HAUSDORFF HYPERSPACES OF EUCLIDEAN SPACES
AND THEIR DENSE SUBSPACES

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Here, we introduce the results obtained in the paper [11] and related problems. We consider metric spaces and their hyperspaces endowed with the Hausdorff metric. Specifically, given a metric space $X = (X, d)$, we shall denote by $\text{Cld}(X)$ and $\text{Bd}(X)$ the hyperspaces consisting of all nonempty closed sets and of all nonempty bounded closed sets in $X$ respectively and we denote by $d_H$ the Hausdorff metric, which is infinite-valued on $\text{Cld}(X)$ if $X$ is unbounded. When $X$ is compact, the space $\text{Cld}(X) (= \text{Bd}(X))$ is equal to the hyperspace $\exp(X)$ of all nonempty compact sets with the Vietoris topology. Even if $X$ is noncompact, on the space $\exp(X)$, the Hausdorff metric topology coincides with the Vietoris topology. However, in case $X$ is noncompact, these topologies are very different on the spaces $\text{Cld}(X)$ and $\text{Bd}(X)$.

Vietoris hyperspaces $\exp(X)$ have been studied extensively for many years. Among the known results, let us mention the theorem of Curtis and Schori [8] (cf. [13, Chapter 8]), saying that $\exp(X)$ is homeomorphic to $(\cong)$ the Hilbert cube $Q = [-1, 1]^\omega$ if and only if $X$ is a Peano continuum, that is, it is compact, connected and locally connected. Later, Curtis [7] characterized non-compact metric spaces $X$ for which $\exp(X)$ is homeomorphic to the Hilbert cube minus a point $Q \setminus 0 (= Q \setminus \{0\})$ or the pseudo-interior $s = (-1, 1)^\omega$ of $Q$. In particular, $\text{Bd}(\mathbb{R}^m) = \exp(\mathbb{R}^m)$ is homeomorphic to $Q \setminus 0$. For more information concerning Vietoris hyperspaces, we refer to the book of Ilanes and Nadler [10].

It is well known that the hyperspace $\exp(X)$ is an ANR (AR) if and only if $X$ is locally connected (and connected). On the other hand, it is proved in [6] that the space $\text{Bd}(X)$ is an ANR (AR) whenever the metric on $X$ is almost convex, that is,

1It is well known that $s$ is homeomorphic to the separable Hilbert space $\ell_2$. 
for every $\alpha > 0$, $\beta > 0$ and for every $x, y \in X$ such that $d(x, y) < \alpha + \beta$, there exists $z \in X$ with $d(x, z) < \alpha$ and $d(z, y) < \beta$. This condition was further weakened in [12], which has turned out to be actually a necessary and sufficient one by Banakh and Voytsitskyy [3]. In the last paper, several equivalent conditions are given, which are too technical to mention them here. We refer to [3] for the details. On the other hand, Cl(X) is not connected whenever $X$ is a metric space which is not totally bounded. For example, Cl(\mathbb{R}) has $2^{\aleph_0}$ many components.

The completion of a metric space $X = (X, d)$ is denoted by $\tilde{X} = (\tilde{X}, d)$. Then Bd(X) can be identified with the subspace of Bd(\tilde{X}), via the isometric embedding $A \mapsto \text{cl}_\tilde{X} A$. Thus we shall often write Bd(X) $\subseteq$ Bd(\tilde{X}), having in mind this identification. In this case, Bd(\tilde{X}) is the completion of Bd(X). By such a reason, we also consider a dense subspace $D$ of a metric space $X = (X, d)$. For each $0 \leq k < m$, let

$$\nu_k^m = \{x = (x_i)_{i=1}^m \in \mathbb{R}^m : x_i \in \mathbb{R} \setminus \mathbb{Q} \text{ except for at most } k \text{ many } i\},$$

which is the universal space for completely metrizable subspaces in $\mathbb{R}^m$ of dim $\leq k$. In case $2k+1 < m$, $\nu_k^m$ is homeomorphic to the $k$-dimensional Nöbeling space $\nu_k^{2k+1}$, which is the universal space for all separable completely metrizable spaces. Note that $\nu_0^m = (\mathbb{R} \setminus \mathbb{Q})^m \cong \mathbb{R} \setminus \mathbb{Q}$.

