数理解析研究所講究録
1526 巻 2006 年 155-163

ERM-SNCP モデリングと不確実性下での交通均衡問題
ERM-SNCP Model for Traffic Equilibrium under Uncertainty

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1 Introduction

In this paper, we consider the *stochastic nonlinear complementarity problem*

$$x \geq 0,\ F(x,\omega) \geq 0,\ x^T F(x,\omega) = 0, \quad \omega \in \Omega, \tag{1.1}$$

where $\omega \in \Omega \subseteq R^m$ is a random vector with given probability distribution $\mathcal{P}$, and $F : R^n \times \Omega \to R^n$ is a given vector-valued function. We denote (1.1) by SNCP$(F(x,\omega))$.

When F is an affine function of x for any $\omega \in \Omega$,

$$F(x,\omega) = M(\omega)x + q(\omega), \quad \omega \in \Omega, \tag{1.2}$$

where $M(\omega) \in R^{n\times n}$ and $q(\omega) \in R^n$, the SNCP$(F(x,\omega))$ reduces to the *stochastic linear complementarity problem* (SLCP), denoted by SLCP$(M(\omega), q(\omega))$, which has been studied recently in [2, 3, 5].

The *expected value*(EV) formulation introduced in [7] and the *expected residual minimization* (ERM) introduced in [2] are two deterministic formulations for SNCP. The EV formulation is to solve a single nonlinear complementarity problem NCP$(E[F(x,\omega)])$. The ERM formulation is to minimize the expected residual of the NCP$(F(x,\omega))$ for all $\omega \in \Omega$. A version of the ERM formulation using NCP functions is to find an optimal solution of

$$\min_{x\in R^n_+} f(x) := E[\|\Phi(x,\omega)\|^2] \tag{1.3}$$

where

$$\Phi(x,\omega) = (\phi(F_1(x,\omega),x_1),\ldots,\phi(F_n(x,\omega),x_n)),$$

and $\phi : R^2 \to R$ is an NCP function, which satisfies

$$\phi(a,b) = 0 \iff a \geq 0,\ \ b \geq 0,\ \ ab = 0.$$

Many NCP functions have been studied for solving nonlinear complementarity problems [4]. In this paper, we study the ERM formulation (1.3) for SNCP with the "min" function $\phi_1(a,b) = \min(a,b)$. Let f_1 be the objective function in (1.3) which is defined by NCP function ϕ_1.

We define a *stochastic* R_0 *function* and show that $F(\cdot,\cdot)$ is a stochastic R_0 function if and only if the objective function f_1 in the ERM formulation (1.3) for the SNCP$(F(x,\omega))$ is coercive. Moreover, we model the traffic equilibrium problem (TEP) under uncertainty as SNCP and show that the involved function F is a stochastic R_0 function.

We will use the following notations. $\langle l,u\rangle$ represents the set $\{l, l+1,\dots,u\}$ for natural numbers l and u with $l < u$, $a_+ = \max(a,0)$ for any given vector a, n_S represents the number of elements in a given finite set S, and $\|\cdot\|$ refers to the Euclidean norm. Throughout the paper, we suppose the following assumption holds:

Assumption I. f is finite, continuous, and $E[\|F(x,\omega)\|^2] < \infty$ at any $x \in R^n_+$.

2 Solution set of ERM for SNCP

Lemma 2.2 in [3] guarantees that the ERM formulation for the SLCP$(M(\omega),q(\omega))$ defined by the "min" function always has a solution when Ω is composed of finite elements. However, the following example tells us that we do not have the same result for the SNCP$(F(x,\omega))$.

Example 2.1 *Let* $F(x,\omega) = (\frac{1}{2} - \frac{3}{2}\omega)e^{-\frac{\omega}{2}x} - \omega$ *where* $\omega \in \Omega = \{\omega^1,\omega^2\}$. *Here* $\omega^1 = 0$, $\omega^2 = 1$, *and* $\mathcal{P}\{\omega^1\} = \mathcal{P}\{\omega^2\} = \frac{1}{2}$.

