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稠密な周期点集合を持つグラフ写像について

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1. INTRODUCTION

The purpose of this announcement is to establish my recent results for graph self-maps for which the set of periodic points is dense. M. Barge and J. Martin [2] showed a structure theorem for maps on the interval with dense periodic points; that is, the twice iterate of such a map is topologically mixing on some countable subintervals and is identical on the other. A similar theorem was proved for tree maps in [7].

We extend the above to graph self-maps (see §3). A motivation for studying graph maps is that higher-dimensional dynamics can often be reduced to one-dimensional dynamics: this is the case in the study of the structure of attractors of a diffeomorphism, the quotient maps generated by maps on manifolds with an invariant foliation of codimension one and the dynamics of pseudo-Anosov homeomorphisms on a surface.

Throughout this paper, by a *graph*, we mean a *connected* compact one-dimensional polyhedron, and a *tree* is a graph which contains no loops. For a graph G, we denote the sets of endpoints and of branch points of G by $\mathrm{E}(G)$ and $\mathrm{B}(G)$, respectively. A *map* f is a continuous function; f^0 is the identity map, and for every $n \geq 0$, $f^{n+1} = f^n \circ f$. We denote by $\mathrm{Fix}(f)$ and $\mathrm{Per}(f)$ the sets of fixed points and of periodic points of f, respectively. A subset K of X is invariant under $f : X \to X$ if $f(K) \subseteq K$, $\mathrm{Int}\, K$ and $\mathrm{Cl}\, K$ denote the interior and closure of K in X, and the orbit of $x \in X$ under f is $\mathrm{Orb}_f(x) = \{f^n(x) \mid n \geq 0\}$.

For a natural number S, N_S denotes the least common multiple of the positive integers less than or equal to S.

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2. Preliminaries

An onto map $f : X \to X$ is called (*topologically*) *transitive* if any of the following equivalent conditions holds.

(i) There exists a point with dense orbit.
(ii) Whenever U, V are non-empty open sets, there exists an $n \geq 1$ such that $f^{-n}(U) \cap V \neq \emptyset$.
(iii) The only closed invariant set K with $\operatorname{Int} K \neq \emptyset$ is $K = X$.

We note that if f^n is transitive for some n, then so is f.

A map f is *totally transitive* if f^n is transitive for all $n \geq 1$. A transitive map is not always totally transitive. On the other hands, it is well known that for a transitive graph map with periodic points, the set of periodic points is dense. Therefore for such a map f, the nth power f^n has dense periodic points for $n \geq 1$.

A map $f : X \to X$ is called *topologically mixing* if for every pair of non-empty open sets U and V, there exists an $N \geq 1$ such that $f^{-n}(U) \cap V \neq \emptyset$ for $n \geq N$. A topological mixing map on a compactum is in general totally transitive. It is also known that a totally transitive graph map with periodic points is topologically mixing.

R. Roe [7] showed a decomposition theorem for tree maps for which the set of periodic points is dense. It is slightly re-worded here. The case of interval maps was proved earlier by Marge-Martin [2].

Theorem 2.1 ([7, Theorem 5]). *Let $f : T \to T$ be a tree map for which the set of periodic points is dense. Let $N_{\mathrm{E}(T)} = \mathrm{LCM}\{2, 3, \ldots, \#\,\mathrm{E}(T)\}$. Then there exists a collection (perhaps finite or empty) $\{J_1, J_2, \cdots\}$ of subtrees of T with disjoint interiors such that*

(i) $f^{N_{\mathrm{E}(T)}}(J_i) = J_i$ *for* $i \geq 1$,
(ii) $f^{N_{\mathrm{E}(T)}}|_{J_i} : J_i \to J_i$ *is totally transitive for* $i \geq 1$, *and*
(iii) $f^{N_{\mathrm{E}(T)}}(x) = x$ *for* $x \in T \setminus \bigcup_i J_i$.

