1. Introduction

The purpose of this résumé is to describe (strong) transitivity properties for graph self-maps in my recent works. W. Parry [7] pointed out a sufficient condition for the existence of a special measure on a symbolic dynamics, which has a close relation to a linearization of the dynamics on intervals. Then, as an application, he introduced the concept of strong transitivity that is one of conditions under which an interval map is conjugate to a uniformly piecewise linear map [7, §5, §6]. E. Coven and I. Mulvey [6, Theorem B and C] stated the relation between transitivity and strong transitivity properties for interval (or circle) self-maps.

We extend the above relation to graph self-maps (see §3). A motivation for studying graph maps is that higher-dimensional dynamics can often be reduced to one-dimensional dynamics.

Throughout this paper, by a graph, we mean a connected compact one-dimensional polyhedron, and a tree is a graph which contains no loops. We also assume that any graph $G$ is endowed with a metric $d$; we define $B(x; \varepsilon), x \in G, \varepsilon > 0$ to be the set of points of $G$ whose distance from $x$ is less than $\varepsilon$. $B(G)$ and $E(G)$ denote the sets of branch points and of endpoints of $G$, respectively. A map $f$ is a continuous function from a space $X$ to itself; $f^0$ is the identity map, and for every $n \geq 0$, $f^{n+1} = f^n \circ f$. We denote by $\text{Fix}(f)$ and $\text{Per}(f)$ the sets of fixed points and of periodic points of $f$, respectively. For a subset $K$ of $X$, $\text{Int} K$ and $\text{Cl} K$ denote the interior and closure of $K$ in $X$.

2. Strong Transitivity

An onto map $f : X \to X$ is called (topologically) transitive if any of the following equivalent conditions holds.
(1) There exists a point with dense orbit.
(2) Whenever $U, V$ are non-empty open sets, there exists an $n \geq 1$ such that $f^{-n}(U) \cap V \neq \emptyset$.
(3) The only closed invariant set $K$ with $\text{Int} K \neq \emptyset$ is $K = X$.

**Remark.** We note that, in the case of a graph map $f : G \to G$, $f$ is transitive if and only if for every pair of non-empty open sets $U$ and $V$ in $G$, there exists a $k \geq 1$ such that $U \cap \text{Int} f^k(V) \neq \emptyset$.

In the study of transitive maps, the subclass of those maps having all iterates transitive plays a significant role. A map $f$ is **totally transitive** if $f^n$ is transitive for all $n \geq 1$ (see [1]); note that a transitive map is not always totally transitive.

A map $f : X \to X$ is called **strongly transitive** if for every non-empty open set $J$ of $X$, there exists an $n$ such that $\bigcup_{k=0}^{n} f^{k}(J) = X$.

We first call a useful proposition which shows a backward structure of a strongly transitive map for each point.

**Proposition 2.1.** Let $f : X \to X$ be a map of $X$ to itself. Then the following are equivalent.

1. For each $x \in X$, $\text{Cl} \bigcup_{n=0}^{\infty} f^{-n}(x) = X$.
2. For every non-empty open set $U$ of $X$, $\bigcup_{n=0}^{\infty} f^{n}(U) = X$.

Furthermore, if $f$ is open, then (1) and (2) are equivalent to

3. If $E \subseteq X$ is a closed set with $f^{-1}(E) \subseteq E$, then $E = \emptyset$ or $X$.

The examples below clarify the difference between transitivity and strong transitivity properties.

**Example 1.** There exists a transitive map of the interval which is not strongly transitive. This example appears in [3, Example 3] to illustrate another property. For completeness, we give a construction of the map here.

Let $\{p_n \mid n \in \mathbb{Z}\}$ be a two-sided sequence of real numbers in $[0, 1]$ such that

$$\cdots < p_{-2} < p_{-1} < p_0 < p_1 < p_2 < \cdots,$$

and $p_n \to 1$ and $p_{-n} \to 0$ when $n \to \infty$. For $n \in \mathbb{Z}$ put $I_n = [p_n, p_{n+1}]$. Define the map $f_n : I_n \to I_{n-1} \cup I_n \cup I_{n+1}$ by $f_n(p_n) = p_n$, $f_n(p_{n+1}) = p_{n+1}$, $f_n\left(\frac{2p_n+p_{n+1}}{3}\right) = p_{n+2}$, $f_n\left(\frac{p_n+2p_{n+1}}{3}\right) = p_{n-1}$, and $f_n$ is linear on the intervals complementary to these points. $f : [0, 1] \to [0, 1]$ is given by $f(0) = 0$, $f(1) = 1$, and $f(x) = f_n(x)$ if $x \in I_n$ (see Figure 2 in [3]).
By Example 1 taken mod 1, we also have

**Example 2.** There exists a transitive map of the circle which is not strongly transitive.

Let $B_n$ be the bouquet with $n$-petals generated by $n$ copies of the unit circle, where $n \geq 1$. Using Example 1 taken mod 1 and a rotation among petals with respect to the origin, we can easily have an example on $B_n$.