2.1 Stochastic R_0 function

Definition 2.1 *[4] Function* $G : R^n \to R^n$ *is called an* R_0 *function on a set* $\mathcal{D}$ *if for every infinite sequence* $\{x^k\} \subseteq \mathcal{D}$ *satisfying*

$$\lim_{k\to\infty} \|x^k\| = \infty,\ \limsup_{k\to\infty} \|(-x^k)_+\| < \infty,\ \limsup_{k\to\infty} \|(-G(x^k))_+\| < \infty, \tag{2.1}$$

there exists $i \in \langle 1,n\rangle$ *such that* $\limsup_{k\to\infty} \min(x^k_i, G_i(x^k)) = \infty$.

Now we define a stochastic R_0 function.

Definition 2.2 $F(\cdot,\cdot) : R^n \times \Omega \to R^n$ *is called a stochastic* R_0 *function on a set* $\mathcal{D}$ *if for every infinite sequence* $\{x^k\} \subseteq \mathcal{D}$ *satisfying*

$$\lim_{k\to\infty} \|x^k\| = \infty,\ \limsup_{k\to\infty} \|(-x^k)_+\| < \infty,\ \limsup_{k\to\infty} \|(-F(x^k,\omega))_+\| < \infty \ \ a.e. \tag{2.2}$$

there exists $i \in \langle 1,n\rangle$ *such that* $\mathcal{P}\{\omega\ :\ \limsup_{k\to\infty} \min(x^k_i, F_i(x^k,\omega)) = \infty\} > 0$.

The following example shows that for a stochastic R_0 function F, it is not necessary to have that $F(\cdot,\tilde{\omega})$ is an R_0 function for some $\tilde{\omega} \in \mathrm{supp}\Omega$. Moreover, $E[F(\cdot,\omega)]$ is not necessary to be an R_0 function.

Example 2.2 *Given a function $F : R^3 \times \Omega \to R^3$ as*

$$F(x,\omega) = \Big((-\omega)_+ e^{x_1},\ \omega_+ e^{x_2},\ \ sign(\omega)x_3\Big)^T,$$

where ω is uniformly distributed in $\Omega = [-1, 1]$.

Proposition 2.1 *Let F be an affine function of x for any $\omega \in \Omega$ defined by (1.2). Then $F(\cdot,\cdot)$ is a stochastic R_0 function on R^n_+ if and only if $M(\cdot)$ is a stochastic R_0 matrix.*

Theorem 2.1 *f_1 is coercive on a set $\mathcal{D} \subseteq R^n$ if and only if $F(\cdot,\cdot)$ is a stochastic R_0 function on $\mathcal{D}$.*

3 ERM-SNCP model for TEP under uncertainty

Let $[\mathcal{N}, \mathcal{A}]$ represent a given transportation network , where $\mathcal{N}$ is the set of nodes, and $\mathcal{A}$ is the set of links. We use $\Omega \subseteq R^m$ to represent a set of random vectors. Each vector $\omega \in \Omega$, corresponding one realization of stochastic factors such as weather, accidents, etc., is of given probability $\mathcal{P}$. For any realization $\omega \in \Omega$, let us denote

$\mathcal{I}$	the set of origin-destination (OD) pairs
$\mathcal{R}_i$	the set of "available" routes, connecting OD pair i (which might, but not necessarily be all paths joining the OD pair)
$h_r(\omega)$	the flow on route r
Δ	the link-route incidence matrix of the network
Γ	the OD pair-route incidence matrix of the network
$u_i(\omega)$	the shortest travel cost function for OD pair i,
$d_i(\omega)$	the demand functon for OD pair i
$C_r(h,\omega)$	the travel cost function for route r

