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Decay Rates of the Derivatives of the Solutions of the Heat Equations and Related Topics

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1 Introduction

In this apaer, we consider the initial-boundary value problem of the heat equation in the exterior domain of a ball,

$$(1.1)\qquad \begin{cases} \dfrac{\partial}{\partial t}u = \Delta u - V(|x|)u & \text{in} \quad \Omega_L \times (0,\infty), \\ \mu u + (1-\mu)\dfrac{\partial}{\partial \nu}u = 0 & \text{on} \quad \partial\Omega_L \times (0,\infty), \\ u(\cdot,0) = \phi(\cdot) \in L^p(\Omega_L), \end{cases}$$

where $0 \le \mu \le 1$, $p \ge 1$, $\Omega_L = \{x \in \mathbf{R}^N : |x| > L\}$, $N \ge 2$, $L > 0$, and ν is the outer unit normal vector to $\partial\Omega_L$. Throughout this paper, we assume that $V = V(|x|)$ satisfies the following condition (V_ω^ℓ) for some $\omega \ge 0$ and $\ell \in \mathbf{N}$:

$$(\mathbf{V}_\omega^\ell)\qquad \begin{cases} \text{(0)} & V = V(|x|) \in C^\ell(\mathbf{R}^N),\ V \ge 0 \text{ in } \mathbf{R}^N, \\ \text{(i)} & \displaystyle\lim_{r\to\infty} r^2 V(r) = \omega, \\ \text{(ii)} & \displaystyle\int_L^\infty r \left| V(r) - \frac{\omega}{r^2} \right| dr < \infty, \\ \text{(iii)} & \displaystyle\sup_{r\ge L} \left| r^{2+j} \left(\frac{d^j}{dr^j} V \right)(r) \right| < \infty, \qquad j = 1,\dots,\ell. \end{cases}$$

The purpose of this paper is to study the decay rates of the derivatives of the solution of (1.1) under the condition (V_ω^ℓ), as $t \to \infty$.

Now, we introduce some notations. For any set A and B, let $f = f(\lambda, \nu)$ and $g = g(\lambda, \mu)$ be maps from $A \times B$ to $(0, \infty)$. Then we say

$$f(\lambda, \mu) \preceq g(\lambda, \mu) \quad \text{for all } \lambda \in A$$

if, for any $\mu \in B$, there exists a positive constant C such that $f(\lambda, \mu) \le Cg(\lambda, \mu)$ for all $\lambda \in A$. Furthermore, we say

$$f(\lambda, \mu) \asymp g(\lambda, \mu) \quad \text{for all } \lambda \in A$$

if $f(\lambda, \mu) \preceq g(\lambda, \mu)$ and $g(\lambda, \mu) \preceq f(\lambda, \mu)$ for all $\lambda \in A$. We put

$$\mathbf{N}_0 = \mathbf{N} \cup \{0\}, \qquad \mathbf{N}_0^N = \{(n_1, \dots, n_N) : n_i \in \mathbf{N}_0, i = 1, \dots, N\}.$$

Furthermore, for any $j = (j_1, \dots, j_N) \in \mathbf{N}_0^N$, we write $|j| = \sum_{i=1}^N j_i$ and $\nabla_x^j = \partial^{|j|}/\partial x_1^{j_1} \cdots \partial x_N^{j_N}$.

To state historical remarks, let Ω be an unbounded domain in $\mathbf{R}^N$. Then, under the suitable assumptions on Ω and V, for any $j \in \mathbf{N}_0^N$, the solution u of (1.1) in the domain Ω satisfies

$$\|(\nabla_x^j u)(\cdot, t)\|_{L^\infty(\Omega)} \preceq t^{-\frac{N}{2p}} \|\phi\|_{L^p(\Omega)} \tag{1.2}$$

for all sufficiently large t. (See Theorem 10.1 of Chapters 3 and 4 in [6].) On the other hand, for the case when $\Omega = \mathbf{R}^N$ (or $\Omega = \mathbf{R}_+^N$) and $V \equiv 0$, the explicit representation of the fundamental solution of the heat equation implies that, for any $j \in \mathbf{N}_0^N$,

$$\|(\nabla_x^j u)(\cdot, t)\|_{L^\infty(\mathbf{R}^N)} \preceq t^{-\frac{N}{2p} - \frac{|j|}{2}} \|\phi\|_{L^p(\mathbf{R}^N)} \tag{1.3}$$

for all $t > 0$. Furthermore, for the case when Ω is a convex domain in $\mathbf{R}^N$ and $V \equiv 0$, Li and Yau [7] studied the behavior of the nonnegative solution of (1.1) with $\mu = 0$, and obtained the inequality

$$\frac{|\nabla_x u|^2}{u^2} - \frac{\partial_t u}{u} \preceq \frac{1}{t}, \quad (x, t) \in \Omega \times (0, \infty). \tag{1.4}$$

Then, by the standard arguments in the parabolic equations, we see that, for any $j \in \mathbf{N}_0^N$ with $|j| \le 1$, the inequality (1.3) holds for all $t > 0$.

On the other hand, Grigor'yan and Saloff-Coste [2] studied the asymptotic behavior of the Green function $G_\mu^V = G_\mu^V(x, y, t)$ of (1.1) for the case when Ω is the exterior domain of a compact set, $\mu = 1$, and $V \equiv 0$. They proved that, for any fixed $x, y \in \Omega$,

$$G_1^V(x,y,t) \asymp t^{-\frac{N}{2}}$$

for all sufficiently large t if $N \geq 3$. This together with the mean value theorem, the Dirichlet boundary condition, and (1.2) implies that

$$\|(\nabla_x G_1^V)(\cdot,\cdot,t)\|_{L^\infty(\Omega\times\Omega)} \asymp t^{-\frac{N}{2}}$$

for all sufficiently large t. So we see that the solution of (1.1) with $\mu = 1$ does not necessarily satisfy the inequality (1.3) even for the case $|j| = 1$. The first author of this paper studied the asymptotic behavior of the solution of the heat equation under the Neumann boundary condition in the exterior domain of a ball in [3]. His results imply that, for the case $\mu = 0$ and $V \equiv 0$ on Ω_L, the inequality (1.3) does not necessarily hold for the case $|j| = 2$. Recently, Shibata and Shimizu [8] studied the decay properties of the Stokes semigroup in the exterior domain of a compact set, under the Neumann boundary condition. Their results are applicable to the heat equation, and we see that the inequality (1.3) holds for the case when $N \geq 3$, Ω is the exterior domain of a compact set, $V \equiv 0$ on Ω, and $\mu = 0$. Our motivation is how the decay rate is affected in the presence of V under various boundary conditions.

Let $u_\mu^V = u_\mu^V(x,t:\phi)$ be a solution of the initial-boundary value problem (1.1) in the exterior domain Ω_L. For any $p \geq 1$ and $t > 0$, put

$$\|\nabla_x^j G_\mu^V(t)\|_{p\to\infty} = \sup\left\{\|(\nabla_x^j u_\mu^V)(\cdot,t:\phi)\|_{L^\infty(\Omega_L)} \ :\ \|\phi\|_{L^p(\Omega_L)} = 1\right\},$$

where $j \in \mathbf{N}_0^N$.

Let $\Delta_{\mathbf{S}^{N-1}}$ be the Laplace-Beltrami operator on $\mathbf{S}^{N-1}$ and $\{\omega_k\}_{k=0}^\infty$ the eigenvalues of

$$-\Delta_{\mathbf{S}^{N-1}} Q = \omega Q \quad \text{on} \quad \mathbf{S}^{N-1}, \qquad Q \in L^2(\mathbf{S}^{N-1}), \tag{1.5}$$

that is,

$$\omega_k = k(N+k-2), \qquad k \in \mathbf{N}_0. \tag{1.6}$$

Furthermore, let $\{Q_{k,i}\}_{i=1}^{l_k}$ and l_k be the orthonormal system and the dimension of the eigenspace corresponding to ω_k, respectively. Let $U_{\mu,L}^V(r)$ be a solution of the initial value problem for the ordinary differential equation,

$$(O_V) \qquad \begin{cases} \partial_r^2 U + \dfrac{N-1}{r}\partial_r U - V(r)U = 0 & \text{in} \quad (L,\infty), \\ (\partial_r U)(L) = \mu, \ \ U(L) = 1-\mu, \end{cases}$$

where $0 \le \mu \le 1$. Put

$$g(t:\omega) = (1+t)^{-\frac{\alpha(\omega)}{2}}. \tag{1.7}$$

Here $\alpha = \alpha(\omega)$ is a nonnegative root of the equation $\alpha(\alpha+N-2) = \omega$, that is,

$$\alpha(\omega) = \frac{-(N-2)+\sqrt{(N-2)^2+4\omega}}{2}. \tag{1.8}$$

Then, under the condition (V_ω^1), we see that

$$g(t:\omega) \asymp [U_{\mu,L}^V(t^{1/2})]^{-1}$$

for all sufficiently large t (see Proposition 2.1).

Now, we give the main results of this paper for the case $N \ge 3$.