**Theorem 1.** Suppose $\langle m, k \rangle = \langle 1, 0 \rangle$ or $0 \leq k < m - 1$. Then,

$$\langle \text{Bd}(\mathbb{R}^m), \text{Bd}(\nu_k^m) \rangle \cong \langle \mathbb{Q} \setminus 0, s \setminus 0 \rangle.$$

Consequently, $\text{Bd}(\nu_k^m) \cong \ell_2$.

This can be derived from the following:

**Theorem 2.** Let $D$ be a dense $G_\delta$ set in $\mathbb{R}^m$ such that $\mathbb{R}^m \setminus D$ is also dense in $\mathbb{R}^m$ and in case $m > 1$ it is assumed that $D = p[D] \times \mathbb{R}$, where $p : \mathbb{R}^m \to \mathbb{R}^{m-1}$ is the projection onto the first $m - 1$ coordinates. Then, $\langle \text{Bd}(\mathbb{R}^m), \text{Bd}(D) \rangle \cong \langle \mathbb{Q} \setminus 0, s \setminus 0 \rangle$.

**Question 1.** In case $m > 1$, under the only assumption that $D \subseteq \mathbb{R}^m$ is a dense $G_\delta$ set and $\mathbb{R}^m \setminus D$ is also dense in $\mathbb{R}^m$, is the pair $\langle \text{Bd}(\mathbb{R}^m), \text{Bd}(D) \rangle$ homeomorphic to $\langle \mathbb{Q} \setminus 0, s \setminus 0 \rangle$? In particular, is the pair $\langle \text{Bd}(\mathbb{R}^m), \text{Bd}(\nu_{m-1}^m) \rangle$ homeomorphic to $\langle \mathbb{Q} \setminus 0, s \setminus 0 \rangle$?

We also consider the following dense subspaces of Bd(X):

- Nwd(X) — all nowhere dense closed sets;
- Perf(X) — all perfect sets;\footnote{I.e., completely metrizable closed sets which are dense in itself.}
• Cantor\( (X)\) — all compact sets homeomorphic to the Cantor set.

In case \( X = \mathbb{R}^m \), we can also consider the following subspace:

• \( \mathfrak{N}(\mathbb{R}^m)\) — all closed sets of the Lebesgue measure zero.

For these spaces, we have the following:

**Theorem 3.** Let \( \mathcal{F} \) be one of the following subspaces of \( \Bd(\mathbb{R}^m) \):

\[ \mathcal{N}(\mathbb{R}^m), \Perf(\mathbb{R}^m), \text{Cantor}(\mathbb{R}^m) \text{ and } \mathfrak{N}(\mathbb{R}^m). \]

Then, \( \langle \Bd(\mathbb{R}^m), \mathcal{F} \rangle \cong \langle \mathbb{Q} \setminus [0, s \setminus 0] \rangle \), hence \( \mathcal{F} \cong \ell_2 \).

To prove Theorems 2 and 3 above, we adopt the characterization of the pseudo-boundary \( \mathbb{Q} \setminus s \) of the Hilbert cube \( \mathbb{Q} \), see [5].

We also study the space \( \text{Cld}(\mathbb{R}) \). It is very different from the hyperspace \( \exp(\mathbb{R}) \). It is not hard to see that \( \text{Cld}(\mathbb{R}) \) has \( 2^{\aleph_0} \) many components, \( \Bd(\mathbb{R}) \) is the only separable one and any other component has weight \( 2^{\aleph_0} \). Applying Toruńczyk's Characterization of Hilbert space [14] (cf. [15]), we can prove

**Theorem 4.** Let \( \mathcal{H} \) be a nonseparable component of \( \text{Cld}(\mathbb{R}) \) which does not contain \( \mathbb{R}, \mathbb{R} \), \( [0, +\infty), (-\infty, 0] \). Then \( \mathcal{H} \cong \ell_2(2^{\aleph_0}) \).

**Question 2.** Does Theorem 4 hold even if \( \mathcal{H} \) contains \( \mathbb{R}, [0, \infty) \) or \( (-\infty, 0] \)?