Moreover, we use $\mathcal{R}$ to represent the set $\{\mathcal{R}_i, i \in \mathcal{I}\}$, and $u(\omega)$, $d(\omega)$, $h(\omega)$, $C(h,\omega)$ to represent the vector composed of $u_i(\omega)$, $d_i(\omega)$, $h_r(\omega)$, $C_r(h,\omega)$ for $i \in \mathcal{I}, r \in \mathcal{R}$, respectively. It is clear that

$$u \text{ and } d : \Omega \to R^{n_\mathcal{I}}_+; \qquad h : \Omega \to R^{n_\mathcal{R}}_+; \qquad C : R^{n_\mathcal{R}}_+ \times \Omega \to R^{n_\mathcal{R}}_+.$$

Here, we suppose the uncertain demand $d(\omega)$ is bounded for almost all $\omega \in \Omega$. We say that the network $[\mathcal{N}, \mathcal{A}]$ is *strongly connected* if for any OD pair $i \in \mathcal{I}$ there is at least one route joining the origin to the destination. Then each row of Γ is nonzero vector. Moreover, since one route connects only one OD pair, Γ has full row-rank.

In a congestion network, drivers have the incentive to compete with each other to select the route of minimal travel cost, at a certain level of travel demand. The Wardrop equilibrium principle [9] states that in the equilibrium state, for any OD pair the travel cost on every used routes equals and any route needs higher travel cost will have no traffic flow. Application of the Wardrop equilibrium for the realization $\omega \in \Omega$ gives

$$\begin{aligned} &C_r(h(\omega),\omega) - u_i(\omega) \geq 0,\ h_r(\omega) \geq 0, \\ &(C_r(h(\omega),\omega) - u_i(\omega))^T h_r(\omega) = 0, \quad i \in \mathcal{I}, r \in \mathcal{R}_i. \end{aligned} \tag{3.1}$$

Moreover, according to the demand conservation, we have $\sum_{r\in\mathcal{R}_i} h_r(\omega) - d_i(\omega) = 0$, which is equivalent to

$$\begin{aligned} &\sum_{r\in\mathcal{R}_i} h_r(\omega) - d_i(\omega) \geq 0,\ u_i(\omega) \geq 0, \\ &(\sum_{r\in\mathcal{R}_i} h_r(\omega) - d_i(\omega))^T u_i(\omega) = 0, \quad i \in \mathcal{I}, r \in \mathcal{R}_i, \end{aligned} \tag{3.2}$$

under some mild assumptions that would be expected to meet always in practice[1]. (3.1)-(3.2) is the NCP formulation of static TEP [1, 6] for each fixed $\omega \in \Omega$. Combining (3.1)-(3.2) with random factors $\omega \in \Omega$, we can reformulate TEP under uncertainty as SNCP

$$x \geq 0,\ F(x,\omega) \geq 0,\ x^T F(x,\omega) = 0, \quad \omega \in \Omega, \tag{3.3}$$

where

$$x = \begin{pmatrix} h \\ u \end{pmatrix}, \quad F(x,\omega) = \begin{pmatrix} C(h,\omega) - \Gamma^T u \\ \Gamma h - d(\omega) \end{pmatrix}. \tag{3.4}$$

In general, we can not find a vector $x = (h,u)$ which is the equilibria for any random vector $\omega \in \Omega$. We have to consider a deterministic formulation of (3.3) such as EV or ERM. The dimension of the vector x is $n = n_{\mathcal{R}} + n_{\mathcal{A}}$, where $n_{\mathcal{R}}$ is the number of routes in the set $\mathcal{R}$ and $n_{\mathcal{A}}$ is the number of arcs in the set $\mathcal{A}$. Note that a solution $x^* = (h^*, u^*)$ of a deterministic formulation of SNCP is in general different from $x_\omega = (h(\omega), u(\omega))$ of (3.2) for any fixed realization of random variable $\omega \in \Omega$. In fact, we take subvector h^* as the flow pattern for TEP under uncertainty and u^* as the average minimum travel cost vector in some sense.