Remark. His proof of Theorem 5 and Lemma 1 in [7] show totally transitivity of $f^{N_{\mathrm{E}(T)}}|_{J_i} : J_i \to J_i$ above.

Let G be a graph, $x \in G$, and U an open connected neighborhood of x in G whose closure is a tree. The number of components of $U \setminus \{x\}$ is called the *valence* of x and is denoted by $\mathrm{v}(x)$ and we set $\mathrm{v}(G) = \max\{\mathrm{v}(x) \mid x \in G\}$. A point of valence ≥ 3 is a branch point and of valence 1 is an endpoint.

The following theorem is a direct generalization of [1, Lemma 2] or [6, Corollary 3.2].

Theorem 2.2 (cf. [6, Corollary 3.2]). *Let $f : G \to G$ be a graph map satisfying* $\mathrm{Fix}(f) \neq \emptyset$ *and* $\mathrm{Cl\,Orb}_f(x) = G$ *for some $x \in X$. Let* $N_{\mathrm{v}(G)} = \mathrm{LCM}\{2, 3, \dots, \mathrm{v}(G)\}$. *Then one of the following occurs:*

(i) $\mathrm{Cl\,Orb}_{f^{N_{\mathrm{v}(G)}}}(x) = G$, *in which case* $\mathrm{Cl\,Orb}_{f^s}(f^k(x)) = G$ *for $s \geq 1$ and $k \geq 0$, i.e., f is totally transitive.*

(ii) $\mathrm{Cl\,Orb}_{f^{N_{\mathrm{v}(G)}}}(x) \neq G$, *in which case there exists a number p, $2 \leq p \leq \mathrm{v}(G)$ such that*

(a) $G = \bigcup_{i=0}^{p-1} \mathrm{Cl\,Orb}_{f^p}(f^i(x))$,

(b) $\mathrm{Cl\,Orb}_{f^p}(f^i(x))$ *is a subgraph of G for $0 \leq i \leq p-1$,*

(c) $\mathrm{Int\,Cl\,Orb}_{f^p}(f^i(x)) \cap \mathrm{Int\,Cl\,Orb}_{f^p}(f^j(x)) = \emptyset$ *for $0 \leq i < j \leq p-1$,*

(d) $f(\mathrm{Cl\,Orb}_{f^p}(f^i(x))) = \mathrm{Cl\,Orb}_{f^p}(f^{i+1}(x)) \pmod p$, *and*

(e) $\mathrm{Cl\,Orb}_{f^{pk}}(f^i(x)) = \mathrm{Cl\,Orb}_{f^p}(f^i(x))$ *for $k \geq 1$ and $0 \leq i \leq p-1$, i.e., $f|_{\mathrm{Cl\,Orb}_{f^p}(f^i(x))}$ is totally transitive for $0 \leq i \leq p-1$.*

3. Results

Here is our main theorem.

Theorem 3.1. *Let $f : G \to G$ be a graph map for which the set of periodic points is dense. Then there exist a natural number N and a collection (perhaps finite or empty) $\{G_1, G_2, \cdots\}$ of subgraphs of G with disjoint interiors such that*

(i) $f^N(G_i) = G_i$ *for $i \geq 1$,*

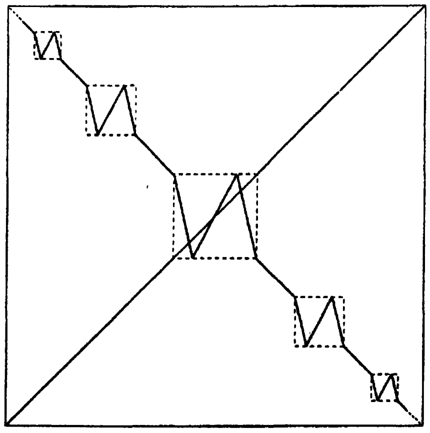
(ii) $f^N|_{G_i} : G_i \to G_i$ *is totally transitive (i.e., topologically mixing) for $i \geq 1$, and*

(iii) $f^N(x) = x$ *for $x \in G \setminus \bigcup_i G_i$.*

Remark. For simplicity, we used the Roe decomposition theorem for tree maps (Theorem 2.1) in our proof. We are able to prove Theorem 3.1 by use of the Barge-Martin decomposition theorem for interval maps [2], [4].

