ON A PROBLEM OF GUTEV, OHTA AND YAMAZAKI  
CONCERNING CONTINUOUS SELECTIONS

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Throughout this note, all spaces are assumed to be $T_1$. For undefined terminology, we refer to [2]. The purpose of this note is to introduce some results of [9] and [10].

Let $X$ be a space and $(X, ||·||)$ a Banach space. By $2^Y$, $F_c(Y)$, $C_c(Y)$ and $C'_c(Y)$ we denote the set of all non-empty subsets of $Y$, the set of all non-empty closed convex subsets of $Y$, the set of all non-empty compact convex subsets of $Y$ and the set $C_c(Y) \cup \{Y\}$, respectively. Then a mapping $\varphi : X \to 2^Y$, which is called a set-valued mapping from $X$ to $Y$, associates each point $x \in X$ with a non-empty subset $\varphi(x)$ of $Y$. For a mapping $\varphi : X \to 2^Y$, a mapping $f : X \to Y$ is called a selection if $f(x) \in \varphi(x)$ for each $x \in X$.

For $K \in F_c(Y)$, a point $y \in K$ is called an extreme point if every open line segment containing $y$ is not contained in $K$. For $K \in F_c(Y)$, the weak convex interior $\text{wci}(K)$ of $K$ ([3]) is the set of all non-extreme points of $K$, that is,

$$\text{wci}(K) = \{y \in K \mid y = \delta y_1 + (1-\delta)y_2 \text{ for some } y_1, y_2 \in K \setminus \{y\} \text{ and } 0 < \delta < 1\}.$$

Our concern of this note is to characterize some topological properties in terms of continuous selections avoiding extreme points. This study is motivated by Problem 3 below posed by V. Gutev, H. Ohta and K. Yamazaki [3].

1 A problem of Gutev, Ohta and Yamazaki

By $w(Y)$ we denote the weight of a space $Y$. A Hausdorff space $X$ is called countably paracompact if every countable open cover of $X$ is refined by a locally finite open cover of $X$. The following insertion theorem due to C. H. Dowker [1, Theorem 4] and M. Katětov [4, Theorem 2] is fundamental.

Theorem 1 (Dowker [1], Katětov [4]). A $T_1$-space $X$ is normal and countably paracompact if and only if for every upper semicontinuous function $g : X \to \mathbb{R}$ and every lower semicontinuous function $h : X \to \mathbb{R}$ with $g(x) < h(x)$ for each $x \in X$, there exists a continuous function $f : X \to \mathbb{R}$ such that $g(x) < f(x) < h(x)$ for each $x \in X$.

The cardinality of a set $S$ is denoted by $\text{Card}S$. For an infinite cardinal number $\lambda$, a $T_1$-space $X$ is called $\lambda$-collectionwise normal if for every discrete collection $\{F_\alpha \mid \alpha \in A\}$ of closed subsets of $X$ with $\text{Card}A \leq \lambda$, there exists a disjoint collection $\{G_\alpha \mid \alpha \in A\}$ of open subsets of $X$ such that $F_\alpha \subset G_\alpha$ for each $\alpha \in A$. A mapping $\varphi : X \to 2^Y$ is called lower semicontinuous (l.s.c. for short) if for every open subset $V$ of $Y$, the set $\varphi^{-1}[V] = \{x \in X \mid \varphi(x) \cap V \neq \emptyset\}$ is open in $X$. Let $\mathbb{R}$ be the space of
real numbers with the usual topology. The space $c_0(\lambda)$ is the Banach space consisting of functions $s : D(\lambda) \to \mathbb{R}$, where $D(\lambda)$ is a set with $\text{Card} \, D(\lambda) = \lambda$, such that for each $\varepsilon > 0$ the set $\{ \alpha \in D(\lambda) \mid |s(\alpha)| \geq \varepsilon \}$ is finite, where the linear operations are defined pointwise and $\|s\| = \sup \{|s(\alpha)| \mid \alpha \in D(\lambda)\}$ for each $s \in c_0(\lambda)$. In order to connect insertion theorems with selection theorems, V. Gutev, H. Ohta and K. Yamazaki [3] introduced lower and upper semicontinuity of a mapping to the Banach space $c_0(\lambda)$ and, with the aid of these concepts, they proved sandwich-like characterizations of paracompact-like properties. Moreover, they introduced generalized $c_0(\lambda)$-spaces for Banach spaces and established the following theorem [3, Theorem 4.5].

**Theorem 2** (Gutev, Ohta and Yamazaki [3]). For a $T_1$-space $X$, the following statements are equivalent.

(a) $X$ is countably paracompact and $\lambda$-collectionwise normal.