In what follows, we let v_a be the travel flow on link a, and v be the link travel flow vector with components $v_a, a \in \mathcal{A}$. We make use of the function $t_a(v,\omega)$ to denote the travel time on link a, and $t(v,\omega)$ to be the link travel time vector with components $t_a(v,\omega)$, $a \in \mathcal{A}$. Clearly, the link travel flow vector v and the route travel flow vector h have the following relationship,

$$v = \Delta h. \tag{3.5}$$

It is pointed out in reference [6] that in many cases the travel cost function is *nonadditive*, which may rise from a variety of transportation polices, nonlinear valuation of travel time, etc. In this paper, we add random factors ω to the general nonadditive travel cost function suggested in [6] as

$$C(h,\omega) = \eta_1 \Delta^T t(\Delta h, \omega) + g(\Delta^T t_a(\Delta h, \omega)) + \Lambda(h,\omega), \tag{3.6}$$

where $\eta_1 > 0$ is the time-based operating costs factor, $g : R_+^{n_\mathcal{R}} \times \Omega \to R_+^{n_\mathcal{R}}$ is the perturbed translation function converting time t to money, and Λ is the perturbed financial cost function (e.g., distance-based operating costs such as maintenance). We call (3.6) the perturbed general nonadditive travel cost function on route r.

Assumption II. There exists a subset $\hat{\Omega} \subseteq \Omega$ with $\mathcal{P}\{\hat{\Omega}\} > 0$, such that for any $\omega \in \hat{\Omega}$,

(i) the travel cost function $C_r(h,\omega)$ on each route is a nondecreasing function of flow h, and finite for any fixed h;

(ii) the travel time function $t_a(v,\omega)$ on each link is a nondecreasing function of flow v, finite for any fixed v, and coercive with flow on this link v_a, i.e., $t_a(v,\omega) \to \infty$ if $v_a \to \infty$.

Assumption II holds in various perturbed travel cost and travel time functions that might be used in practice.

Proposition 3.1 *Suppose the network $[\mathcal{N},\mathcal{A}]$ is strongly connected, and Assumption II holds, then $F(\cdot,\cdot)$ in (3.4) is a stochastic R_0 function on R_+^n.*

4 Evaluation of ERM-SNCP model for TEP under uncertainty

The reliability concerns the safety of the feasible pattern, that is, its capacity of dealing with the perturbed traffic demand. Clearly, the reliability of a feasible flow pattern h with a tolerance $\epsilon \geq 0$ can be measured by

$$rel_\epsilon(h) := \mathcal{P}\{\omega \; : \; \Gamma h - d(\omega) \geq -\epsilon\}. \tag{4.1}$$

Notice that $\Gamma h - d(\omega) \geq 0$ manifests that all the demand can be delivered in the traffic flow pattern h.

Moreover, for any feasible flow pattern h, the average ratio of the possible delivered demand to the total demand of the system manifests the reliability of a feasible flow pattern, which is given by

$$dr(h) := E[\frac{1}{n_\mathcal{I}} \sum_{i \in \mathcal{I}} \frac{\min((\Gamma h)_i, d_i(\omega))}{d_i(\omega)}]. \tag{4.2}$$

Clearly $0 \leq dr(h) \leq 1$ and the nearer $dr(h)$ is to 1, the more reliability the solution earns in practice.