THEOREM **1.1** *Let $N \ge 3$ and consider the initial-boundary value problem (1.1) under the condition (V_ω^ℓ) with $\omega \ge 0$ and $\ell \in \mathbf{N}$. Let $p \ge 1$. Assume either*

$$\mu \ne \frac{2n'}{2n'+L} \quad \text{or} \quad V(r) \not\equiv \frac{\omega_{2n'}}{r^2} \quad \text{on} \quad [L,\infty) \tag{1.9}$$

for any $n' \in \mathbf{N}_0$ with $2n' \le \ell+1$. Then, for any $j \in \mathbf{N}_0^N$ with $|j| \le \ell+1$,

$$\|\nabla_x^j G_\mu^V(t)\|_{p\to\infty} \preceq t^{-\frac{N}{2p}-\frac{|j|}{2}} \quad \text{if} \quad |j| \le \alpha(\omega), \tag{1.10}$$

$$\|\nabla_x^j G_\mu^V(t)\|_{p\to\infty} \asymp t^{-\frac{N}{2p}-\frac{\alpha(\omega)}{2}} \quad \text{if} \quad |j| > \alpha(\omega) \tag{1.11}$$

for all sufficiently large t.

If, for some $n' \in \mathbf{N}_0$, the equalities hold in (1.9), we have another decay property.

THEOREM **1.2** *Let $N \ge 3$ and consider the initial-boundary value problem (1.1). Assume that there exists a natural number n' such that*

$$n = 2n', \qquad V(r) \equiv \frac{\omega_n}{r^2} \quad \text{on} \quad [L,\infty), \qquad \mu = \frac{n}{n+L}. \tag{1.12}$$

Let $p \ge 1$. Then, for any $j \in \mathbf{N}_0^N$,

$$\|\nabla_x^j G_\mu^V(t)\|_{p\to\infty} \preceq t^{-\frac{N}{2p}-\frac{|j|}{2}} \quad \text{if} \quad |j| \le n, \tag{1.13}$$

$$\|\nabla_x^j G_\mu^V(t)\|_{p\to\infty} \asymp t^{-\frac{N}{2p}-\frac{\alpha(\omega_n+\omega_1)}{2}} \quad \text{if} \quad |j| > n \tag{1.14}$$

for all sufficiently large t.

Here we remark that, under the condition (1.12), V satisfies the condition (V_ω^ℓ) for all $\ell \in \mathbf{N}$ and $\alpha(\omega) = \alpha(\omega_n) = n$. Furthermore, as a corollary of Theorems 1.1 and 1.2, we have

COROLLARY **1.1** *Let $N \geq 3$ and $u_\mu^V = u_\mu^V(x, t : \phi)$ be a solution of the initial-boundary value problem* (1.1) *with $\phi \in L^p(\Omega_L)$, under the condition (V_ω^ℓ) with $\omega \geq 0$ and $\ell \in \mathbf{N}$. Let $p \geq 1$ and $j \in \mathbf{N}_0^N$ with $|j| \leq \ell + 1$. Then there exist positive constants C and T such that*

$$\|(\nabla_x^j u_\mu^V)(\cdot, t : \phi)\|_{L^\infty(\Omega_L)} \leq C t^{-\frac{N}{2p} - \frac{|j|}{2}} \|\phi\|_{L^p(\Omega_L)}$$

for all $t \geq T$ and all $\phi \in L^p(\Omega_L)$ if and only if, either $\omega \geq \omega_{|j|}$ or

$$|j| = 1, \qquad V(r) \equiv 0 \quad on \quad [L, \infty), \qquad \mu = 0.$$

According to Corollary 1.1, we may say that results in [7] and [8] are exceptional cases.

For the decay rates of the derivatives of the solution for case $N = 2$, similar results and peculiar results are both obtained although we will not give any proofs to the results for $N = 2$.

We first consider the cases either

$$N = 2 \quad \text{and} \quad \omega > 0 \tag{1.15}$$

or

$$N = 2, \qquad \mu = 0, \quad \text{and} \quad V \equiv 0 \quad \text{on} \quad [L, \infty). \tag{1.16}$$

THEOREM **1.3** *Assume either* (1.15) *or* (1.16). *Then Theorems* 1.1 *and* 1.2 *hold true.*

Next, we consider the cases either

$$(N, \omega) = (2, 0) \quad \text{and} \quad \mu > 0 \tag{1.17}$$

or

$$(N, \omega, \mu) = (2, 0, 0) \quad \text{and} \quad V \not\equiv 0 \quad \text{on} \quad [L, \infty). \tag{1.18}$$

Then we see that

$$U_{\mu,L}^0(r) = 1 - \mu + \mu \log\left(\frac{r}{L}\right).$$

THEOREM **1.4** *Let $N = 2$ and consider the initial-boundary value problem* (1.1) *under the condition* $(\tilde{V}^{\ell}_{\omega})$ *with $\omega = 0$ and $\ell \in \mathbf{N}$. Let $p \geq 1$, and $R > L$. Assume either* (1.17) *or* (1.18). *Then, for any $j \in \mathbf{N}_0^N$ with $|j| \leq \ell + 1$,*

$$\begin{aligned} \|\nabla_x^j G_\mu^V(t)\|_{p\to\infty} &\asymp \|\nabla_x^j G_\mu^V(t)\|_{R:p\to\infty} \asymp t^{-\frac{1}{p}}(\log t)^{-1}, \\ \|r^{-|j|}\nabla_\theta^j G_\mu^V(t)\|_{p\to\infty} &\preceq t^{-\frac{1}{p}-\frac{|j|}{2}} \end{aligned}$$

for all sufficiently large t.

In Section 2, we give fundamental lemmas and propositions without proofs. For their proofs, readers consult Sections 2 and 3 of [5]. Section 3 is devoted to the large time behavior of a radial solution to (1.1) with a radial initial value and its derivatives. Upper estimates for proofs of Theorems 1.1 and 1.2 are given in Section 4 and their proofs are provided in Section 5. As concluding remarks, some related topics are stated in Section 6.

2 Preliminaries

In this section, we give preliminary lemmas, whose proofs can be seen in Section 2 of [5], in order to study the decay rates of the derivatives of the solution (1.1) for the case $N \geq 3$.

For any $\mu \in [0,1]$, $R \geq L$, and $\omega \geq 0$, let $U^\omega_{\mu,R}$ be the solution of

$$(O_\omega) \qquad \begin{cases} \partial_r^2 U + \dfrac{N-1}{r}\partial_r U - \dfrac{\omega}{r^2}U = 0 & \text{in} \quad (R,\infty), \\ (\partial_r U)(R) = \mu, \ \ U(R) = 1-\mu. \end{cases}$$

Put

$$U_+^\omega(r) = \left(\frac{r}{L}\right)^{\alpha(\omega)}, \quad U_-^\omega(r) = \left(\frac{r}{L}\right)^{-\beta(\omega)}, \tag{2.1}$$

where $\beta(\omega) = N - 2 + \alpha(\omega)$. Then the functions $U_+^\omega(r)$ and $U_-^\omega(r)$ are solutions of the ordinary differential equation

$$\partial_r^2 U + \frac{N-1}{r}\partial_r U - \frac{\omega}{r^2}U = 0 \quad \text{in} \quad (0,\infty), \tag{2.2}$$

and $U_+^\omega(r) \not\equiv U_-^\omega(r)$ on $(0,\infty)$. So, by the uniqueness of the solution of (O_ω), there exist constants c_1 and c_2 such that

$$U^\omega_{\mu,R}(r) = c_1 U_+^\omega(r) + c_2 U_-^\omega(r), \quad r \geq R.$$

Therefore, by $U^{\omega}_{\mu,R}(R) = 1-\mu$ and $\partial_r U^{\omega}_{\mu,R}(R) = \mu$, we obtain

$$U^{\omega}_{\mu,R}(r) = \frac{\alpha - \mu\alpha - R\mu}{\alpha+\beta}\left(\frac{r}{R}\right)^{-\beta} + \frac{R\mu - \beta\mu + \beta}{\alpha+\beta}\left(\frac{r}{R}\right)^{\alpha} \tag{2.3}$$

where $\alpha = \alpha(\omega)$ and $\beta = \beta(\omega)$. In what follows, we put

$$U^{\omega,k}_{\mu,R}(r) = U^{\omega+\omega_k}_{\mu,R}(r), \quad U^{\omega,k}_{+}(r) = U^{\omega+\omega_k}_{+}(r), \quad U^{\omega,k}_{-}(r) = U^{\omega+\omega_k}_{-}(r),$$

for simplicity. Then we have the following lemma on $U^{\omega}_{\mu,R}$.

LEMMA **2.1** *Let $L \le R < S$ and $a, b \ge 0$. Assume $N \ge 3$. Then*

$$U^{a,k}_{\mu,R}(r) \asymp U^{b,k}_{\mu,R}(r) \tag{2.4}$$

for all $r \in [R,S]$, $\mu \in [0,1]$, and $k \in \mathbf{N}_0$,

$$U^{a,k}_{\mu,R}(r) \asymp \left[\frac{\mu}{k+1} + 1 - \mu\right]\left(\frac{r}{R}\right)^{\alpha(a+\omega_k)} \tag{2.5}$$

for all $r \ge S$, $\mu \in [0,1]$, and $k \in \mathbf{N}_0$, and

$$U^{a,k}_{0,R}(r) \asymp U^{a,k}_{+}(r) \tag{2.6}$$

for all $r \ge R$ and $k \in \mathbf{N}_0$. Furthermore

$$0 \le \frac{d}{dr}U^{a,k}_{\mu,R}(r) \preceq \frac{\mu + (k+1)(1-\mu)}{R}\left(\frac{r}{R}\right)^{\alpha(a+\omega_k)-1}, \tag{2.7}$$

$$0 < U^{a,k}_{\mu,R}(r) \preceq \left[\frac{\mu}{k+1} + 1 - \mu\right]\left(\frac{r}{R}\right)^{\alpha(a+\omega_k)}, \tag{2.8}$$

for all $r > R$, $0 \le \mu \le 1$, and $k \in \mathbf{N}_0$.

