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ON A PROBLEM OF GUTEV, OHTA AND YAMAZAKI CONCERNING CONTINUOUS SELECTIONS

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Throughout this note, all spaces are assumed to be T_1. For undefined terminology, we refer to [2]. The purpose of this note is to introduce some results of [9] and [10].

Let X be a space and $(Y, \|\cdot\|)$ a Banach space. By 2^Y, $\mathcal{F}_c(Y)$, $\mathcal{C}_c(Y)$ and $\mathcal{C}_c'(Y)$ we denote the set of all non-empty subsets of Y, the set of all non-empty closed convex subsets of Y, the set of all non-empty compact convex subsets of Y and the set $\mathcal{C}_c(Y) \cup \{Y\}$, respectively. Then a mapping $\varphi : X \to 2^Y$, which is called a set-valued mapping from X to Y, associates each point $x \in X$ with a non-empty subset $\varphi(x)$ of Y. For a mapping $\varphi : X \to 2^Y$, a mapping $f : X \to Y$ is called a *selection* if $f(x) \in \varphi(x)$ for each $x \in X$.

For $K \in \mathcal{F}_c(Y)$, a point $y \in K$ is called an *extreme point* if every open line segment containing y is not contained in K. For $K \in \mathcal{F}_c(Y)$, the *weak convex interior* $\mathrm{wci}(K)$ of K ([3]) is the set of all non-extreme points of K, that is,

$$\mathrm{wci}(K) = \{y \in K \mid y = \delta y_1 + (1-\delta) y_2 \text{ for some } y_1, y_2 \in K \setminus \{y\} \text{ and } 0 < \delta < 1\}.$$

Our concern of this note is to characterize some topological properties in terms of continuous selections avoiding extreme points. This study is motivated by Problem 3 below posed by V. Gutev, H. Ohta and K. Yamazaki [3].

1 A problem of Gutev, Ohta and Yamazaki

By $w(Y)$ we denote the weight of a space Y. A Hausdorff space X is called *countably paracompact* if every countable open cover of X is refined by a locally finite open cover of X. The following insertion theorem due to C. H. Dowker [1, Theorem 4] and M. Katětov [4, Theorem 2] is fundamental.

Theorem 1 (Dowker [1], Katětov [4]). *A T_1-space X is normal and countably paracompact if and only if for every upper semicontinuous function $g : X \to \mathbf{R}$ and every lower semicontinuous function $h : X \to \mathbf{R}$ with $g(x) < h(x)$ for each $x \in X$, there exists a continuous function $f : X \to \mathbf{R}$ such that $g(x) < f(x) < h(x)$ for each $x \in X$.*

The cardinality of a set S is denoted by $\mathrm{Card}\, S$. For an infinite cardinal number λ, a T_1-space X is called *λ-collectionwise normal* if for every discrete collection $\{F_\alpha \mid \alpha \in A\}$ of closed subsets of X with $\mathrm{Card}\, A \leq \lambda$, there exists a disjoint collection $\{G_\alpha \mid \alpha \in A\}$ of open subsets of X such that $F_\alpha \subset G_\alpha$ for each $\alpha \in A$. A mapping $\varphi : X \to 2^Y$ is called *lower semicontinuous* (*l.s.c.* for short) if for every open subset V of Y, the set $\varphi^{-1}[V] = \{x \in X \mid \varphi(x) \cap V \neq \emptyset\}$ is open in X. Let $\mathbf{R}$ be the space of real numbers with the usual topology. The space $c_0(\lambda)$ is the Banach space consisting of functions $s : D(\lambda) \to \mathbf{R}$, where $D(\lambda)$ is a set with $\operatorname{Card} D(\lambda) = \lambda$, such that for each $\varepsilon > 0$ the set $\{\alpha \in D(\lambda) \mid |s(\alpha)| \geq \varepsilon\}$ is finite, where the linear operations are defined pointwise and $\|s\| = \sup\{|s(\alpha)| \mid \alpha \in D(\lambda)\}$ for each $s \in c_0(\lambda)$. In order to connect insertion theorems with selection theorems, V. Gutev, H. Ohta and K. Yamazaki [3] introduced lower and upper semicontinuity of a mapping to the Banach space $c_0(\lambda)$ and, with the aid of these concepts, they proved sandwich-like characterizations of paracompact-like properties. Moreover, they introduced generalized $c_0(\lambda)$-spaces for Banach spaces and established the following theorem [3, Theorem 4.5].