Guaranteed a certain degree of reliability, the fairness each individual user feels is especially desirable in the state of equilibrium. If serious unfairness occurs, the user who spends longer travel cost would change his decision to get rid of such unfairness. For each fixed $\omega \in \Omega$, the essence of Wardrop equilibria is the fairness to all users with the same OD pair. However, in the case involved uncertain factors, the travel cost for any feasible flow pattern of each route connecting the same OD pair is not necessarily the same. For a fixed $\omega \in \Omega$, the *unfairness* of a feasible flow pattern for an OD pair $i \in \mathcal{I}$ [8] is measured by

$$C_i^{\text{unfair}}(h, \omega) = \frac{C_i^{\max}(h, \omega)}{C_i^{\min}(h, \omega)}$$

where $C_i^{\max}(h, \omega)$ and $C_i^{\min}(h, \omega)$ are the largest and smallest travel cost of routes being used, which connecting OD pair i. Thus, the average unfairness of the decision for the whole system under uncertainty can be measured by

$$\text{unf}(h) := E[\frac{1}{n_{\mathcal{I}}} \sum_{i \in \mathcal{I}} C_i^{\text{unfair}}(h, \omega)] = E\Big[\frac{1}{n_{\mathcal{I}}} \sum_{i \in \mathcal{I}} \frac{C_i^{\max}(h, \omega)}{C_i^{\min}(h, \omega)}\Big], \tag{4.3}$$

which provides an effective measure for robustness.

In the view of administrator who mainly considers optimization of the system, the flow pattern h that leads to the smallest total travel cost

$$\text{tc}(h) := E[h^T C(h, \omega)] \tag{4.4}$$

is the best choice.

Example 4.1. The transportation network shown in Figure 1 is adopted from[10], which has 13 nodes, 19 links and 4 OD pairs ($1 \to 2$, $1 \to 3$, $4 \to 2$, $4 \to 3$), with the network characters t_a^0 and c_a^0. We suppose the simple case that the perturbed travel cost function is *additive*, defined as

$$C(h, \omega) = \Delta^T t(\Delta h, \omega), \quad \omega \in \Omega,$$

where the perturbed travel time function, derived from the Bureau of Public Road link travel time function (1964), can be written as

$$t_a(v, \omega) := t_a^0 \left(1 + 0.15\Big(\frac{v_a}{c_a(\omega)}\Big)^4\right), \quad a \in \mathcal{A}. \tag{4.5}$$

Here $t_a^0 > 0$ is the travel time in the network without congestion, and $c_a(\omega) \geq 0$ represents perturbed link capacity with $\mathcal{P}\{\omega \ : \ c_a(\omega) > 0\} > 0$ for all $a \in \mathcal{A}$.

Case 1. Suppose $c(\omega) \equiv c^0$, and $d(\omega) = \omega = (\omega_1, \omega_2, \omega_3, \omega_4)$ where $\omega_1, \omega_2, \omega_3, \omega_4$ follow the independent truncated normal distributions, respectively:

$$\omega_1 \sim 300 \leq N(400, 2500) \leq 500, \quad \omega_2 \sim 600 \leq N(800, 2500) \leq 1000,$$
$$\omega_3 \sim 400 \leq N(600, 2500) \leq 800, \quad \omega_4 \sim 100 \leq N(200, 900) \leq 300.$$

Case 2. Based on Case 1, we suppose some great changes of capacity of the link for $a = 5$ may happen due to the weather and road condition, as

$$\mathcal{P}\{\omega : c_5(\omega) \equiv \frac{1}{4}c_5^0\} = \frac{1}{2}, \qquad \mathcal{P}\{\omega : c_5(\omega) \equiv c_5^0\} = \frac{1}{2}.$$

Case 3. Based on Case 1, we extend the range of ω_1, ω_2 as

$$\omega_1 \sim 200 \leq \mathcal{N}(400, 2500) \leq 600, \quad \omega_2 \sim 400 \leq \mathcal{N}(800, 2500) \leq 1200.$$

We report computation results for the performance of solutions of the EV formulation and the ERM formulation of the SNCP to the original transportation equilibrium in Table 4.1 with parameter $\epsilon = 0$. Figures 2-4 show the travel flow pattern of the ERM-SNCP model for the three cases, respectively. Here the width of each link in Figures 2-4 is proportional to the amount of travel flow on that link.