Next we recall the following two lemmas on the decay rate of the solutions of the initial-boundary value problem (1.1) under the condition (V^{ℓ}_{ω}).

LEMMA **2.2** *Let u^V_{μ} be a solution of* (1.1) *under the condition* (V^1_{ω}) *with $\omega \ge 0$. Let $1 \le p \le q \le \infty$ and $i = 1, 2, \ldots$. Then there exists a positive constant C, independent of V, such that*

$$\|u^V_{\mu}(\cdot,t)\|_{L^q(\Omega_L)} \le Ct^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})}\|\phi\|_{L^p(\Omega_L)} \tag{2.9}$$

for all $t > 0$.

LEMMA **2.3** *Let u_μ^V be a solution of* (1.1) *under the condition* (V_ω^ℓ) *with* $\omega \ge 0$ *and* $\ell \ge 1$. *Then, for any* $\epsilon \in (0,1)$ *and* $p \ge 1$, *there exists a positive constant* C *such that*

$$|(\partial_t^i \nabla_x^j u_\mu^V)(x,t)| \le C t^{-\frac{N}{2p}-\frac{|j|}{2}-i} \|\phi\|_{L^p(\Omega_L)}, \tag{2.10}$$

for all $(x,t) \in \Omega_L \times (0,\infty)$ *with* $|x| \ge \epsilon t^{1/2} > L+2$ *and all* $i \in \mathbf{N}_0$ *and* $j \in \mathbf{N}_0^N$ *with* $2i + |j| \le \ell + 1$.

Next, we study the behavior of the solution $U_{\mu,L}^V(r)$ of (O_V) under the assumption (V_ω^ℓ). Put

$$V_k(r) = V(r) + \frac{\omega_k}{r^2}, \qquad k \in \mathbf{N}_0.$$

In what follows, for $k \in \mathbf{N}_0$ and $\lambda \in \mathbf{R}$, we put

$$\alpha_k = \alpha(\omega + \omega_k), \quad \beta_k = N - 2 + \alpha_k, \quad h_\lambda(r) = V(r) - \frac{\lambda}{r^2}$$

for simplicity. We first prove the following lemma.

LEMMA **2.4** *Let* $R \ge L$, $a \ge 0$, *and* $k \in \mathbf{N}_0$. *For any* $g \in C([R,\infty))$, *put*

$$H_R^{a,k}[g](r) = U_-^{a,k}(r) \int_R^r s^{1-N}[U_-^{a,k}(s)]^{-2} \left(\int_R^s \tau^{N-1} U_-^{a,k}(\tau) g(\tau) d\tau \right) ds.$$

Then

(i) $H_R^{a,k}[g](r)$ *is a solution of the ordinary differential equation*

$$U'' + \frac{N-1}{r} U' - \frac{a+\omega_k}{r^2} U = g \quad in \quad (R,\infty),$$

with $U(R) = U'(R) = 0$. *In particular,*

$$U_{\mu,R}^{V_k}(r) = U_{\mu,R}^{a,l}(r) + H_R^{a,l}[h_{\omega_l + a - \omega_k} U_{\mu,R}^{V_k}](r)$$

for all $r \ge R$, $k \in \mathbf{N}_0$, *and* $l = 0, \dots, k$.

(ii) *If* $g(r) \ge 0$ *on* $[R, R_1]$ *with* $R_1 > R$, *then*

$$H_R^{a,k}[g](r) \ge 0, \quad H_R^{a,k}[g]'(r) \ge 0, \quad R \le r \le R_1. \tag{2.11}$$

(iii) *Assume that there exists a positive constant* A *such that*

$$|g(r)| \le A |h_a(r)| U_{\mu,R}^{a,k}(r), \quad r \ge R. \tag{2.12}$$

Then there exist positive constants C_1 and C_2, independent of R and k, such that

$$
\begin{aligned}
(2.13)\qquad |H_R^{a,k}[g]'(r)| &\le C_1 A r^{-1} U_{\mu,R}^{a,k}(r) \int_R^r \tau |h_a(\tau)| d\tau, \\
(2.14)\qquad |H_R^{a,k}[g](r)| &\le C_2 A U_{\mu,R}^{a,k}(r) \int_R^r \tau |h_a(\tau)| d\tau
\end{aligned}
$$

for all $r \ge R$.

In view of Lemma 2.4, we have the following proposition on the behavior of $U_{\mu,L}^{V_k}(r)$ as $r \to \infty$, by using the function $U_{\mu,L}^{\omega,k}(r) = U_{\mu,L}^{\alpha(\omega+\omega_k)}(r)$.

PROPOSITION 2.1 *Assume (V_ω^1) with $\omega \ge 0$ and $N \ge 3$. Then*

$$
(2.15)\qquad 0 \le (\partial_r U_{\mu,L}^{V_k})(r) \preceq (k+1) \left(\frac{r}{L}\right)^{\alpha_k - 1}
$$

for all $r > L$, $0 \le \mu \le 1$, and $k \in \mathbf{N}_0$. Furthermore

$$
\begin{aligned}
(2.16)\qquad U_{\mu,L}^{V_k}(r) &\asymp U_{\mu,L}^{\omega,k}(r), \qquad 0 \le \mu \le 1, \\
(2.17)\qquad U_{0,L}^{V_k}(r) &\asymp U_{+}^{\omega,k}(r)
\end{aligned}
$$

for all $r \ge L$ and $k \in \mathbf{N}_0$. In particular,

$$
(2.18)\qquad U_{\mu,L}^{V_k}(r) \asymp \left[\frac{\mu}{k+1} + 1 - \mu\right] U_{+}^{\omega,k}
$$

for all sufficiently large r, $0 \le \mu \le 1$, and $k \in \mathbf{N}_0$.

Furthermore, by Proposition 2.1, we have the following proposition.

PROPOSITION 2.2 *Assume (V_ω^1) with $\omega \ge 0$ and $N \ge 3$. For any $g \in C([L,\infty))$, put*

$$
F_L^V[g](r) = U_{0,L}^V(r) \int_L^r s^{1-N} [U_{0,L}^V(s)]^{-2} \left(\int_L^s \tau^{N-1} U_{0,L}^V(\tau) g(\tau)\, d\tau \right) ds.
$$

Then, for any $k \in \mathbf{N}_0$, $F_L^{V_k}[g](r)$ is a solution of

$$
(2.19)\qquad \begin{cases} U'' + \dfrac{N-1}{r} U' - V_k(r) U = g \quad \text{in} \quad (L,\infty), \\ U(L) = U'(L) = 0. \end{cases}
$$

If there exist constants $A > 0$ *such that*

$$|g(r)| \leq AU_{0,L}^{V_k}(r), \qquad r \geq L,$$

then there exists a positive constant C*, independent of* k*, such that*

$$|F_L^{V_k}[g](r)| \leq CA(k+1)^{-1}r^2U_{0,L}^{V_k}(r), \tag{2.20}$$

$$|F_L^{V_k}[g]'(r)| \leq CArU_{0,L}^{V_k}(r), \tag{2.21}$$

for all $r \geq L$*.*

Next, we consider the case (1.12).

PROPOSITION 2.3 *Assume* (V_ω^ℓ) *with* $\omega \geq 0$ *and* $\ell \in \mathbf{N}$*. Furthermore assume that there exists a multi-index* $J \in \mathbf{N}_0^N$ *with* $|J| = n+1 \leq \ell+2$ *such that*

$$\begin{array}{l} (\nabla_x^j U_{\mu,L}^V)(|x|) \not\equiv 0 \text{ in } \Omega_L, \text{ for all } j \in \mathbf{N}_0^N \text{ with } |j| \leq n, \\ (\nabla_x^J U_{\mu,L}^V)(|x|) \equiv 0 \text{ in } \Omega_L. \end{array} \tag{2.22}$$

Then there exists a nonnegative integer n' *such that* (1.12),

$$U_{\mu,L}^V(|x|) = \frac{1-\mu}{L^n}(x_1^2 + \cdots x_N^2)^{n'} = \frac{1-\mu}{L^n}|x|^n, \qquad x \in \Omega_L, \tag{2.23}$$

and

$$(\nabla_x^j U_{\mu,L}^V)(|x|) \equiv 0 \quad in \quad \Omega_L \tag{2.24}$$

hold for all $j \in \mathbf{N}_0^N$ *with* $|j| \geq n+1$*.*

3 Derivatives of the solutions of (P_μ^k)

In this section, we consider the radial solution v of the initial-boundary value problem

$$(P_\mu^k) \qquad \begin{cases} \partial_t v = \Delta v - V_k(|x|)v & \text{in} \quad \Omega_L \times (0,\infty), \\ \mu v - (1-\mu)\partial_r v = 0 & \text{on} \quad \partial\Omega_L \times (0,\infty), \\ v(\cdot,0) = \psi(\cdot) \in L^p(\Omega_L), \end{cases}$$

where $0 \leq \mu \leq 1$, $p \geq 1$, $k \in \mathbf{N}_0$, and ψ is a radial function in Ω_L. For any positive ϵ and T, put

$$\begin{aligned} D_\epsilon(T) &= \left\{(x,t) \in \Omega_L \times (T,\infty) \,:\, |x| < \epsilon(1+t)^{1/2}\right\}, \\ \Gamma_\epsilon(T) &= \left\{(x,t) \in \Omega_L \times (T,\infty) \,:\, |x| = \epsilon(1+t)^{1/2}\right\} \\ &\qquad \cup \left\{(x,T) \,: x \in \Omega_L,\, |x| \leq \epsilon(1+T)^{1/2}\right\}. \end{aligned}$$

We will construct a super-solution of (P^k_μ) in $D_\epsilon(T)$ for some positive constants ϵ and T, and give some estimates on the derivatives of the solution v^k_μ of (P^k_μ) in $D_\epsilon(T)$. In what follows, under the assumption (V^ℓ_ω), we put

$$U_k(r) = U^{V_k}_{0,L}(r), \qquad g_k(t) = g(t : \omega + \omega_k)$$

for simplicity. We first construct a super-solution of (P^k_μ).