4. Examples

Example 1. Let $f : [0,1] \to [0,1]$ be the map whose graph appears below, where copies of the small square converge to $\{(0,1)\}$ or $\{(1,0)\}$, $f(0) = 1$, and $f(1) = 0$. Then the closed intervals J_i which are the projective images of those squares to the first coordinate have that $f^2(J_i) = J_i$, $f^2|_{J_i}$ is totally transitive, and $f^2(x) = x$ for $x \in [0,1] \setminus \bigcup_i J_i$.



Example 2. Let S^1 be the unit circle on the complex plane. Using the map $f : [0,1] \to [0,1]$ in Example 1, we define the continuous map $g : S^1 \to S^1$ by $g(e^{2\pi i\theta}) = e^{2\pi i f(\theta)}$, where $0 \leq \theta \leq 1$. Put $H_i = \{e^{2\pi i\theta} \mid \theta \in J_i\}$. Then we have that $g^2(H_i) = H_i$, $g^2|_{H_i}$ is totally transitive, and $g^2(x) = x$ for $x \in S^1 \setminus \bigcup_i H_i$.

Example 3. Let B_3 be the bouquet defined by the one-point union on the origins of the three copies S_0, S_1 and S_2 of the unit circle S^1. Here we may write any element of B_3 by the productive coordinate $(e^{2\pi i\theta}, j)$, $0 \leq \theta \leq 1$, $j = 0,1,2$. Using $g : S^1 \to S^1$ in Example 2, define $h : B_3 \to B_3$ by $h((e^{2\pi i\theta}, j)) = (g(e^{2\pi i\theta}), j + 1 \ (\text{mod } 3))$, where $j = 0,1,2$. Put $K_i^j = H_i \times \{j\}$, where H_i as in Example 2, $i \geq 1$, and $j = 0,1,2$. Then we see that $h^6(K_i^j) = K_i^j$, $h^6|_{K_i^j}$ is totally transitive, and $h^6(x) = x$ for $x \in B_3 \setminus \bigcup_{j=0,1,2} \bigcup_i K_i^j$.

Decomposition theorem does not always hold for general spaces.

Example 4. Let B_n be the bouquet with n-petals generated by the unit circle for $n \geq 1$. Define $h_n : B_n \to B_n$ like h in Example 3. Attach, for each $n \geq 1$, the origin of B_n to the point $\{n\}$ of the half real line $\mathbb{R}^{\geq 0}$, and the one-dimensional locally finite (non-compact) polyhedron is denoted by B. The map $\hat{h} : B \to B$ is defined by $\hat{h}|_{B_n} = h_n$ for $n \geq 1$ and $\hat{h}(x) = x$ for $x \in B \setminus \bigcup_{n \geq 1} B_n$. Then the map has no decomposition in the conclusion of our theorem.

Example 5. Let $h_n : B_n \to B_n$, $n \geq 1$, be as in Example 3. Attach, for each $n \geq 1$, the origin of B_n to the point $\{1/n\}$ of the unit interval $[0,1]$ on condition that the diameter of B_n is less than or equal to $1/n$, and the one-dimensional Peano continuum is denoted by C. The map

$\check{h} : C \to C$ is defined by $\check{h}|_{B_n} = h_n$ for $n \geq 1$ and $\check{h}(x) = x$ for $x \in C \setminus \bigcup_{n\geq 1} B_n$. Then the map also has no decomposition in the conclusion of our theorem.

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