Theorem 2 (Gutev, Ohta and Yamazaki [3]). *For a T_1-space X, the following statements are equivalent.*

- (a) *X is countably paracompact and λ-collectionwise normal.*
- (b) *For every generalized $c_0(\lambda)$-space Y and every l.s.c. mapping $\varphi : X \to \mathcal{C}'_c(Y)$ with $\operatorname{Card}\varphi(x) > 1$ for each $x \in X$, there exists a continuous selection $f : X \to Y$ of φ such that $f(x) \in \operatorname{wci}(\varphi(x))$ for each $x \in X$.*
- (c) *For every closed subset A of X and every two mappings $g, h : A \to c_0(\lambda)$ such that g is upper semicontinuous, h is lower semicontinuous and $g(x) < h(x)$ for each $x \in A$, there exists a continuous mapping $f : X \to c_0(\lambda)$ such that $g(x) < f(x) < h(x)$ for each $x \in A$.*

Concerning this theorem, they posed the following problem [3, Problem 4.7]:

Problem 3 (Gutev, Ohta and Yamazaki [3]). *Can "every generalized $c_0(\lambda)$-space Y" in condition (b) of Theorem 2 be replaced by "every Banach space Y with $w(Y) \leq \lambda$"?*

It is proved in [9] that the answer of Problem 3 is affirmative.

Theorem 4 ([9]). *A T_1-space X is countably paracompact and λ-collectionwise normal if and only if for every Banach space Y with $w(Y) \leq \lambda$ and every l.s.c. mapping $\varphi : X \to \mathcal{C}'_c(Y)$ with $\operatorname{Card}\varphi(x) > 1$ for each $x \in X$, there exists a continuous selection $f : X \to Y$ of φ such that $f(x) \in \operatorname{wci}(\varphi(x))$ for each $x \in X$.*

In particular, we have the following.

Corollary 5. *A T_1-space X is countably paracompact and collectionwise normal if and only if for every Banach space Y and every l.s.c. mapping $\varphi : X \to \mathcal{C}'_c(Y)$ with $\operatorname{Card}\varphi(x) > 1$ for each $x \in X$, there exists a continuous selection $f : X \to Y$ of φ such that $f(x) \in \operatorname{wci}(\varphi(x))$ for each $x \in X$.*

Comparing Corollary 5 with selection theorems due to E. Michael [6] and S. Nedev [7], it is natural to ask whether other topological properties such as paracompactness can be characterized analogously. In the next section, we present some characterizations in terms of continuous selections avoiding extreme points.

2 Characterizations in terms of continuous selections avoiding extreme points

For an infinite cardinal number λ, a Hausdorff space X is called *λ-paracompact* if every open cover $\mathcal{U}$ of X with $\mathrm{Card}\,\mathcal{U} \leq \lambda$ is refined by a locally finite open cover of X. The following theorem is a λ-paracompact analogue of Theorems 2 and 4.

Theorem 6 ([9]). *A T_1-space X is normal and λ-paracompact if and only if for every Banach space Y with $w(Y) \leq \lambda$ and every l.s.c. mapping $\varphi : X \to \mathcal{F}_c(Y)$ with $\mathrm{Card}\,\varphi(x) > 1$ for each $x \in X$, there exists a continuous selection $f : X \to Y$ of φ such that $f(x) \in \mathrm{wci}(\varphi(x))$ for each $x \in X$.*

Thus we have the following variation of [6, Theorem 3.2″].

Corollary 7. *A T_1-space X is paracompact if and only if for every Banach space Y and every l.s.c. mapping $\varphi : X \to \mathcal{F}_c(Y)$ such that $\mathrm{Card}\,\varphi(x) > 1$ for each $x \in X$, there exists a continuous selection $f : X \to Y$ of φ such that $f(x) \in \mathrm{wci}(\varphi(x))$ for each $x \in X$.*

For an infinite cardinal number λ, a space X is *λ-PF-normal* if every point-finite open cover $\mathcal{U}$ of X with $\mathrm{Card}\,\mathcal{U} \leq \lambda$ is normal. A space X is called *PF-normal* if X is λ-PF-normal for every infinite cardinal λ. Every λ-collectionwise normal space is λ-PF-normal, and ω-PF-normality coincides with normality ([5, Theorem 2], [8, Theorem 3.2]). Note that PF-normality is not hereditary to closed subsets ([3, p.506], [8, p. 409]), but it is hereditary to open F_σ-subsets.