LEMMA **3.1** *Assume $N \geq 3$ and (V^ℓ_ω) with $\omega \geq 0$ and $k \in \mathbf{N}_0$. Let $\gamma > 0$. Then there exist positive constants T, ϵ, and C, which are independent of k, and a function $W = W(x,t)$ in $\Omega_L \times (0,\infty)$ such that*

$$\partial_t W \geq \Delta W - V_k(|x|)W \quad \textit{in} \quad D_\epsilon(T), \tag{3.1}$$

$$\mu W(x,t) + (1-\mu)\frac{\partial}{\partial \nu}W(x,t) \geq 0 \quad \textit{on} \quad \partial\Omega_L \times (T,\infty), \tag{3.2}$$

$$W(x,t) \geq C^{-\alpha_k}(1+t)^{-\gamma} \quad \textit{on} \quad \Gamma_\epsilon(T), \tag{3.3}$$

and

$$0 < W(x,t) \leq (1+t)^{-\gamma} g_k(t) U_k(|x|) \quad \textit{in} \quad D_\epsilon(T). \tag{3.4}$$

PROOF. Let A and ϵ be constants to be chosen later such that $A > 0$ and $0 < \epsilon < 1$. Let T_ϵ be a positive constant such that $\epsilon(1+T_\epsilon)^{1/2} = L + 1$. Put

$$W(x,t) = (1+t)^{-\gamma} g_k(t)\left[U_k(|x|) - A(1+k)(1+t)^{-1} F^{V_k}_L[U_k](|x|)\right]$$

for all $(x,t) \in \Omega_L \times (T_\epsilon, \infty)$. Then, there exists a constant $C_1 = C_1(\gamma)$ such that

$$\begin{aligned} \partial_t W &\geq \left[-\gamma(1+t)^{-\gamma-1} g_k(t) + (1+t)^{-\gamma} g'_k(t)\right] U_k(|x|) \\ &\geq -C_1(1+k)(1+t)^{-\gamma-1} g_k(t) U_k(|x|) \end{aligned} \tag{3.5}$$

and by (2.19), we have

$$\Delta W - V_k(|x|)W = -A(1+k)(1+t)^{-\gamma-1} g_k(t) U_k(|x|) \tag{3.6}$$

in $\Omega_L \times (T_\epsilon, \infty)$. Let $A = C_1$. Then, by (3.5) and (3.6), we have

$$\partial_t W \geq \Delta W - V_k(|x|)W \quad \text{in} \quad \Omega_L \times (T_\epsilon, \infty). \tag{3.7}$$

On the other hand, by Proposition 2.2, there exists a positive constant C_2, independent of ϵ, such that

$$0 \leq A(1+k)(1+t)^{-1} F^{V_k}_L[U_k](|x|) \leq C_2 A \epsilon U_k(|x|) \tag{3.8}$$

for all $(x,t) \in D_\epsilon(T_\epsilon)$. Let $0 < \epsilon \le \min\{1, 1/2C_2A\}$. Then we have

$$(3.9) \qquad \frac{1}{2} g_k(t)U_k(|x|) \le (1+t)^\gamma W(x,t) \le g_k(t)U_k(|x|)$$

for all $(x,t) \in D_\epsilon(T_\epsilon)$. Then, by the definition of W, we have

$$(3.10) \qquad \mu W + (1-\mu)\frac{\partial}{\partial \nu} W = \mu W \ge 0 \quad \text{on} \quad \partial\Omega_L \times (0,\infty).$$

By Proposition 2.1 and (1.7), we see that

$$(3.11)\, U_k(\epsilon(1+t)^{1/2}) \asymp U^{\omega,k}_{\mu,L}(\epsilon(1+t)^{1/2}) \asymp (k+1)^{-1}\left(\frac{\epsilon}{L}\right)^{\alpha_k}[g_k(t)]^{-1}$$

for all $t \ge T_\epsilon$ and $k \in \mathbf{N}_0$. By (3.9) and (3.11), there exists a positive constant C_3 such that

$$(3.12) \qquad \begin{aligned} (1+t)^\gamma W(x,t) &\ge \frac{1}{2} g_k(t)U_k(|x|) = \frac{1}{2} g_k(t)U_k(\epsilon(1+t)^{1/2}) \\ &\ge C_3^{-1}(k+1)^{-1}\left(\frac{\epsilon}{L}\right)^{\alpha_k} \end{aligned}$$

for all $(x,t) \in \Gamma_\epsilon(T_\epsilon)$ with $t > T_\epsilon$. Furthermore, by (2.15), (3.9), and $\epsilon(1+T_\epsilon)^{1/2} = L+1$, there exists a positive constant C_4 such that

$$(3.13) \quad \begin{aligned} W(x,T_\epsilon) &\ge \frac{1}{2}(1+T_\epsilon)^{-\gamma-\frac{\alpha_k}{2}} U_k(L) = \frac{1}{2}(1+T_\epsilon)^{-\gamma}\left(\frac{\epsilon}{L+1}\right)^{\alpha_k} \\ &\ge C_4^{-\alpha_k}(1+T_\epsilon)^{-\gamma} \end{aligned}$$

for all $(x,T_\epsilon) \in \Gamma_\epsilon(T_\epsilon)$ and $k \in \mathbf{N}_0$. By (3.7), (3.10), (3.12), and (3.13), we have (3.1)–(3.4), and the proof of Lemma 3.1 is complete. □

Next we give the following lemmas on the estimates of derivatives of v^k_μ. First, we estimate v and its time derivatives.

LEMMA **3.2** *Assume that ψ is a radial function in Ω_L such that $\|\psi\|_{L^p(\Omega_L)} = 1$ with $p \ge 1$. Let $N \ge 3$ and v be a solution of (P^k_μ) with $v(\cdot,0) = \psi(\cdot)$ under the condition (V^ℓ_ω) with $\omega \ge 0$. Put*

$$w(x,t) = F^{V_k}_L[(\partial_t v)(\cdot,t)](|x|).$$

Then there exist positive constants T, ϵ, and η, independent of k, such that

$$(3.14) \qquad |\partial^i_t v(x,t)| \le \eta^{\alpha_k} t^{-\frac{N}{2p}-i} g_k(t) U^{\omega,k}_+(|x|),$$

$$(3.15) \qquad |\partial^i_t w(x,t)| \le \eta^{\alpha_k} t^{-\frac{N}{2p}-1-i} g_k(t)|x|^2 U^{\omega,k}_+(|x|)$$

for all $(x,t) \in D_\epsilon(T)$ and all $i \in \mathbf{N}_0$ with $2i \le \ell + 1$.

PROOF. Let $i \in \mathbf{N}_0$ and put $v_i = \partial_t^i v$. Let T and ϵ be positive constants given in Lemma 3.1. Let W be the function constructed in Lemma 3.1 with $\gamma = N/2p + i$. For any $\eta_1 > 0$, we put

$$\overline{v}_i(x,t) = \eta_1^{\alpha_k} W(x,t)$$

for all $(x,t) \in D_\epsilon(T)$. Then, taking a sufficiently large T and η_1 if necessary, by Lemma 2.3, we have

$$|v_i(x,t)| \le \overline{v}_i(x,t) \qquad \text{on} \quad \Gamma_\epsilon(T).$$

So, by the comparison principle, we have

$$|v_i(x,t)| \le \overline{v}_i(x,t) \qquad \text{in} \quad D_\epsilon(T).$$

This inequality together with (2.8), (2.16), and (3.4) implies

$$|v_i(x,t)| \le \eta_1^{\alpha_k} t^{-\frac{N}{2p}-i} g_k(t) U_k(|x|) \preceq \eta_1^{\alpha_k} t^{-\frac{N}{2p}-i} g_k(t) U_+^{\omega,k}(|x|)$$

for all $(x,t) \in D_\epsilon(T)$, and we obtain the inequality (3.14). On the other hand, since

$$(\partial_t^i w)(x,t) = F_L^{V_k}[(\partial_t^{i+1} v)(\cdot,t)](|x|) \tag{3.16}$$

for all $(x,t) \in \Omega_L \times (0,\infty)$, by (2.17), (2.20) and (3.14), we have (3.15), and the proof of Lemma 3.2 is complete. □

Furthermore we have the following lemma on the time derivatives of $\partial_r v$ and $\partial_r w$.