**Question 3.** For \( m > 1 \), is \( \text{Cld}(\mathbb{R}^m) \setminus \Bd(\mathbb{R}^m) \) an \( \ell_2(2^{\aleph_0}) \)-manifold?

Now, we consider the subspaces \( \mathfrak{N}(\mathbb{R}), \mathcal{N}(\mathbb{R}), \Perf(\mathbb{R}) \) and \( \text{Cld}(\mathbb{R} \setminus \mathbb{Q}) \) of \( \text{Cld}(\mathbb{R}) \).

Similarly to \( \Bd(\mathbb{R}) \), it can be shown that those complements are \( Z_\sigma \)-sets in \( \text{Cld}(\mathbb{R}) \).

Due to Negligibility Theorem ([1], [9]), if \( M \) is an \( \ell_2(2^{\aleph_0}) \)-manifold and \( A \) is a \( Z_\sigma \)-set in \( M \) then \( M \setminus A \cong M \). Thus, the following follows from Theorem 4:

**Corollary 5.** Let \( \mathcal{H} \) be a nonseparable component of \( \text{Cld}(\mathbb{R}) \) which does not contain \( \mathbb{R}, [0, +\infty), (-\infty, 0] \). Then, the following spaces are homeomorphic to \( \ell_2(2^{\aleph_0}) \):

\[ \mathcal{H} \cap \mathfrak{N}(\mathbb{R}), \mathcal{H} \cap \mathcal{N}(\mathbb{R}), \mathcal{H} \cap \Perf(\mathbb{R}) \text{ and } \mathcal{H} \cap \text{Cld}(\mathbb{R} \setminus \mathbb{Q}). \]

**Borel classes.** Given a metric space \( \langle X, d \rangle \), let \( \langle \bar{X}, d \rangle \) be its completion. Then, the hyperspace \( \Bd(\bar{X}) \) is the completion of the hyperspace \( \Bd(X) \). Concerning Borel classes of hyperspaces, the following are also shown in the paper [11]:

1. \( \Bd(X) \) is \( F_\sigma \) in \( \Bd(\bar{X}) \) if \( X \) is \( \sigma \)-compact.
2. \( \Bd(X) \) is \( G_\delta \) in \( \Bd(\bar{X}) \) if \( X \) is Polish.\(^3\)

\(^3\)I.e., separable and completely metrizable
(3) \( \text{Bd}(X) \) is Polish for every Polish space \( X \) in which bounded sets are totally bounded.

(4) \( \text{Nwd}(X) \) is \( G_\delta \) in \( \text{Bd}(X) \) for every separable metric space \( X \).

(5) \( \text{Perf}(X) \) is \( G_\delta \) in \( \text{Bd}(X) \) if \( X \) is separable and locally compact.

(6) \( \text{Perf}(X) \) is \( F_{\sigma\delta} \) in \( \text{Bd}(X) \) for every Polish space \( X \).

(7) \( \text{Bd}(X) \) is analytic for every analytic metric space \( X \) in which bounded sets are totally bounded.

Fix a dense set \( X \) in a separable Banach space \( E \) which admits the metric \( d \) induced from the norm of \( E \). Then \( (X, d) \) is an almost convex metric space and therefore by a result of [6] the space \( \text{Bd}(X) \) is an AR. In case \( X \) is \( G_\delta \), the space \( \text{Bd}(X) \) is completely metrizable by (2). If additionally \( E \) is finite-dimensional then \( \text{Bd}(X) \) is Polish by (3). In case \( X \) is \( \sigma \)-compact, by (1), \( \text{Bd}(X) \) is absolutely \( F_{\sigma\delta} \).

Remarks. Recently, Banakh and Voytsitskyy [4] proved that the space \( \text{Cld}(X) \) (resp. \( \text{Bd}(X) \)) is homeomorphic to \( \ell_2 \) if and only if \( X \) is a completely metrizable nowhere locally compact metric space such that each (resp. bounded) subset of \( X \) is totally bounded and the completion \( \tilde{X} \) of \( X \) is connected and locally connected.

REFERENCES