(b) For every generalized $c_0(\lambda)$-space $Y$ and every l.s.c. mapping $\varphi : X \to C'_e(Y)$ with $\text{Card} \, \varphi(x) > 1$ for each $x \in X$, there exists a continuous selection $f : X \to Y$ of $\varphi$ such that $f(x) \in \text{wci}(\varphi(x))$ for each $x \in X$.

(c) For every closed subset $A$ of $X$ and every two mappings $g, h : A \to c_0(\lambda)$ such that $g$ is upper semicontinuous, $h$ is lower semicontinuous and $g(x) < h(x)$ for each $x \in A$, there exists a continuous mapping $f : X \to c_0(\lambda)$ such that $g(x) < f(x) < h(x)$ for each $x \in A$.

Concerning this theorem, they posed the following problem [3, Problem 4.7]:

**Problem 3** (Gutev, Ohta and Yamazaki [3]). Can “every generalized $c_0(\lambda)$-space $Y$” in condition (b) of Theorem 2 be replaced by “every Banach space $Y$ with $w(Y) \leq \lambda$”?

It is proved in [9] that the answer of Problem 3 is affirmative.

**Theorem 4** ([9]). A $T_1$-space $X$ is countably paracompact and $\lambda$-collectionwise normal if and only if for every Banach space $Y$ with $w(Y) \leq \lambda$ and every l.s.c. mapping $\varphi : X \to C'_e(Y)$ with $\text{Card} \, \varphi(x) > 1$ for each $x \in X$, there exists a continuous selection $f : X \to Y$ of $\varphi$ such that $f(x) \in \text{wci}(\varphi(x))$ for each $x \in X$.

In particular, we have the following.

**Corollary 5.** A $T_1$-space $X$ is countably paracompact and collectionwise normal if and only if for every Banach space $Y$ and every l.s.c. mapping $\varphi : X \to C'_e(Y)$ with $\text{Card} \, \varphi(x) > 1$ for each $x \in X$, there exists a continuous selection $f : X \to Y$ of $\varphi$ such that $f(x) \in \text{wci}(\varphi(x))$ for each $x \in X$.

Comparing Corollary 5 with selection theorems due to E. Michael [6] and S. Nedev [7], it is natural to ask whether other topological properties such as paracompactness can be characterized analogously. In the next section, we present some characterizations in terms of continuous selections avoiding extreme points.
2 Characterizations in terms of continuous selections avoiding extreme points

For an infinite cardinal number $\lambda$, a Hausdorff space $X$ is called $\lambda$-paracompact if every open cover $\mathcal{U}$ of $X$ with $\text{Card}\mathcal{U} \leq \lambda$ is refined by a locally finite open cover of $X$. The following theorem is a $\lambda$-paracompact analogue of Theorems 2 and 4.

**Theorem 6** ([9]). A $T_1$-space $X$ is normal and $\lambda$-paracompact if and only if for every Banach space $Y$ with $w(Y) \leq \lambda$ and every l.s.c. mapping $\varphi : X \to \mathcal{F}_c(Y)$ with $\text{Card}\varphi(x) > 1$ for each $x \in X$, there exists a continuous selection $f : X \to Y$ of $\varphi$ such that $f(x) \in \text{wci}(\varphi(x))$ for each $x \in X$.

Thus we have the following variation of [6, Theorem 3.2”].

**Corollary 7.** A $T_1$-space $X$ is paracompact if and only if for every Banach space $Y$ and every l.s.c. mapping $\varphi : X \to \mathcal{F}_c(Y)$ such that $\text{Card}\varphi(x) > 1$ for each $x \in X$, there exists a continuous selection $f : X \to Y$ of $\varphi$ such that $f(x) \in \text{wci}(\varphi(x))$ for each $x \in X$.

For an infinite cardinal number $\lambda$, a space $X$ is $\lambda$-PF-normal if every point-finite open cover $\mathcal{U}$ of $X$ with $\text{Card}\mathcal{U} \leq \lambda$ is normal. A space $X$ is called PF-normal if $X$ is $\lambda$-PF-normal for every infinite cardinal $\lambda$. Every $\lambda$-collectionwise normal space is $\lambda$-PF-normal, and $\omega$-PF-normality coincides with normality ([5, Theorem 2], [8, Theorem 3.2]). Note that PF-normality is not hereditary to closed subsets ([3, p.506], [8, p. 409]), but it is hereditary to open $F_\sigma$-subsets.