LEMMA **3.3** *Assume the same assumptions as in Lemma 3.2. Then there exist positive constants T, η, and ϵ, independent of k, such that*

$$|\partial_t^i \partial_r v(x,t)| \le \eta^{\alpha_k} t^{-\frac{N}{2p}-i} g_k(t) |x|^{-1} U_+^{\omega,k}(|x|), \tag{3.17}$$

$$|\partial_t^i \partial_r w(x,t)| \le \eta^{\alpha_k} t^{-\frac{N}{2p}-1-i} g_k(t) |x| U_+^{\omega,k}(|x|) \tag{3.18}$$

for all $(x,t) \in D_\epsilon(T)$ and all $i \in \mathbf{N}_0$ with $2i \le \ell + 1$.

PROOF. By (2.17), (2.21), (3.14), and (3.16), we have (3.18). So we prove (3.17). Put $v_i = \partial_t^i v$ and $w_i = \partial_t^i w$. Then v_i and w_i satisfy

$$\partial_t v_i = \Delta w_i - V_k(|x|) w_i$$

by the definition of $F_L^{V_K}$. By the uniqueness of the initial value problem for the ordinary differential equation, there exists a function $\zeta(t)$ in $(0,\infty)$ such that

$$v_i(x,t) = \zeta(t) U_{\mu,L}^{V_k}(|x|) + w_i(x,t) \tag{3.19}$$

for all $(x,t) \in \Omega_L \times (0,\infty)$. Furthermore, by (2.17), (2.20), (3.14), (3.15), and (3.19), there exist constants C_1, C_2, T, η_1, and ϵ such that

$$\begin{aligned}|\zeta(t)|U_k(\epsilon(1+t)^{1/2}) &\le |v_i(x,t)|\Big|_{|x|=\epsilon(1+t)^{1/2}} + |w_i(x,t)|\Big|_{|x|=\epsilon(1+t)^{1/2}} \\ &\le C_1 t^{-\frac{N}{2p}-i} + C_2\eta_1^{\alpha_k} t^{-\frac{N}{2p}-i} g_k(t) U_+^{\omega,k}(\epsilon(1+t)^{1/2})\end{aligned}$$

for all $t \ge T$. This together with (3.11) implies that there exists a constant η_2 such that

$$|\zeta(t)| \le \eta_2^{\alpha_k} t^{-\frac{N}{2p}-i} g_k(t), \qquad t \ge T,\ k \in \mathbf{N}_0. \tag{3.20}$$

In addition, by (2.15), (3.18), and (3.19), there exists a constant η_3 such that such that

$$\begin{aligned}|(\partial_r v_i)(x,t)| &\le |\zeta(t)|(\partial_r U_{\mu,L}^{V_k})(|x|) + |\partial_r w_i(|x|,t)| \\ &\le \eta_3^{\alpha_k} t^{-\frac{N}{2p}-i} g_k(t) U_+^{\omega,k}(|x|)|x|^{-1}\end{aligned}$$

for all $(x,t) \in D_\epsilon(T)$ and $k \in \mathbf{N}_0$. So we obtain (3.17), and the proof of Lemma 3.3 is complete. □

We give upper estimates on the spatio-temporal derivatives of v and w and its proof is done in the similar way to the proofs of Lemmas 3.2 and 3.3.

LEMMA **3.4** *Assume the same assumptions as in Lemma 3.2. Then there exist positive constants T, η, and ϵ, independent of k, such that*

$$|\partial_t^i\partial_r^j v(x,t)| \le \eta^{\alpha_k} t^{-\frac{N}{2p}-i} g_k(t)|x|^{-j} U_+^{\omega,k}(|x|), \tag{3.21}$$

$$|\partial_t^i\partial_r^j w(x,t)| \le \eta^{\alpha_k} t^{-\frac{N}{2p}-1-i} g_k(t)|x|^{2-j} U_+^{\omega,k}(|x|) \tag{3.22}$$

for all $(x,t) \in D_\epsilon(T)$, $i \in \mathbf{N}_0$ with $2(i+1) \le \ell+1$, and $j = 2,\dots,\ell+2$.

Finally, we give estimates on the derivatives of v for the case (1.12).

LEMMA **3.5** *Assume that ψ is a radial function such that $\|\psi\|_{L^p(\Omega_L)} = 1$ with $p \ge 1$. Let v be the solution of (P_μ^k) with $v(\cdot,0) = \psi(\cdot)$ and $k = 0$, under the condition* (1.12). *Then, for any $j \in \mathbf{N}_0^N$ with $|j| \ge n+1$ and $i \in \mathbf{N}_0$, there exist positive constants C, T, and ϵ such that*

$$|\partial_t^i \nabla_x^j v(x,t)| \le C t^{-\frac{N}{2p}-\frac{1}{2}-i-\frac{n}{2}} \tag{3.23}$$

for all $(x,t) \in D_\epsilon(T)$.

PROOF. By (1.12), we have

$$U^V_{\mu,L}(x) = c\left(\sum_{i=1}^N x_i^2\right)^{n'}, \quad U^\omega_+(r) = \left(\frac{r}{L}\right)^n, \quad g(t:\omega) = (1+t)^{-\frac{n}{2}},$$

where $n = 2n'$ and c is a positive constant. (See also Proposition 2.3). Put $v_i(x,t) = \partial_t^i v(x,t)$ and $w_i(x,t) = F_L^V[v_{i+1}](|x|)$. Let $j \in \mathbf{N}_0^N$ with $|j| \geq n+1$. Then $\nabla_x^j U^V_{\mu,L}(|x|) \equiv 0$ in Ω_L, and by (3.19), we have $\nabla_x^j v_i(x,t) = \nabla_x^j w_i(x,t)$ for all $(x,t) \in \Omega_L \times (0,\infty)$. Therefore, by the radial symmetry of w_i and the inequality (3.22) with $k=0$, there exist positive constants T and ϵ such that

$$\begin{aligned} \left|(\nabla_x^j v_i)(x,t)\right| &\preceq \sum_{m=1}^{|j|} \frac{|(\partial_r^m w_i)(x,t)|}{|x|^{|j|-m}} \preceq t^{-\frac{N}{2p}-1-i-\frac{n}{2}}|x|^{n+2-|j|} \\ &\preceq t^{-\frac{N}{2p}-1-i-\frac{n}{2}}|x| \preceq t^{-\frac{N}{2p}-\frac{1}{2}-i-\frac{n}{2}} \end{aligned}$$

for all $(x,t) \in D_\epsilon(T)$, and the proof of lemma 3.5 is complete. □

REMARK **3.1** If the L^p-norm of the initial value is not 1, then all the right-hand terms in the estimates in Lemmas 3.2, 3.3 and 3.4 must be multiplied by $\|\psi\|_{L^p(\Omega_L)}$.

4 Upper bounds of derivatives of solutions

In this section, we prove the following two propositions, which are mentioned in Section 1 as upper estimates, by using lemmas given in the previous sections.

PROPOSITION 4.1 *Assume the same assumptions as in Theorem 1.1. Then, for any $p \geq 1$ and $j \in \mathbf{N}_0^N$ with $|j| \leq \ell + 1$,*

$$\|\nabla_x^j G_\mu^V(t)\|_{p\to\infty} \preceq t^{-\frac{N}{2p}-\frac{\min\{\alpha(\omega),|j|\}}{2}} \tag{4.1}$$

for all sufficiently large t.

PROPOSITION 4.2 *Assume the same assumptions as in Theorem 1.2. Then, for any $p \geq 1$ and $j \in \mathbf{N}_0^N$ with $|j| \geq n+1$,*

$$\|\nabla_x^j G_\mu^V(t)\|_{p\to\infty} \preceq t^{-\frac{N}{2p}-\frac{\alpha(\omega_n+\omega_1)}{2}} \tag{4.2}$$

for all sufficiently large t.