Preliminary numerical results of traffic equilibrium problems under uncertainty indicate that the flow pattern drawing from a solution x_{ERM} of the ERM formulation has high reliability and delivered rate.

Table 4.1 Reliability, unfairness and total travel time of x_{EV} and x_{ERM}

x_{EV}	Case 1	Case 2	Case 3
Reliability $rel_\epsilon(h)$	0.0626	0.0626	0.0625
Delivered rate $\mathrm{dr}(h)$	93.24%	93.24%	91.20%
Total travel cost $\mathrm{tc}(h)$	7.93e+4	8.47e+4	7.93e+4
Unfairness $\mathrm{unf}(h)$	1.25	1.56	1.25
x_{ERM}	Case 1	Case 2	Case 3
Reliability $rel_\epsilon(h)$	0.5301	0.4602	0.5436
Delivered rate $\mathrm{dr}(h)$	99.42%	99.18%	99.28%
Total travel cost $\mathrm{tc}(h)$	1.09e+5	1.20e+5	1.35e+5
Unfairness $\mathrm{unf}(h)$	1.38	1.71	1.47

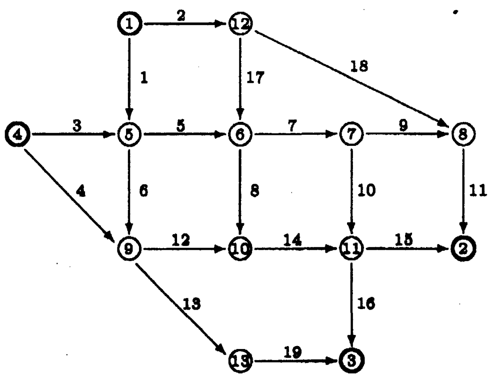

Figure 1: An example network

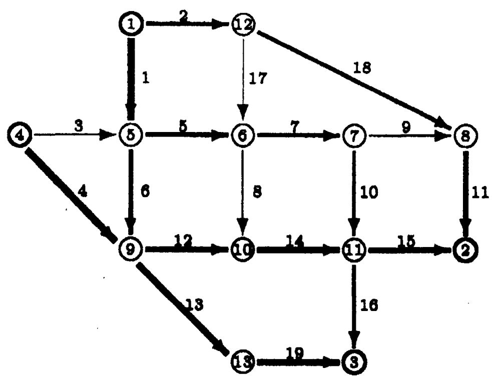

Figure 2: Travel flow pattern of ERM-SNCP in case 1

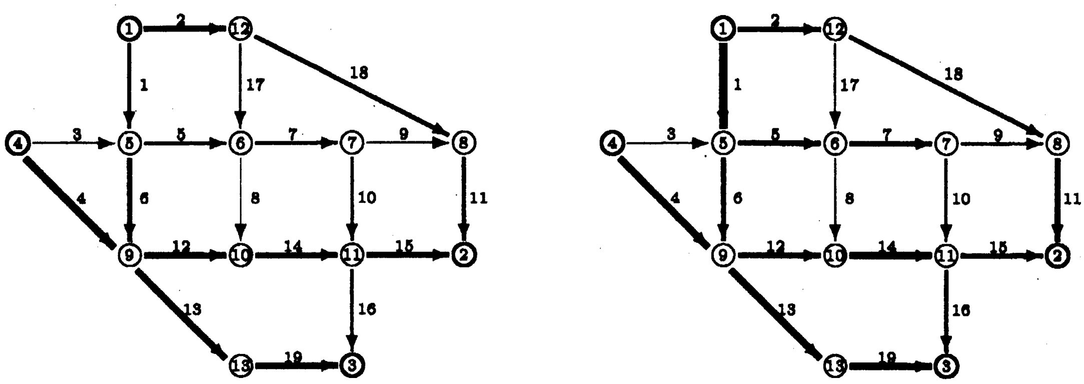

Figure 3: Travel flow pattern of ERM-SNCP in case 2 (left) and in case 3 (right)

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