**Example 3.** There exists a transitive map of $B_n$ which is not strongly transitive.

**Example 4.** Since the map $f$ in Example 1 is actually totally transitive as stated in [3, Example 3], we have a totally transitive interval map which is not strongly transitive. On the other hand, the interval map $g$ below is strongly transitive, but not totally transitive. $g(x) = 2x + 1/2, (0 \leq x \leq 1/4); -2x + 3/2, (1/4 \leq x \leq 3/4); 2x - 3/2, (3/4 \leq x \leq 1)$.

### 3. Main Results

Here is our main result.

**Theorem 3.1.** Let $f : G \to G$ be a graph map with $\# \text{Fix}(f^k) < \infty$ for each $k \geq 1$. If $f$ is transitive, then it is strongly transitive.

A map $f$ on a graph $G$ is *piecewise monotone* if there is a finite set $A$ in $G$ such that $f$ is monotone on each component of $G \setminus A$.

**Corollary 3.2.** Let $f : G \to G$ be a piecewise monotone graph map. If $f$ is transitive, then it is strongly transitive.

*Remark.* The interval case of the corollary above was proved by Coven-Mulvey [6].

**Example 5.** Let $f : [0, 1] \to [0, 1]$ be the map whose graph appears below. Then $f$ is transitive and the set of fixed points of $f^k$ is finite for each $k \geq 1$. Therefore $f$ is strongly transitive, in fact, for each non-degenerate subinterval $J$ of $[0, 1]$, there exists an $n$ such that $f^n(J) = [0, 1]$. 

\[36\]
Proposition 3.3. Let $f : T \to T$ be a totally transitive tree map. Then $f$ is strongly transitive if and only if for every non-degenerate connected set $J$ of $T$, there exists an $M$ such that for any $m \geq M$, $f^m(J) = T$.

The following generalizes the result for interval maps of Coven-Mulvey [6] to one for tree maps.

Theorem 3.4. Let $f : T \to T$ be an onto tree map. Let $v(T)$ be the maximum order of any branch point in $T$ and $N_{v(T)}$ the least common multiple of $\{2, \ldots, v(T)\}$. Then the following are equivalent.

(1) $f$ is transitive and has a point of period which is prime to $2, \ldots, v(T)$.
(2) $f^{N_{v(T)}}$ is transitive.
(3) $f$ is totally transitive.
(4) $f$ is topologically mixing.

Furthermore, if $\# \text{Fix}(f^k)$ is finite for each $k \geq 1$, then the following is equivalent to

(5) for every non-degenerate connected set $J$ of $T$, there exists an $M$ such that for any $m \geq M$, $f^m(J) = T$.

Remark. The equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) are well-known [8, Theorem 4.1], [1].

4. REMARKS

(1): It is useful to investigate the relation between the dynamics of a graph map and the dynamics of the induced self-homeomorphism of the inverse limit space [2], [3].

Let $f : X \to X$ be an onto map. Associated with $f$ is the inverse limit space $(X, f) = \{(x_0, x_1, \ldots) \mid x_i \in X, \text{ and } f(x_{i+1}) = x_i\}$, and
the induced homeomorphism $\hat{f} : (X, f) \to (X, f)$ (which is called the shift homeomorphism), given by $\hat{f}((x_0, x_1, \ldots)) = (f(x_0), x_0, x_1, \ldots)$.

**Proposition 4.1.** Let $f : X \to X$ be an onto map of a metrizable compact space $X$. If the shift homeomorphism $\hat{f} : (X, f) \to (X, f)$ is strongly transitive, then $f$ is strongly transitive.

Unfortunately, the shift homeomorphism of a strongly transitive graph map is not always strongly transitive. In fact, we have the following.

**Proposition 4.2.** Let $G$ be a non-degenerate graph and $f : G \to G$ be an onto map. Then the shift homeomorphism $\hat{f} : (G, f) \to (G, f)$ is strongly transitive if and only if $G$ is the circle and $f$ is conjugate to an irrational rotation.

(II): We note that statement (2) in Proposition 2.1, which was introduced by Parry [7], implies strong transitivity for tree maps.

**Proposition 4.3.** Let $f : T \to T$ be an onto tree map. Then $f$ is strongly transitive if and only if for every non-empty open set $U$ of $T$, $\bigcup_{n=0}^{\infty} f^n(U) = T$.

However, it is not always true for a general graph map.

**Example 6.** Let $f : [0,1] \to [0,1]$ be the map whose graph appears below. Using this map $f$, we define the circle map $g : S^1 \to S^1$ by $g(e^{2\pi i \theta}) = e^{2\pi i f(\theta)}$, where $0 \leq \theta \leq 1$. Then the map $g$ is transitive and satisfies statement (2) in Proposition 2.1, but is not strongly transitive.

---

**REFERENCES**


DEPARTMENT OF MATHEMATICS, SHIMANE UNIVERSITY, MATSUE, 690-8504, JAPAN

E-mail address: yokoi@riko.shimane-u.ac.jp