PROOF OF PROPOSITION 4.1. Let u_μ^V be the solution of (1.1) with $\phi \in C_0(\Omega_L)$. By the same arguments as in [3] and [4], ϕ can be expanded in the Fourier series, that is, there exist radial functions $\{\phi_{k,i}\} \subset L^2(\Omega_L)$ such that

$$\phi(x) = \sum_{k=0}^{\infty}\sum_{i=1}^{l_k} \phi_{k,i}(|x|)Q_{k,i}\left(\frac{x}{|x|}\right) \quad \text{in} \quad L^2(\Omega_L). \tag{4.3}$$

Let $u_\mu^{k,i}$ be a solution of (1.1) with the initial data $\phi_{k,i}(|x|)Q_{k,i}(x/|x|)$ and $v_\mu^{k,i}$ a radial solution of (P_μ^k) with the initial data $\phi_{k,i}$. By the uniqueness of the solution of (1.1), we see that

$$u_\mu^{k,i}(x,t) = v_\mu^{k,i}(x,t)Q_{k,i}\left(\frac{x}{|x|}\right), \quad (x,t) \in \Omega_L \times (0,\infty), \tag{4.4}$$

where $k \in \mathbf{N}_0$ and $i = 1, \dots, l_k$. On the other hand, by the standard elliptic regularity theorem and $\|Q_{k,i}\|_{L^2(\mathbf{S}^{N-1})} = 1$, for any $n \in \mathbf{N}$, we have

$$\|Q_{k,i}\|_{C^{2n}(\mathbf{S}^{N-1})} \preceq (1+\omega_k)^{n+1} \asymp (k+1)^{2n+2} \tag{4.5}$$

for all $k \in \mathbf{N}_0$ and $i = 1, \dots, l_k$. Furthermore the eigenspace of $\Delta_{\mathbf{S}^{N-1}}$ corresponding to ω_ℓ is spanned by the functions $\nabla_x^j |x|$ for $j \in \mathbf{N}_0^N$ with $|j| = \ell$, and we have

$$l_k \le N^k. \tag{4.6}$$

By the orthogonality of $\{Q_{k,i}\}_{k,i}$, we have

$$\int_{\Omega_L} u_\mu^{k_1,i_1}(x,t)u_\mu^{k_2,i_2}(x,t)dx = 0 \tag{4.7}$$

for all $t \ge 0$ if $(k_1,i_1) \ne (k_2,i_2)$. On the other hand, for any $t > 0$,

$$u_\mu^V(x,t) = \lim_{m\to\infty} \sum_{k=0}^{m}\sum_{i=1}^{l_k} v_\mu^{k,i}(x,t)Q_{k,i}\left(\frac{x}{|x|}\right) \tag{4.8}$$

holds uniformly for all $x \in \Omega_L$. Hence we have

$$\begin{aligned}\int_{\partial B(0,|x|)} u_\mu^V(x,t)Q_{k,i}\left(\frac{x}{|x|}\right)d\sigma &= v_\mu^{k,i}(x,t)\int_{\partial B(0,|x|)}\left|Q_{k,i}\left(\frac{x}{|x|}\right)\right|^2 d\sigma \\ &= |x|^{N-1}v_\mu^{k,i}(x,t)\end{aligned}$$

for all $(x,t) \in \Omega_L \times (0,\infty)$. Then, by (4.5) and the Jensen inequality, we have

$$|x|^{N-1}|v_\mu^{k,i}(x,t)|^p \preceq (k+1)^{2p}\int_{\partial B(0,|x|)} |u_\mu^V(x,t)|^p d\sigma$$

for all $(x,t)\in\Omega_L\times(0,\infty)$ and $k\in\mathbf{N}_0$. So, by (2.9), we have

$$
\begin{aligned}
(4.9)\qquad \|v_\mu^{k,i}(\cdot,t)\|_{L^p(\Omega_L)} &\preceq \left(\int_L^\infty r^{N-1}|v_\mu^{k,i}(r,t)|^p dr\right)^{1/p}\\
&\preceq (k+1)^2\|u_\mu^V(\cdot,t)\|_{L^p(\Omega_L)} \preceq (k+1)^2\|\phi\|_{L^p(\Omega_L)}
\end{aligned}
$$

for all $t>0$ and $k\in\mathbf{N}_0$.

Let $j\in\mathbf{N}_0^N$ with $|j|\le \ell+1$. Let $k\in\mathbf{N}$ and $i=1,\dots,l_k$. By (1.6), (4.4), and (4.5), we have

$$
(4.10)\ |\nabla_x^j u_\mu^{k,i}(x,t)| \preceq (k+1)^{\ell+3}\sum_{m=0}^{|j|}\frac{|\partial_r^m v_\mu^{k,i}(x,t)|}{|x|^{|j|-m}},\quad (x,t)\in\Omega_L\times(0,\infty).
$$

Since $D_{\epsilon_1}(T)\subset D_{\epsilon_2}(T)$ if $\epsilon_1\le\epsilon_2$, by Lemmas 3.2, 3.3, 3.4, Remark 3.1 and (4.9), there exist positive constants η_1, η_2, η_3, T_*, and ϵ_* such that

$$
\begin{aligned}
\frac{|\partial_r^m v_\mu^{k,i}(x,t+t_0)|}{|x|^{|j|-m}} &\preceq \eta_1^{\alpha_k}t^{-\frac{N}{2p}}g_k(t)U_+^{\omega,k}(|x|)|x|^{-|j|}\|v_\mu^{k,i}(\cdot,t_0)\|_{L^p(\Omega_L)}\\
&\preceq (k+1)^2\epsilon^{[\alpha_k-|j|]_+}\eta_3^{\alpha_k}t^{-\frac{N}{2p}-\frac{\min\{\alpha_k,|j|\}}{2}}\|\phi\|_{L^p(\Omega_L)}
\end{aligned}
$$

for all $(x,t)\in D_\epsilon(T_*)$ with $0<\epsilon\le\epsilon_*$, $t_0>0$, and $m=0,1,\dots,|j|$, where $\alpha_k=\alpha(\omega+\omega_k)$. Letting $t_0\to 0$, we obtain

$$
\frac{|\partial_r^m v_\mu^{k,i}(x,t)|}{|x|^{|j|-m}} \preceq (k+1)^2\epsilon^{[\alpha_k-|j|]_+}\eta_3^{\alpha_k}t^{-\frac{N}{2p}-\frac{\min\{\alpha_k,|j|\}}{2}}\|\phi\|_{L^p(\Omega_L)}
$$

for all $(x,t)\in D_\epsilon(T_*)$ with $0<\epsilon\le\epsilon_*$ and $m=0,1,\dots,|j|$. This inequality together with (4.10) implies that

$$
(4.11)\quad |\nabla_x^j u_\mu^{k,i}(x,t)| \preceq (k+1)^{\ell+5}\epsilon^{[\alpha_k-|j|]_+}\eta_3^{\alpha_k}t^{-\frac{N}{2p}-\frac{\min\{\alpha_k,|j|\}}{2}}\|\phi\|_{L^p(\Omega_L)}
$$

for all $(x,t)\in D_\epsilon(T_*)$ with $0<\epsilon\le\epsilon_*$. Let $0<\epsilon\le\epsilon_*$ and T_ϵ be a positive constant such that $T_\epsilon>T_*$ and $\epsilon(1+T_\epsilon)^{1/2}\ge L+2$. By (4.11), taking a sufficiently small ϵ if necessary, we see

$$
(4.12)\qquad |\nabla_x^j u_\mu^{k,i}(x,t)| \preceq \frac{1}{2^kN^k}t^{-\frac{N}{2p}-\frac{\min\{\alpha_k,|j|\}}{2}}\|\phi\|_{L^p(\Omega_L)}
$$

for all $(x,t)\in D_\epsilon(T_\epsilon)$, $k\in\mathbf{N}$, and $i=1,\dots,l_k$. Similarly, for the case $k=0$, we have

$$
\begin{aligned}
(4.13)\qquad |\nabla_x^j u_\mu^{0,1}(x,t)| &= |\nabla_x^j v_\mu^{0,1}(x,t)| \preceq \sum_{m=1}^{|j|}\frac{|(\partial_r^m v_\mu^{0,1})(x,t)|}{|x|^{|j|-m}}\\
&\preceq t^{-\frac{N}{2p}-\frac{\min\{\alpha_0,|j|\}}{2}}\|\phi\|_{L^p(\Omega_L)}
\end{aligned}
$$

for all $(x,t) \in D_\epsilon(T_\epsilon)$. By (4.6), (4.12), and (4.13), we obtain

$$(4.14)\qquad \begin{aligned} |(\nabla_x^j u_\mu^V)(x,t)| &\le \limsup_{m\to\infty} \sum_{k=0}^{m}\sum_{i=1}^{l_k} |(\nabla_x^j u_\mu^{k,i})(x,t)| \\ &\preceq t^{-\frac{N}{2p}-\frac{\min\{\alpha_0,|j|\}}{2}} \|\phi\|_{L^p(\Omega_L)} \end{aligned}$$

for all $(x,t) \in D_\epsilon(T_\epsilon)$. On the other hand, by Lemma 2.3, we have

$$(4.15)\qquad |(\nabla_x^j u_\mu^V)(x,t)| \preceq t^{-\frac{N}{2p}-\frac{|j|}{2}} \|\phi\|_{L^p(\Omega_L)}$$

for all $(x,t) \notin D_\epsilon(T_\epsilon)$. Therefore, by (4.14) and (4.15), we obtain

$$(4.16)\qquad |(\nabla_x^j u_\mu^V)(x,t)| \preceq t^{-\frac{N}{2p}-\frac{\min\{\alpha(\omega),|j|\}}{2}} \|\phi\|_{L^p(\Omega_L)}$$

for all $(x,t) \in \Omega_L$ with $t \ge T_\epsilon$, where $\phi \in C_0(\Omega_L)$. Since $C_0(\Omega_L)$ is a dense subset of $L^p(\Omega_L)$, the inequality (4.16) holds for all $\phi \in L^p(\Omega_L)$, and the proof of Proposition 4.1 is complete. □

PROOF OF PROPOSITION 4.2. By (1.12), V satisfies the condition (V_ω^ℓ) with $\omega = \omega_n$ and $\ell = 0, 1, 2, \ldots$. Let $j \in \mathbf{N}_0^N$ with $|j| \ge n+1 = 2n'+1$. Let u_μ^V be the solution of (1.1) with $\phi \in C_0(\Omega_L)$ and $u_\mu^{k,i}$ a function given in the proof of Proposition 4.1. By the same argument as in the proof of (4.13) and Lemma 3.5, for any sufficiently small $\epsilon > 0$, there exists a positive constant T_ϵ such that

$$(4.17)\qquad |(\nabla_x^j u_\mu^{0,1})(x,t)| \preceq t^{-\frac{N}{2p}-\frac{n+1}{2}} \|\phi\|_{L^p(\Omega_L)}$$

for all $(x,t) \in D_\epsilon(T_\epsilon)$.

