-invariant $F_{\mu\nu}$ and A_L by the method of crossed product based upon a ***co-action*** of $\widehat{\mathcal{G}_B}$ on the gauge-invariant observable algebra, whose general essence can be simplified very much owing to the classical nature of A_L^c;

(2) owing to the trivial commutativity $[Q_\Lambda, A_L^c] = 0$ with the gauge transformation at the ***operator level***, the ***condensed classical component*** A_L^c as an order parameter is a ***physical mode*** without causing any problem of negative metric, whereas the corresponding quantum one A_L^q (as particle mode) is ***unphysical***: $[B(x), A_L^q(y)] \neq 0$ (as a relation in the indefinite inner product space). The same contrast is seen also between the classical Cooper pair χ^c and its confined quantum component χ^q;

(3) thus, the gauge-non-invariant $\hat{A}_\mu$ can safely be formulated in the Hilbert space with a positive definite inner product in such a form as $\hat{A}_\mu = \hat{A}_\mu^{transverse} + A_L^c$, where $\hat{A}_\mu^{transverse}$ is the quantum part of $\hat{A}_\mu$ reducing to the transverse modes in the limit of asymptotic states and A_L^c denotes the *classical* longitudinal mode [$\because$ an algebraic version of "Coulomb gauge"?].

Here the mutual relation between particle modes and condensates in non-particle modes can be understood naturally in parallel with the situations encountered in the representations of non-compact groups such as the Lorentz group; while the appearance of indefinite inner products is unavoidable in the representations with *finite multiplets*, one can attain unitary representations with positive-definite inner products if the representation Hilbert spaces are allowed to be *infinite-dimensional*. The former case corresponds to the situations with *particle modes*, and the latter to those with *non-particle modes*. This kind of contrast arises from the level differences of the levels of our focus points at which the group $\mathcal{G}_B$ of local gauge transformations and its dual $\widehat{\mathcal{G}_B}$ are treated: while the question as to whether a quantity A is gauge invariant or not should be answered by its behaviour under the *algebraic* gauge transformation τ_Λ, $\tau_\Lambda(A) = A$ or not, the problem as to whether A is *physical or not* in a given situation should be judged by means of the gauge transformation at the ***operator level***, $[Q_\Lambda, A] = 0$ or not, in each relevant representation. In spite of their gauge dependence, the Coulomb tail A_L^c and the Cooper pairs χ^c as c-number condensates become physical quantities owing this commutativity without the necessity of indefinite inner products. Thus, if the variables in $\widehat{\mathcal{G}_B}$ such as the field $A_L(x)$ appear in particles modes, their non-commutativity with $B(x)$ requires an indefinite inner product, whose negative-norm contributions are already known to be kinematically confined. In contrast, the condensation of such unphysical modes as A_L^c or χ^c occurs in the sectors totally disjoint to the particle-like sectors.

In the general situation, the application of the Fourier duality in the above will require us to use the white-noise fields [4] for treating the infinite-dimensional Heisenberg group in the absence of Haar measures on it, but, in the present context, however, it can be avoided owing to the above mechanism. Even if the above conclusion (3) may appear, at first sight, to repeat simply the standard discussion in the heuristic non-covariant formulations, the mathematical and conceptual meanings are, therefore, quite different: here, the covariant formalism describes the microscopic quantum level in the fibres and the non-covariant formalism appears at the level of a total bundle space providing a unified description of quantum and classical aspects.

C) Treatment of the minimal coupling $-j^\mu A_\mu$:

A_μ in the coupling term $-j^\mu A_\mu$ of $\alpha) = \alpha'')$ as in the Aharonov-Bohm effect can be described in terms of $F_{\mu\nu}$, j_μ and the *classical* A_L^c (without involving negative metric) when the relevant contexts of "macro-ization" are suitably taken into account. In fact, this term can be reformulated as

$$
\begin{aligned}
-\int j^\mu A_\mu d^4x &= \int [\frac{1}{2}F_{\nu\mu}F^{\nu\mu} + \partial_\nu(F^{\nu\mu}A_\mu)]d^4x \quad (\text{in } H_{phys} = \mathcal{V}_{phys}/\mathcal{V}_0), \\
\text{or} \quad &= \int [\frac{1}{2}F_{\nu\mu}F^{\nu\mu} - B\partial A + \partial_\nu(F^{\nu\mu}A_\mu + BA^\nu)]d^4x \quad (\text{in } \mathcal{V}),
\end{aligned}
$$

which is gauge invariant except for the coboundary term $\int \partial^{\nu}(F_{\nu\mu}A^{\mu} + BA^{\nu})d^4x = \int(F_{\nu\mu}A^{\mu} + BA^{\nu})dS^{\nu}$. This last term can have macroscopic "topological" contributions only on the sphere at the infinity where A^{μ} can be replaced by the classical Coulomb tail A_L^c in such contexts as Aharonov-Bohm effect, Berry phase, and Coulomb tails.

Finally, along the present line of thoughts based upon the duality of $\mathcal{G}$ and $\widehat{\mathcal{G}}$, we can reformulate the intrinsic problem to any gauge theories between the gauge constraints on the dynamics and the introduction of gauge-fixing conditions to resolve it at the cost of breaking gauge invariance, which will shed new lights on the spontaneous breakdown of Lorentz invariance due to the Coulomb tails and on the mutual relation between the (inhomogeneous) Cooper pair condensates and the Meissner effect. This will be discussed elsewhere.

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