Theorem 8 ([10]). *A T_1-space X is countably paracompact and λ-PF-normal if and only if for every Banach space Y with $w(Y) \leq \lambda$ and every l.s.c. mapping $\varphi : X \to \mathcal{C}_c(Y)$ with $\mathrm{Card}\,\varphi(x) > 1$ for each $x \in X$, there exists a continuous selection $f : X \to Y$ of φ such that $f(x) \in \mathrm{wci}(\varphi(x))$ for each $x \in X$.*

Corollary 9. *A T_1-space X is countably paracompact and PF-normal if and only if for every Banach space Y and every l.s.c. mapping $\varphi : X \to \mathcal{C}_c(Y)$ with $\mathrm{Card}\,\varphi(x) > 1$ for each $x \in X$, there exists a continuous selection $f : X \to Y$ of φ such that $f(x) \in \mathrm{wci}(\varphi(x))$ for each $x \in X$.*

Theorems 6 and 8 provide the following variation of [6, Theorem 3.1″].

Corollary 10. *For a T_1-space X, the following statements are equivalent.*

(a) *X is normal and countably paracompact.*

(b) *For every separable Banach space Y and every l.s.c. mapping $\varphi : X \to \mathcal{F}_c(Y)$ with $\mathrm{Card}\,\varphi(x) > 1$ for each $x \in X$, there exists a continuous selection $f : X \to Y$ of φ such that $f(x) \in \mathrm{wci}(\varphi(x))$ for each $x \in X$.*

(c) *For every separable Banach space Y and every l.s.c. mapping $\varphi : X \to \mathcal{C}_c(Y)$ with $\mathrm{Card}\,\varphi(x) > 1$ for each $x \in X$, there exists a continuous selection $f : X \to Y$ of φ such that $f(x) \in \mathrm{wci}(\varphi(x))$ for each $x \in X$.*

Applying Theorem 2, V. Gutev, H. Ohta and K. Yamazaki [3, Theorem 4.6] proved that a T_1-space X is perfectly normal and λ-collectionwise normal if and only if for every generalized $c_0(\lambda)$-space Y and every l.s.c. mapping $\varphi : X \to \mathcal{C}'_c(Y)$, there exists a continuous selection $f : X \to Y$ of φ such that $f(x) \in \mathrm{wci}(\varphi(x))$ for each $x \in X$ with $\mathrm{Card}\,\varphi(x) > 1$. By applying Theorem 4, instead of Theorem 2, to the proof of [3, Theorem 4.6], we have the following corollary.

Corollary 11. *A T_1-space X is perfectly normal and λ-collectionwise normal if and only if for every Banach space Y with $w(Y) \leq \lambda$ and every l.s.c. mapping $\varphi : X \to \mathcal{C}'_c(Y)$, there exists a continuous selection $f : X \to Y$ of φ such that $f(x) \in \mathrm{wci}(\varphi(x))$ for each $x \in X$ with* $\mathrm{Card}\,\varphi(x) > 1$.

Analogously, we have the following.

Corollary 12. *A T_1-space X is perfectly normal and λ-paracompact if and only if for every Banach space Y with $w(Y) \leq \lambda$ and every l.s.c. mapping $\varphi : X \to \mathcal{F}_c(Y)$, there exists a continuous selection $f : X \to Y$ of φ such that $f(x) \in \mathrm{wci}(\varphi(x))$ for each $x \in X$ with* $\mathrm{Card}\,\varphi(x) > 1$.

Corollary 13. *A T_1-space X is perfectly normal and λ-PF-normal if and only if for every Banach space Y with $w(Y) \leq \lambda$ and every l.s.c. mapping $\varphi : X \to \mathcal{C}_c(Y)$, there exists a continuous selection $f : X \to Y$ of φ such that $f(x) \in \mathrm{wci}(\varphi(x))$ for each $x \in X$ with* $\mathrm{Card}\,\varphi(x) > 1$.

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