On the other hand, by the same argument as in the proof of (4.14), taking a sufficiently small $\epsilon > 0$ if necessary, we have

$$(4.18)\qquad \limsup_{m\to\infty} \sum_{k=1}^{m}\sum_{i=1}^{l_k} |(\nabla_x^j u_\mu^{k,i})(x,t)| \preceq t^{-\frac{N}{2p}-\frac{\min\{\alpha(\omega_n+\omega_1),|j|\}}{2}} \|\phi\|_{L^p(\Omega_L)}$$

for all $(x,t) \in D_\epsilon(T_\epsilon)$. We note that $\alpha(\omega_n+\omega_1) \le \alpha(\omega_n)+1 = n+1$. Therefore, by (4.17), (4.18), and $|j| \ge n+1$, we have

$$(4.19)\qquad |(\nabla_x^j u_\mu^V)(x,t)| \preceq t^{-\frac{N}{2p}-\frac{\alpha(\omega_n+\omega_1)}{2}} \|\phi\|_{L^p(\Omega_L)}$$

for all $(x,t) \in D_\epsilon(T_\epsilon)$. Furthermore, by (4.15) and (4.19), taking a sufficiently small ϵ if necessary, we have

$$(4.20)\qquad |(\nabla_x^j u_\mu^V)(x,t)| \preceq t^{-\frac{N}{2p}-\frac{\alpha(\omega_n+\omega_1)}{2}} \|\phi\|_{L^p(\Omega_L)}$$

for all $(x,t) \in \Omega_L \times (T_\epsilon, \infty)$, where $\phi \in C_0(\Omega_L)$. Furthermore, since $C_0(\Omega_L)$ is a dense subset of $L^p(\Omega_L)$, we have the inequality (4.20) for all $\phi \in L^p(\Omega_L)$, and the proof of Proposition 4.2 is complete. □

5 Proofs of Theorems 1.1 and 1.2

In this section we consider the asymptotic behavior of the derivatives of the radial solution v of (1.1) for some initial data $\psi \in C_0(\Omega_L)$ and complete proofs of Theorems 1.1 and 1.2.

PROPOSITION 5.1 *Let $R > 0$, $\omega \geq 0$, and $\psi(\not\equiv 0)$ be a nonnegative, radial function belonging to $C_0(\Omega_R)$. Let v be a radial solution of*

$$\begin{cases} \partial_t v = \Delta v - \dfrac{\omega}{|x|^2} v & \text{in} \quad \Omega_R \times (0,\infty), \\ v(x,t) = 0 & \text{on} \quad \partial\Omega_R \times (0,\infty), \\ v(x,0) = \psi(x) & \text{in} \quad \Omega_R. \end{cases} \tag{5.1}$$

Then, for any $p \in [1,\infty]$,

$$\|v(\cdot,t)\|_{L^p(\Omega_R)} \asymp t^{-\frac{N}{2}(1-\frac{1}{p})-\frac{\alpha(\omega)}{2}} \tag{5.2}$$

holds for all sufficiently large t. Furthermore there exists a positive constant ϵ_* such that, for any $0 < \epsilon \leq \epsilon_*$,

$$v(x,t)\Big|_{|x|=\epsilon(1+t)^{1/2}} \asymp \epsilon^{\alpha(\omega)} t^{-\frac{N+\alpha(\omega)}{2}}, \qquad t > T \tag{5.3}$$

holds with suitably chosen $T = T(\epsilon)$.

PROOF. Put

$$z(y,s) = (1+t)^{\frac{N+\alpha}{2}} v(x,t), \quad y = (1+t)^{-\frac{1}{2}} x, \quad s = \log(1+t), \tag{5.4}$$

where $\alpha = \alpha(\omega)$. Then the function z satisfies

$$\begin{cases} \partial_s z = \dfrac{1}{\rho}\operatorname{div}(\rho \nabla_y z) + \dfrac{N+\alpha}{2} z - \dfrac{\omega}{|y|^2} z & \text{in} \quad W, \\ z = 0 & \text{on} \quad \partial W, \\ z(y,0) = \psi(y) & \text{in} \quad \Omega_R, \end{cases} \tag{5.5}$$

where $\rho(y) = \exp(|y|^2/4)$ and

$$\Omega(s) = e^{-s/2}\Omega_R, \; W = \bigcup_{0<s<\infty} (\Omega(s) \times \{s\}), \; \partial W = \bigcup_{0<s<\infty} (\partial\Omega(s) \times \{s\}).$$

Put

$$\varphi(y) = c_0|y|^{\alpha(\omega)}\exp(-|y|^2/4),$$

where c_0 is a positive constant such that $\|\varphi\|_{L^2(\mathbf{R}^N,\rho dy)} = 1$. Then, since

$$\int_{\Omega_R} v(x,t)U^\omega_{1,R}(|x|)dx = \int_{\Omega_R} \phi(x)U^\omega_{1,R}(|x|)dx > 0, \qquad t \geq 0,$$

by the same argument as in the proof of Lemma 6.1 in [4], we see that

$$a \equiv \int_{\Omega_R} \phi(x)U^\omega_{1,R}(|x|)dx = \lim_{s\to\infty} \int_{\Omega(s)} z(y,s)\varphi(y)\rho(y)dy > 0. \tag{5.6}$$

Furthermore, by the same argument as in the proof of Lemmas 3.3 and 3.4 in [4], for any r_1 and r_2 with $0 < r_1 < r_2$, we have

$$\sup_{s>0} \|z(\cdot,s)\|_{L^2(\Omega(s),\rho dy)} < \infty, \tag{5.7}$$

$$\sup_{s>0} \|z(\cdot,s)\|_{L^\infty(\{y\,:\,|y|\geq r_1\})} < \infty, \tag{5.8}$$

$$\lim_{s\to\infty} \|z(\cdot,s) - a\varphi\|_{C(\{y\,:\,r_1\leq|y|\leq r_2\})} = 0. \tag{5.9}$$

By (5.6), (5.7) and (5.9), we have $\|z(\cdot,s)\|_{L^1(\Omega(s))} \asymp 1$ for all sufficiently large s. So, by (5.8) and (5.9), for any $p \in [1,\infty]$, we have $\|z(\cdot,s)\|_{L^p(\Omega(s))} \asymp 1$ for all sufficiently large s, and obtain (5.2).

On the other hand, by the same argument as in (3.19), there exists a function ζ in $(0,\infty)$ such that

$$v(x,t) = \zeta(t)U^V_{0,R}(|x|) + F^V_L[(\partial_t v)(\cdot,t)](|x|) \tag{5.10}$$

for all $(x,t) \in \Omega_R \times (0,\infty)$ with $V = \omega/r^2$. By (5.2) with $p = \infty$, we may apply the same arguments as in the proof of Lemma 3.2 with $\gamma = (N+\alpha(\omega))/2$ to v. Then we see that there exist positive constants ϵ_* and T_* such that

$$|F^V_L[(\partial_t v)(\cdot,t)](|x|)| \preceq t^{-\frac{N}{2}-\alpha(\omega)-1}|x|^{\alpha(\omega)+2} \tag{5.11}$$

for all $(x,t) \in D_{\epsilon_*}(T_*)$. Therefore, by (2.18), (5.9), (5.10), (5.11), and the same arguments as in the deduction of (3.20), we may take a sufficiently small $\tilde\epsilon$ so that

$$\begin{aligned}\zeta(t) &= [U^V_{0,R}(\tilde\epsilon(1+t)^{1/2})]^{-1}\left[v(x,t) - F^V_L[\partial_t v](|x|)\right]\Big|_{|x|=\tilde\epsilon(1+t)^{1/2}} \\ &\asymp \tilde\epsilon^{-\alpha}t^{-\frac{\alpha}{2}}\left[t^{-\frac{N+\alpha}{2}} + O(\tilde\epsilon^{\alpha+2})t^{-\frac{N+\alpha}{2}}\right] \asymp t^{-\frac{N}{2}-\alpha}\end{aligned} \tag{5.12}$$

for all sufficiently large t. Then, by (5.10)–(5.12) and the similar argument as in (5.12), we have (5.3), and the proof of Proposition 5.1 is complete. □

PROOF OF THEOREM 1.1. Assume (V_ω^ℓ). Let $\tilde\omega$ be a constant such that $\tilde\omega > \omega$ and

$$\alpha(\tilde\omega) < \alpha(\omega) + 1. \tag{5.13}$$

Then, by (V_ω^ℓ)-(i), we may take a sufficiently large R so that

$$V(r) \le \frac{\tilde\omega}{r^2}, \qquad r \ge R.$$

Let $p \ge 1$ and $\psi(\not\equiv 0)$ be a nonnegative, radial function belonging to $C_0(\Omega_R)$. Let v be a solution of (5.1) with ω replaced by $\tilde\omega$. For any $T > 0$, let u_T^V be a solution of (1.1) with the initial data $\phi(\cdot) = v(\cdot, T)/\|v(\cdot, T)\|_{L^p(\Omega_R)}$. Here we remark that

$$\|u_T^V(\cdot, 0)\|_{L^p(\Omega_L)} = 1. \tag{5.14}$$

By the comparison principle, (5.2), and (5.3), for any sufficiently small $\epsilon > 0$, there exists a positive constant T_ϵ such that

$$\begin{aligned} u_T^V(x,T) &\ge \frac{v(x,2T)}{\|v(\cdot,T)\|_{L^p(\Omega_R)}} \asymp T^{\frac{N}{2}(1-\frac{1}{p})+\frac{\alpha(\tilde\omega)}{2}} v(x,2T) \\ &\succeq \epsilon^{\alpha(\tilde\omega)} T^{-\frac{N}{2p}} \end{aligned} \tag{5.15}$$

for all $(x,T) \in \Omega_L \times (T_\epsilon, \infty)$ with $|x| = \epsilon(1+2T)^{1/2} > \max\{R, 2L+2\}$.