**Theorem 8** ([10]). A $T_1$-space $X$ is countably paracompact and $\lambda$-PF-normal if and only if for every Banach space $Y$ with $w(Y) \leq \lambda$ and every l.s.c. mapping $\varphi : X \to \mathcal{F}_c(Y)$ with $\text{Card}\varphi(x) > 1$ for each $x \in X$, there exists a continuous selection $f : X \to Y$ of $\varphi$ such that $f(x) \in \text{wci}(\varphi(x))$ for each $x \in X$.

**Corollary 9.** A $T_1$-space $X$ is countably paracompact and PF-normal if and only if for every Banach space $Y$ and every l.s.c. mapping $\varphi : X \to \mathcal{F}_c(Y)$ with $\text{Card}\varphi(x) > 1$ for each $x \in X$, there exists a continuous selection $f : X \to Y$ of $\varphi$ such that $f(x) \in \text{wci}(\varphi(x))$ for each $x \in X$.

Theorems 6 and 8 provide the following variation of [6, Theorem 3.1”].

**Corollary 10.** For a $T_1$-space $X$, the following statements are equivalent.

(a) $X$ is normal and countably paracompact.

(b) For every separable Banach space $Y$ and every l.s.c. mapping $\varphi : X \to \mathcal{F}_c(Y)$ with $\text{Card}\varphi(x) > 1$ for each $x \in X$, there exists a continuous selection $f : X \to Y$ of $\varphi$ such that $f(x) \in \text{wci}(\varphi(x))$ for each $x \in X$.

(c) For every separable Banach space $Y$ and every l.s.c. mapping $\varphi : X \to \mathcal{F}_c(Y)$ with $\text{Card}\varphi(x) > 1$ for each $x \in X$, there exists a continuous selection $f : X \to Y$ of $\varphi$ such that $f(x) \in \text{wci}(\varphi(x))$ for each $x \in X$.  

\[\text{Corollary } 10.\]  
For a $T_1$-space $X$, the following statements are equivalent.

(a) $X$ is normal and countably paracompact.

(b) For every separable Banach space $Y$ and every l.s.c. mapping $\varphi : X \to \mathcal{F}_c(Y)$ with $\text{Card}\varphi(x) > 1$ for each $x \in X$, there exists a continuous selection $f : X \to Y$ of $\varphi$ such that $f(x) \in \text{wci}(\varphi(x))$ for each $x \in X$.

(c) For every separable Banach space $Y$ and every l.s.c. mapping $\varphi : X \to \mathcal{F}_c(Y)$ with $\text{Card}\varphi(x) > 1$ for each $x \in X$, there exists a continuous selection $f : X \to Y$ of $\varphi$ such that $f(x) \in \text{wci}(\varphi(x))$ for each $x \in X$.  

Applying Theorem 2, V. Gutev, H. Ohta and K. Yamazaki [3, Theorem 4.6] proved that a $T_1$-space $X$ is perfectly normal and $\lambda$-collectionwise normal if and only if for every generalized $c_0(\lambda)$-space $Y$ and every l.s.c. mapping $\varphi : X \to C'_c(Y)$, there exists a continuous selection $f : X \to Y$ of $\varphi$ such that $f(x) \in wci(\varphi(x))$ for each $x \in X$ with $\text{Card} \varphi(x) > 1$. By applying Theorem 4, instead of Theorem 2, to the proof of [3, Theorem 4.6], we have the following corollary.

**Corollary 11.** A $T_1$-space $X$ is perfectly normal and $\lambda$-collectionwise normal if and only if for every Banach space $Y$ with $w(Y) \leq \lambda$ and every l.s.c. mapping $\varphi : X \to C'_c(Y)$, there exists a continuous selection $f : X \to Y$ of $\varphi$ such that $f(x) \in wci(\varphi(x))$ for each $x \in X$ with $\text{Card} \varphi(x) > 1$.

Analogously, we have the following.

**Corollary 12.** A $T_1$-space $X$ is perfectly normal and $\lambda$-paracompact if and only if for every Banach space $Y$ with $w(Y) \leq \lambda$ and every l.s.c. mapping $\varphi : X \to F_c(Y)$, there exists a continuous selection $f : X \to Y$ of $\varphi$ such that $f(x) \in wci(\varphi(x))$ for each $x \in X$ with $\text{Card} \varphi(x) > 1$.

**Corollary 13.** A $T_1$-space $X$ is perfectly normal and $\lambda$-PF-normal if and only if for every Banach space $Y$ with $w(Y) \leq \lambda$ and every l.s.c. mapping $\varphi : X \to C_c(Y)$, there exists a continuous selection $f : X \to Y$ of $\varphi$ such that $f(x) \in wci(\varphi(x))$ for each $x \in X$ with $\text{Card} \varphi(x) > 1$.

**References**


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