On the other hand, there exists a function $\zeta_V(t)$ such that

$$u_T^V(x,t) = \zeta_V(t) U_{\mu,L}^V(|x|) + F_L^V[\partial_t u_T^V](|x|) \tag{5.16}$$

for all $x \in \Omega_L$. By Lemmas 3.2–3.4 and (5.14), taking a sufficiently small ϵ and sufficiently large T_ϵ if necessary, we have

$$\left|\partial_r^j F_L^V[\partial_t u_T^V](|x|)\right| \preceq t^{-\frac{N}{2p}-1-\frac{\alpha(\omega)}{2}} |x|^{2-|j|+\alpha(\omega)} \tag{5.17}$$

for all $(x,t) \in D_\epsilon(T_\epsilon)$ and $j \in \mathbf{N}_0$ with $|j| \le \ell + 2$. Furthermore, by (5.13) and (5.15)–(5.17), there exist positive constants C_1 and C_2 such that

$$\begin{aligned} \zeta_V(T) U_{\mu,L}^V(|x|) &\ge u_T^V(x,T) - \left|F_L^V[\partial_t u_T^V](|x|)\right| \\ &\ge C_1 \epsilon^{\alpha(\tilde\omega)} T^{-\frac{N}{2p}} - C_2 \epsilon^{\alpha(\omega)+2} T^{-\frac{N}{2p}} \succeq \epsilon^{\alpha(\omega)+1} T^{-\frac{N}{2p}} \end{aligned}$$

for all $x \in \Omega_L$ with $L + 1 < |x| = \epsilon(1+2T)^{1/2}/2 < \epsilon(1+T)^{1/2}$ and $T \ge T_\epsilon$. Therefore, by (2.5) and (2.16), we have

$$\zeta_V(T) \succeq T^{-\frac{N}{2p}-\frac{\alpha(\omega)}{2}} \tag{5.18}$$

for all sufficiently large T. Therefore, by (5.16)–(5.18), there exist positive constants C_3 and C_4 such that

$$(5.19)\qquad \begin{aligned}&|\nabla_x^j u_T^V(x,T)|\\ &\ge C_4 T^{-\frac{N}{2p}-\frac{\alpha(\omega)}{2}}|\nabla_x^j U_{\mu,L}^V(x)| - C_4 T^{-\frac{N}{2p}-1-\frac{\alpha(\omega)}{2}}|x|^{2+\alpha(\omega)-|j|}\end{aligned}$$

for all $L<|x|\le\epsilon(1+T)^{1/2}$, $T\ge T_\epsilon$, and $j\in\mathbf{N}_0^N$ with $|j|\le\ell$.

Let $j\in\mathbf{N}_0^N$ with $|j|\le\ell$. By the assumption of Theorem 1.1 and Proposition 2.3, there exists a point $x_0\in\Omega_L$ such that $(\nabla_x^j U_{\mu,L}^V)(x_0)\neq 0$. Then, by (5.19), there exist positive constants C_5 and C_6 such that

$$(5.20)\quad |(\nabla_x^j u_T^V)(x_0,T)| \ge C_5 T^{-\frac{N}{2p}-\frac{\alpha(\omega)}{2}} - C_6 T^{-\frac{N}{2p}-\frac{\alpha(\omega)}{2}-1} \succeq T^{-\frac{N}{2p}-\frac{\alpha(\omega)}{2}}$$

for all sufficiently large T. This inequality together with (5.14) implies

$$(5.21)\qquad \|\nabla_x^j G_\mu^V(T)\|_{p\to\infty} \succeq T^{-\frac{N}{2p}-\frac{\alpha(\omega)}{2}}$$

for all sufficiently large T. This together with Proposition 4.1 implies (1.10) and (1.11), and the proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. Let $u_T^{V_1}$ be a function given in the proof of Theorem 1.1 with $V(r)=(\omega_n+\omega_1)/r^2$. Put

$$\tilde{u}_T^V(x,t) = u_T^{V_1}(x,t)\frac{x_1}{|x|}.$$

Then $\tilde{u}_T^V$ is a solution of (1.1) with $V(r)=\omega_n/r^2$.

Let $j=(j_1,\dots,j_N)\in\mathbf{N}_0^N$ with $|j|\ge n+1$. Put $j'=(j_1+1,j_2,\dots,j_N)$ and

$$\tilde{U}_{\mu,L}^{\omega_n+\omega_1}(r) = \int_L^r U_{\mu,L}^{\omega_n+\omega_1}(s)ds$$

Then, by (2.5), we see that $\tilde{U}_{\mu,L}^{\omega_n+\omega_1}(r)\asymp r^{\alpha(\omega_n+\omega_1)+1}$ for all sufficiently large r. If $\nabla_x^{j'}\tilde{U}_\mu^{\omega_n+\omega_1}(|x|)\equiv 0$ in Ω_L, then, we see that $\tilde{U}_{\mu,L}^{\omega_n+\omega_1}(r)$ is a polynomial. This contradicts $\alpha(\omega_n+\omega_1)\notin\mathbf{N}$ if $n\ge 1$. If $n=0$, by (1.12),

$$U_{\mu,L}^{\omega_n+\omega_1}(r) = U_{0,L}^{\omega_1}(r) = \frac{1}{N}\left(\frac{r}{L}\right)^{-(N-1)} + \frac{N-1}{LN}r,$$

and $\tilde{U}_\mu^{\omega_n+\omega_1}(r)$ is not a polynomial. So we have

$$\nabla_x^{j'}\tilde{U}_\mu^{\omega_n+\omega_1}(|x|) = \nabla_x^j\left[U_\mu^{\omega_n+\omega_1}(|x|)\frac{x_1}{|x|}\right]\not\equiv 0 \quad\text{in}\quad \Omega_L.$$

By the similar arguments in (5.16)–(5.20) and $\omega = \omega_n + \omega_1$, there exist positive constants C_1 and C_2 such that

$$\begin{aligned} |(\nabla_x^j \tilde{u}_T^V)(x_0, T)| &\geq C_1 T^{-\frac{N}{2} - \frac{\alpha(\omega_n+\omega_1)}{2}} - C_2 T^{-\frac{N}{2} - \alpha(\omega_n+\omega_1) - 1} \\ &\succeq T^{-\frac{N}{2} - \frac{\alpha(\omega_n+\omega_1)}{2}} \end{aligned}$$

for all sufficiently large T. Furthermore, since $\|\tilde{u}_T^V(\cdot, 0)\|_{L^p(\Omega_L)} \asymp 1$, we obtain

$$\|\nabla_x^j G_\mu^V(T)\|_{p\to\infty} \succeq T^{-\frac{N}{2p} - \frac{\alpha(\omega_n+\omega_1)}{2}}$$

for all sufficiently large T. Therefore, this inequality together with Propositions 4.1 and 4.2 imply (1.13) and (1.14), and the proof of Theorem 1.2 is complete. □

6 Concluding remarks

As concluding remarks, we state some related topics. In the previous sections, we treat the exterior of a ball, however, we can treat the whole space and we can argue the movement of hot spots (the maximum points of a solution) with a potential V. According to the decay order of V, the bahavior of hots spots varies. Such works are now in progress and we will discuss these topics later.

References

[1] A. Grigor'yan, Upper bounds of derivatives of the heat kernel on an arbitrary complete manifold, J. Funct. Anal., **127** (1995), 363-389.

[2] A. Grigor'yan and L. Saloff-Coste, Dirichlet heat kernel in the exterior of a compact set, Comm. Pure Appl. Math., **55** (2002), 93–133.

[3] K. Ishige, Movement of hot spots on the exterior domain of a ball under the Neumann boundary condition, J. Differential Equations., **212** (2005), 394-431.

[4] K. Ishige, Movement of hot spots on the exterior domain of a ball under the Dirichlet boundary condition, preprint.

[5] K. Ishige and Y.Kabeya, Decay rates of the derivatives of the solutions of the heat equations in the exterior domain of a ball, preprint.

[6] O. A. Ladyzenskaja, V. A. Solonnikov and N. N. Ural'ceva, "Linear and Quasilinear Equations of Parabolic Type", Amer. Math. Soc., Providence, 1968.

[7] P. Li and S. T. Yau, On the parabolic kernel of the Schrödinger operator, Acta Math., **156** (1986), 153-201.

[8] Y. Shibata and S. Shimizu, Decay properties of the Stokes semigroup in the exterior domains with Neumann boundary condition, preprint.